MANAGEMENT SCIENCE
Vol. 13, No. 11, July, 1967
Printed in U.S.A.

COMPETITIVE BIDDING WITH ASYMMETRIC INFORMATION*[1]

ROBERT B. WILSON

Stanford University

This paper analyzes the problem of competitive bidding (via sealed tenders) under uncertainty when one of the parties knows the value of the prize with certainty. A real problem of this type has been reported in a case study by Woods. The equilibrium strategies are derived and characterized for computational purposes in terms of the solution to a related differential equation. An example is solved in detail and found to be rather surprising in terms of the relative strength it imputes to the position of the party with better information.

1. Introduction

In his dissertation [3], Donald H. Woods has studied an interesting real instance of competitive bidding under uncertainty with asymmetrical information. The purpose of this paper is to formulate the problem reported by Woods in terms of a non-cooperative, variable-sum game, characterize the equilibrium strategies, and then analyze the structure of an example (with rather surprising results).

Woods' case study describes the deliberations of two major oil companies bidding via sealed tender for rights to an offshore parcel. One company already owns oil rights to a contiguous parcel and therefore, by drilling offset control wells on the boundary, has obtained nearly complete information on the value of the rights. The other company, however, has only imperfect seismic and other information on which to act.

In the end, the latter decided not to bid; on the grounds that, were the competitor merely to bid the value of the rights, it could win the bidding only by losing money on the rights. On the other hand, the company with perfect information bid for a nominal profit and won. But we shall not take these answers as final; indeed, at the very least it is clear that both strategies ignore the variable-sum aspect of the game.

Woods has summarized the problem succinctly: if the value of a prize is uncertain, how much should one bid against an opponent who knows it for certain? It is worth mentioning that the problem is essentially the same if the opponent has only better, but still imperfect, information derived from an experimental observation which is correlated with the true value of the rights.[2]

* Received October 1966 and revised January 1967.

[1] This study was supported in part by funds made available by the Ford Foundation to the Graduate School of Business, and in part by funds made available by the Atomic Energy Commission to the Operations Research Program, Stanford University. However, the conclusions, opinions, and other statements in this publication are those of the author and are not necessarily those of the Ford Foundation, nor of the Atomic Energy Commission.

[2] Cf. Wilson, R. B., "Competitive Bidding with Disparate Information", Graduate School of Business, Stanford University, Working Paper No. 114, October 1966.

The analytical methodology we shall employ is similar to Vickrey's in his work on auction markets [2].

2. Formulation

The problem will be formulated as a two-person, non-cooperative, variable-sum game. To begin, it is clear that the party with imperfect information (say, party, *1*) must employ a randomized strategy; but, the party with perfect information (say, party *2*) can employ pure strategies, since its bid is effectively randomized from *1*'s point of view by the intrinsic uncertainty about the value of the prize.[3]

Let v denote the value of the prize, and assume that *1* has assessed a probability density function $f(v)$ for v over the domain $(-\infty, +\infty)$, with a corresponding distribution function $F(v)$. Further, suppose that *1* will choose its bid p via a randomized strategy with density function $g(p)$, and corresponding distribution function $G(p)$, to be determined. For notational convenience, we shall often employ the related functions $\varphi(v) = F(v)/f(v)$ and $\psi(p) = G(p)/g(p)$.

On the other hand, *2*'s strategy is to select a bid $q(v)$ as a function of the true value v of the prize. We shall assume that *2*'s bid function $q(v)$ is differentiable and strictly monotonic, so that the inverse function $v(\cdot)$ exists and satisfies $v(q(v)) = v$.

The Nash equilibrium strategies [1] are of exclusive interest in the following analysis. That is, conditional on his opponent's strategy, each party is presumed to choose his own strategy to maximize his expected utility. Further, each party is assumed to have a utility function which is linear in his net gain (the difference between the prize and his bid if he wins, zero if he loses).

