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Prices and the winner's curse

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We usually assume that increases in supply, allocation by rationing, and exclusion of potential buyers reduce prices. But all these activities raise the expected price in an important set of cases when common-value assets are sold. Furthermore, when we make the assumptions needed to rule out these "anomalies" for symmetric buyers, small asymmetries among the buyers necessarily cause the anomalies to reappear. Our results help explain rationing in initial public offerings and outcomes of spectrum auctions. We illustrate our results in the "Wallet Game" and in another new game we introduce, the "Maximum Game."

1. Introduction

■ "Increases in supply lower prices." "It is never profitable to commit to rationing at a price at which there is surely excess demand." "Excluding potential buyers cannot raise prices."

Although economists from Veblen (1899) to Becker (1991) have shown counterexamples, these statements are still often taken almost for granted. However, this article shows why it is perfectly reasonable for these statements to be false in auction markets, without any special assumptions, and when this is likely to happen.

To understand our results, it is important to understand how a bidder determines the maximum he will be willing to pay for an asset. If a buyer's estimate of an asset's value is affected only by his own perceptions and not by the perceptions of others, he should be willing to pay up to his valuation. This is the Adam Smith world, where a buyer can easily maximize his utility given any set of prices, and a firm can easily maximize its profits. In this sort of "private value" model, the statements in the first paragraph are generally true.[1]

But in many important markets, others' perceptions are informative. The extreme cases are "common-value" assets, or assets all buyers would value equally if they shared the same information. Financial assets held by noncontrol investors may be the best example; oil fields are commonly cited. Most assets have both a private and a common value element, particularly if imperfect substitutes exist. For example, a house's value will have both common and idiosyncratic (private) elements.

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[1] They are always true for ascending auctions, on which we focus. Counterexamples can be constructed for first-price sealed-bid auctions with affiliated private values. See Pinkse and Tan (2000).

With common values, buyers may find it prudent to exit an ascending price auction at more or less than their pre-auction estimate of the value, so the statements in the first paragraph are often false in common-value auctions.

The reason is the "winner's curse." Buyers must bid more conservatively the more bidders there are, because winning implies a greater winner's curse. This effect can more than compensate for the increase in competition caused by more bidders, so more bidders can *lower* expected prices. Conversely, adding more supply, and/or rationing, creates more winners, so reduces the bad news learned by winning, and so may raise bids enough to *increase* expected prices. This article shows when this happens and why it is surprisingly often.

A good example is provided by the market for initial public offerings (IPOs). Rather than being priced to clear the market, many IPOs are made at prices that guarantee excess demand. By pricing low enough so that everyone will want to buy, potential shareowners are absolved of the winner's curse of being buyers only when they are among the most optimistic investors. This allows the pooling price to be quite high and, under quite reasonable conditions, as high or higher than the expected price in a standard auction. Our analysis similarly shows why restaurateurs, theater owners, and football teams may profitably choose to price at levels that will likely require rationing.

Likewise, in a descending-price (or Dutch) auction, in which the price is lowered continuously until a bidder accepts the current price, selling to *more* bidders can *lower* the expected price, that is, can mean waiting *longer* on average until someone jumps in to buy. The result is analogous to the phenomenon that giving a seminar to a larger audience can yield fewer questions: the more listeners, the more each one wonders, "If my question is so good, why hasn't someone else asked it?" With more bidders, each worries more that "If the price is fair, why hasn't someone else bought it?" These effects can dominate both the competitive effect that more competition leads (*ceteris paribus*) to more aggressive behavior, and the direct effect that (even holding individual behavior constant) more people results in faster bidding and a faster flow of questions.

Our results are especially likely in asymmetric "almost-common-value" ascending auctions in which some competitors have a small advantage, because the other bidder(s) then face an exacerbated winner's curse. Again this point is similar to the observation that the presence of a single audience member who is thought to be smarter than her colleagues can significantly dampen everyone else's questioning.

