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Competitive Bidding With Disparate Information*,†

This paper analyzes competitive bidding via sealed tenders for the case in which the bidders have different sources and amounts of information available about the value of the prize. The equilibrium pure strategies are characterized and computational methods are derived. A simple example is solved completely.

1. Introduction

Formulations in the theory of games are often faulty as realistic decision models because they lead to randomized strategies, a qualitative characteristic which is rarely evident in practice. It is interesting, therefore, that enriching the formulation with appropriate sorts of uncertainty frequently leads to the existence of equilibrium pure strategies. An instance of this phenomenon is analyzed in this paper. In common sense terms, it appears that an opponent's uncertainty about the information available to one often provides a sufficient disguise, or effective randomization, to permit one to employ a pure strategy. This feature lends special significance to the recently developed general theory of games under uncertainty; *e.g.*, see the formulation of Harsanyi [2], of which the present application is a special case. For the analysis of a related application in which the availability of strictly superior information to one of the players requires the other to use a randomized strategy, the reader can refer to an earlier paper [5].

Numerous extensions of the present results have been obtained in a rigorous general exposition by Ortega-Reichert [3], and a special case of some related interest has been studied by Griesmer, Levitan, and Shubik [1], and by Vickrey [4].

We consider the problem of competitive bidding via sealed tenders for the case in which the bidders have different sources and amounts of information available about the value of the prize. (Cf. [5] in which one bidder has perfect information.) The problem is formulated as a two-person, variable-sum, non-cooperative game.

2. Formulation

Suppose that two parties, called *1* and *2*, will bid for a prize of monetary value v which is not known with certainty by either party. For simplicity, assume that both parties initially assess the same prior probability density, $g(v)$, for the value of the prize v. Then, before the bidding, each party i observes an outcome θ_i of a random variable $\tilde{\theta}_i$ distributed with the conditional density $h_i(\theta_i \mid v)$. We assume that conditional on v, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are independent. Applying Bayes' Rule, each party can obtain a posterior density for the value of the prize, say

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$$g_i(v \mid \theta_i) = h_i(\theta_i \mid v)g(v) \Big/ \int_{-\infty}^{\infty} h_i(\theta_i \mid v)g(v)\, dv; \tag{1}$$

as well as a posterior marginal density for his opponent's observation, say

$$f_{ji}(\theta_j \mid \theta_i) = \int_{-\infty}^{\infty} h_j(\theta_j \mid v)g_i(v \mid \theta_i)\, dv. \tag{2}$$

For notational convenience, denote the posterior marginal distribution functions by

$$F_{ji}(\theta_j \mid \theta_i) = \int_{-\infty}^{j\theta} f_{ji}(\xi \mid \theta_i)\, d\xi; \tag{3}$$

and define

$$\varphi_{ji}(\theta_j \mid \theta_i) = F_{ji}(\theta_j \mid \theta_i)/f_{ji}(\theta_j \mid \theta_i). \tag{4}$$

Also, let $\bar{v}(\theta_1, \theta_2)$ be the expected value of the prize v conditional on both of the observations θ_1 and θ_2; say,

$$\bar{v}(\theta_1, \theta_2) = \int_{-\infty}^{\infty} v[h_2(\theta_2 \mid v)g_1(v \mid \theta_1)/f_{21}(\theta_2 \mid \theta_1)]\, dv. \tag{5}$$

The bracketed expression in (5) is one of the formulae for the conditional density of $(v \mid \theta_1, \theta_2)$.

Each of the functions (1)–(5) is assumed to be known to both parties, but *1* does not know θ_2 and *2* does not know θ_1.

Our aim is to identify the equilibrium pure strategies when they exist; say, $p_i(\theta_i)$ is the bid to be made by party i if he observes θ_i. Supposing that for each party the posterior distribution (1) of the prize v is stochastically ordered by the observation θ_i, it can be shown that each pure strategy function $p_i(\theta_i)$ is monotonic; consequently, an inverse function $\Pi_i(p_i)$ exists satisfying $\Pi_i(p_i(\theta_i)) = \theta_i$. We will assume further that each inverse function Π_i is differentiable.