3. Party *2*'s Strategy

If *2* knows that *1* will employ a randomized strategy with distribution function $G(p)$; then, given the value v of the prize, he will choose his bid q to maximize $[v - q]G(q)$, since $G(q)$ is his probability of winning. The differential necessary condition for the maximum is

$$[v - q]g(q) - G(q) = 0; \tag{1}$$

hence, the bidding function $q(v)$ for *2* is determined in terms of its inverse function $v(\cdot)$ by $v = v(q(v))$ where

$$v(\xi) = \psi(\xi) + \xi, \tag{2}$$

provided it is monotonic.

4. Party *1*'s Strategy

If *1* knows that *2*'s strategy will be determined by the condition (1), then for any bid p that he might make his expected return is $E_{-\infty}^{v(p)}(\mathbf{v} - p)$, where E_a^b is the partial expectation operator for the random variable $\mathbf{v}$ over the domain

[3] If this is not clear, it will become evident in the ensuing analysis.

(a, b). Thus, *1*'s randomized strategy is determined from the following variational problem:

$$\text{(3)} \quad \text{Maximize} \qquad \int_{-\infty}^{\infty} E_{-\infty}^{v(p)}(\mathbf{v} - p)g(p)\,dp$$

$$\text{Subject to} \qquad g(p) \geqq 0, \qquad \int_{-\infty}^{\infty} g(p)\,dp = 1.$$

The results of Zahl [5] can be applied to obtain a differential necessary condition for the solution to this problem. Let λ be a Lagrange multiplier determined by the condition $\int_{-\infty}^{\infty} g(p)\,dp = 1$; then, at a constrained maximum,

$$\text{(4)} \qquad \lambda \geqq E_{-\infty}^{v(p)}(\mathbf{v} - p) + \int_{p}^{\infty} [v(\xi) - \xi]f(v(\xi))\,d\xi,$$

or equivalently,

$$\text{(5)} \qquad \lambda \geqq E_{-\infty}^{v(p)}(\mathbf{v} - p) + E_{v(p)}^{\infty}(q'(\mathbf{v})[\mathbf{v} - q(\mathbf{v})]),$$

for all p, with strict inequality obtaining only if $g(p) = 0$. The derivation of (4) depends upon the formal property

$$\text{(6)} \qquad \begin{aligned} \partial v(\xi)/\partial g(p) &= 0 && \text{if } \xi < p, \\ &= 1/g(\xi) && \text{if } \xi > p, \end{aligned}$$

and (5) is derived from (4) by changing the variable of integration.

The condition (5) has an interesting interpretation: over the set of possible bids $\{p \mid g(p) > 0\}$ by *1*, the sum of the expected returns to the two parties would be constant (i.e., λ) only if for each prize v *2*'s return $[v - q(v)]$ were weighted by $q'(v)$. Thus, the situation differs from a constant or zero-sum game via a dependence on a variational property of *2*'s strategy.

It is worth remarking before proceeding further that the conditions (4) and (5) implicitly assume a zero probability for ties, and in particular, are not precise analogues of the case in which $g(p)$ must, or can, be a mass function. The use of (6) depends upon this restriction.

5. Characterization of the Equilibrium Strategies

In most cases the conditions (4) and (5) can be considerably simplified by eliminating the Lagrange multiplier λ; cf. Yaari [4]. Here we shall pursue one approach for accomplishing the simplification.

Assume in (5) that the density $g(p)$ is differentiable and everywhere positive in a neighborhood of p; then, (5) can be construed as an identity and differentiating both sides with respect to p yields

$$\text{(7)} \qquad 0 = [v(p) - p]\,[1 - q'(v(p))]f(v(p))v'(p) - F(v(p)).$$

Now, $q'(v(p)) = 1/v'(p)$, so from (2) it follows that (7) can be put in the form,

$$\text{(8)} \qquad \varphi(\psi(p) + p) = \psi(p)\psi'(p).$$

This condition is our major result; for, having determined $\psi(p)$ from the differential equation (8), we can then determine the two parties' strategies from the further conditions,