For a more significant application, consider the A and B band spectrum ascending auction held in 1994–1995 by the U.S. Federal Communications Commission. Pacific Telesis was the natural buyer of the single Los Angeles license available for sale[2] and was able to acquire it very cheaply. Markets where two licenses were sold generally yielded more competitive prices relative to their demographic characteristics.[3] Even where one bidder had an advantage, the prices of both licenses were determined by aggressive competition for the second license. So prices were higher, even though the third-highest bid set the price in these markets, while the second-highest bid set the price in Los Angeles.[4]

The recent European "third-generation" mobile-phone license auctions also yielded low prices in ascending auctions when the number of clearly advantaged incumbent bidders equalled the number of licenses. The license prices were over 100 euros per capita each in the United

[2] AT&T was ineligible to bid, and PacTel benefited from its name recognition and experience in California, as well as its familiarity with the California wireless market in which it was a duopolist before its spinoff of its cellular subsidiary, Airtouch. No one knew what PCS licenses were really worth, but it is fair to say that the Los Angeles license was worth more to PacTel than to anyone else.

[3] For example, Chicago yielded about $31 per head of population for each of the two licenses, compared with less than $26 per head of population for Los Angeles' single license, in spite of Chicago's inferior demographics. (The famous long commutes of Angelenos and the area's population density make it a particularly desirable place to own a wireless telephone franchise.) The single New York license yielded only $17 per head of population. See Klemperer and Pagnozzi (2002).

[4] In our model, one bidder has a small private-value advantage. Bulow, Huang, and Klemperer (1996) analyze a similar almost-common-value model in which one bidder has a small ownership advantage and show that in this context, too, a small advantage can lead to large profits for the advantaged bidder and to low revenues for the seller.

Kingdom, where there were five licenses and just four incumbent bidders, and in Germany, where there were six licenses and four incumbents, but they were less than 35 euros per capita in the Netherlands, where there were equal numbers (five) of licenses and incumbents.[5,6]

Our results for the asymmetric case apply even when the differences between players' actual valuations are arbitrarily small. However, we do assume that the advantaged player is much the most likely to have the highest actual valuation, so in this sense our asymmetric results do depend on a particular firm being clearly advantaged—as in the examples above of the Los Angeles and European spectrum license auctions.

Section 2 sets up a simple model of a standard ascending auction among bidders with "almost" common values.[7] Section 3 develops the intuition behind our results. Section 4 shows when higher prices are associated with selling more units in the symmetric case.[8] Section 5 shows that the results are dramatically different when bidders are asymmetric: greater supply *raises* price precisely when it does not with symmetric bidders!

Section 6 shows when rationing, as in IPOs, is an *optimal* mechanism. Section 7 shows when restricting participation in an auction can raise expected revenues.[9] Section 8 considers first-price auctions.

Section 9 extends the model to more bidders and more units, and to more general value functions. In particular, we introduce another natural structure of valuation function, the "Maximum Game," which yields extreme "perverse" results, independent of the distribution of bidder types: if n symmetric bidders have a common value for an object equal to the maximum of their signals, then in a pure ascending auction every bidder bids up to his own signal, and the price equals the actual second-highest of all n signals. But if the object is simply rationed at a fixed price, every bidder (even one with the lowest possible signal) will be willing to pay the expected highest of the other $n-1$ signals, which is *greater* in expectation. Other "perverse" results of prices decreasing in demand and increasing in supply in the symmetric case are equally easy to obtain. Our main model uses an additive value function, as in the "Wallet Game,"[10] but the Maximum Game structure underlies models such as Matthews (1984), Harstad and Bordley (1996), Levin (2001), and Parlour and Rajan (2001), which explains why they have obtained some results related to ours.[11] Section 10 concludes.