Finally a utility function for money which is linear in money is assumed for each party, so that each chooses his bidding strategy to maximize his expected net gain if he should win, given his opponent's strategy.

3. The Equilibrium Conditions

Suppose *2* were to choose $p_2(\theta_2)$ as his pure strategy function. Then, given θ_1, *1* should choose his bid price $p_1(\theta_1)$ to

$$\text{Maximize } (p_1): \int_{-\infty}^{\Pi_2(p_1)} [\bar{v}(\theta_1, \theta_2) - p_1] f_{21}(\theta_2 \mid \theta_1)\, d\theta_2 . \tag{6}$$

Symmetrically, the choice of a pure strategy function $p_1(\theta_1)$ by *1* poses a similar problem for *2*. The differential necessary conditions for the simultaneous solution of these two problems are the following:

$$\begin{aligned} 0 &= [\bar{v}(\theta_1, \Pi_2(p_1)) - p_1] f_{21}(\Pi_2(p_1) \mid \theta_1)\Pi_2'(p_1) - F_{21}(\Pi_2(p_1) \mid \theta_1), \\ 0 &= [\bar{v}(\Pi_1(p_2), \theta_2) - p_2] f_{12}(\Pi_1(p_2) \mid \theta_2)\Pi_1'(p_2) - F_{12}(\Pi_1(p_2) \mid \theta_2). \end{aligned} \tag{7}$$

Equivalently, substituting $\Pi_i(p_i)$ for θ_i and using a common variable $p = p_1 = p_2$ converts (7) into a pair of functional equations for $\Pi_1(p)$ and $\Pi_2(p)$:

$$\varphi_{ji}(\Pi_j(p) \mid \Pi_i(p)) = [\bar{v}(\Pi_1(p), \Pi_2(p)) - p]\Pi_j'(p), \tag{8}$$

for $(i, j) - (1, 2)$ and $(2, 1)$. This condition can be further simplified by eliminating the common bracketed factor to obtain

$$\Pi'_2(p)/\Pi'_1(p) = \varphi_{21}(\Pi_2(p) \mid \Pi_1(p))/\varphi_{12}(\Pi_1(p)/\Pi_2(p)), \tag{9}$$

and then construing Π_2 as function of Π_1 yields

$$d\Pi_2/d\Pi_1 = \varphi_{21}(\Pi_2 \mid \Pi_1)/\varphi_{21}(\Pi_1 \mid \Pi_2). \tag{10}$$

This is the essence of the matter for computational purposes, since solution of the differential equation (10) yields $\Pi_2(p)$ in terms of $\Pi_1(p)$ and with this relation in hand one can then solve (8) with $(i, j) = (2, 1)$ as a differential equation for $\Pi_1(p)$. Constants of integration must be determined to ensure the equilibrium conditions (6) or (7) for all (θ_1, θ_2).

4. An Example

Suppose that the two parties have the same types of information available, meaning that h_1 and h_2 are identical functions and, therefore, that φ_{12} and φ_{21} are identical functions. In this case, a solution to (10) is $\Pi_2(p) = \Pi_1(p)$. Further, if the common prior assessment g is a diffuse Normal density and the observations θ_i are each Normally distributed with mean v, then each posterior marginal density f_{ji} is a Normal density function with mean Π_i and a common standard deviation, say σ; hence, $F_{ji}(\Pi_1 \mid \Pi_1) = \frac{1}{2}$, $f_{ji}(\Pi_1 \mid \Pi_1) = 1/\sigma\sqrt{2\pi}$, $\varphi_{ji}(\Pi_1 \mid \Pi_1) = \sigma\sqrt{\pi/2}$, and $\bar{v}(\Pi_1, \Pi_1) = \Pi_1$. The solution to (8) is therefore $\Pi_1(p) = \Pi_2(p) = p + \sigma\sqrt{\pi/2}$ and the optimal strategy functions are $p_i(\theta_i) = \theta_i - \sigma\sqrt{\pi/2}$, $i = 1, 2$. The strategy amounts to bidding the 0.038 fractile of one's posterior distribution for v.

Additional examples are given by Ortega-Reichert [3].

Robert B. Wilson
Stanford University

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