$$(9) \qquad G(p)/G'(p) = \psi(p)$$

subject to

$$(10) \qquad G'(p) \geqq 0, \qquad G(-\infty) = 0, \qquad G(+\infty) = 1,$$

and

$$(11) \qquad v - q(v) = \psi(q(v)).$$

Note that the two endpoint conditions in (10) together determine the total of two constants of integration in (8) and (9). Of course, if instead of $(-\infty, +\infty)$ the range of **v** is restricted to $[a, b]$, then (8) and (11) will induce a range for p of $[c, d]$ where

$$(12) \qquad a - c = \psi(c), \qquad b - d = \psi(d),$$

in which case the endpoint conditions take the form

$$(13) \qquad G(c) = 0, \qquad G(d) = 1;$$

this procedure is exemplified in Section 6 below. The differential equation (9) can be integrated directly to yield

$$(14) \qquad G(p) = K \cdot \exp\left\{ \int_{-\infty}^{p} [1/\psi(\pi)]\, d\pi \right\}, \qquad K > 0,$$

where K is a constant of integration, which is more useful if ψ can be determined analytically from (8).

In general, (8)–(11) must be solved by numerical methods, but in the next section a simple example is examined which is amenable to analytical solution. In any case, the validity of (8) depends upon the assumptions used to derive (7) from (5) and consequently care must be exercised to guard against extraneous solutions derived from (8): an example is given in the last section.

6. An Easy Example

Suppose that **v** has the uniform distribution over [0, 1], so that $f(v) = 1$, $F(v) = v$, and $\varphi(v) = v$. Then (8) yields $\psi(p) = \alpha p$ where $\alpha = (1 + 5^{1/2})/2$, (11) yields $q(v) = (1 + \alpha)^{-1}v$, and (12) yields $[c, d] = [0, (1 + \alpha)^{-1}]$; hence, (9) or (14) yields $G(p) = Kp^{1/\alpha}$, where $K = (1 + \alpha)^{1/\alpha}$, and from (4) or (5) one obtains $\lambda = \frac{1}{2}\alpha^{-3}$. Further computation shows that the expected return to *1* is $\frac{1}{2}\alpha^{-6}$, and to *2* it is v^{α}/α; also, marginally with respect to *1*'s prior distribution on **v**, the expected return to *2* is $1/\alpha^3$.

The features of the solution are perhaps surprising. The marginal expected return to *2* is $\lambda^{-1} = 8.5$ times the expected return to *1*; indeed, *2* bids to take an $\alpha^{-1} = 62\%$ profit if he should win! (This might be called a "golden section" strategy, since in fact α^{-1} is the well-known limit of the ratio of successive Fibonacci numbers.) Note that although *2*'s bid is uniformly distributed over

$[0, (1 + \alpha)^{-1}]$, *1*'s randomized strategy is moderately biased in favor of small bids.

7. An "Ornery" Example

Suppose that $-\mathbf{v}$ has the exponential distribution over $(0, \infty)$, so that $f(v) = \theta e^{\theta v}$, $F(v) = e^{\theta v}$, and $\varphi(v) = \theta^{-1}$ for $v < 0$. Then one solution of (8) is $\psi(p) = (2p/\theta)^{1/2}$, which must be extraneous since $p < 0$, $\theta > 0$. Whether this is due to the fact that φ is not strictly monotonic, or to the optimality of a mass function for $g(p)$ is not clear.

References

1. Luce, R. D., and H. Raiffa, *Games and Decisions*, John Wiley and Sons, Inc., New York, 1957.
2. Vickrey, W., "Counterspeculation, Auctions, and Competitive Sealed Tenders," *J. of Finance*, Vol. 16, No. 1, March 1961, pp. 8–37.
3. Woods, D. H., *Decision Making under Uncertainty in Heirarchial Organizations*, dissertation, Harvard Business School, Boston, 1965.
4. Yaari, M., "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," *Rev. of Economic Studies*, Vol. 32, No. 2, pp. 137–150.
5. Zahl, S., "An Allocation Problem with Applications to Operations Research and Statistics," *Operations Research*, Vol. 11, No. 3, May–June, 1963, pp. 426–441.