2. The model

■ We use the simplest possible model to make our points: each of three risk-neutral potential bidders observes a private signal t_i independently and identically distributed according to the distribution $F(t_i)$, $i = 1, 2, 3$. We assume $F(\cdot)$ has a strictly positive continuous finite derivative

[5] Our formal model does not allow for more than one advantaged bidder, and it is unclear how far our results generalize in this direction (see Section 9). But the results of the model would probably be reinforced by incorporating entry and bidding costs, which were probably significant in the European auctions, and, of course, our results would be reinforced by larger advantages than those we consider. (See Avery (1998), Daniel and Hirshleifer (1995), Hirshleifer (1995), and Klemperer (2000) for a general discussion of bidding costs.)

[6] Klemperer was the principal auction theorist advising the U.K. government on its auction. Bulow was also an advisor. See Binmore and Klemperer (2002) and Klemperer (2002a, 2002b).

[7] Recent articles that use models similar to ours are Avery and Kagel (1997) and de Frutos and Rosenthal (1998), but these works address concerns very different from ours. See also Bikhchandani and Riley (1991). Klemperer (1998) and de Frutos and Pechlivanos (2002) build on the current article.

[8] In our model, the ascending auction always allocates a single object to the highest valuer—see Maskin (1992) for more general conditions under which this holds—but does not always allocate multiple objects efficiently.

[9] Externalities between bidders (see Jehiel and Moldovanu (1996, 2000) and Caillaud and Jehiel (1998)) provide a different reason why an auctioneer may want to limit the participation of bidders. In Compte and Jehiel's (forthcoming) mixed private/common value setting, bidders have different amounts of information, and adding bidders may hurt revenues.

[10] The Wallet Game was introduced in Bulow and Klemperer (1997) and Klemperer (1998).

[11] Matthews focuses on first-price auctions and obtains results related to our Section 7. Harstad and Bordley and Parlour and Rajan examine rationing mechanisms and obtain results related to our Section 6. All these articles use more complex models with affiliation. Levin obtains results similar to our Section 4 results, while Krishna and Morgan (2001) obtain results equivalent to the symmetric case of our Section 7. Our article shows why results like ours and theirs can arise, and it shows they are surprisingly likely and that affiliation is not important for them.

$f(\cdot)$ everywhere on its range, and the lowest possible signal is $\underline{t} > 0$, so $F(\underline{t}) = 0$. Conditional on all the signals, the expected value, v_i, of a unit to i is

$$v_1 = (1+\alpha_1)t_1 + t_2 + t_3$$
$$v_2 = t_1 + (1+\alpha_2)t_2 + t_3$$
$$v_3 = t_1 + t_2 + (1+\alpha_3)t_3.$$

That is, each unit has a *common value*, $v = \sum_{i=1}^{3} t_i$, to all the bidders, plus a *private value*, $\alpha_i t_i$, to each bidder i. We will focus on two cases, "the symmetric case" in which $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$ and "the asymmetric case" in which $\alpha_1 > \alpha_2 = \alpha_3 = \alpha > 0$. In the latter case we will refer to bidder 1 as the "advantaged" bidder and to bidders 2 and 3 as "disadvantaged" bidders. To make our points most starkly and straightforwardly, we consider the case in which the private-value components, that is, the α_i's, are all small, and we consider the asymmetric case in which α/α_1 is also small; we state our results throughout for the limits in which $\alpha_i \to 0, \forall_i$, so $v_1 \approx v_2 \approx v_3 \approx v$ and, for the asymmetric case, $\alpha/\alpha_1 \to 0$. Thus the limit of the symmetric game is just the "Wallet Game," in our previous terminology.[12]

Since α_1 is small, bidder 1's private-value "advantage" in the asymmetric case is small. But because α/α_1 is also small, bidder 1 is much the most likely to have the highest actual value for a unit (and as $\alpha/\alpha_1 \to 0$, bidder 1 always has the highest actual value). So bidder 1's advantage is not small in this sense.

No bidder wants more than one unit. We consider two cases: the auctioneer has one unit to sell, and the auctioneer has two units to sell. (The number of units is common knowledge.)

We assume a conventional ascending-bid "English" auction in which the price, p, starts at zero and rises continuously until the number of bidders still willing to pay the current price equals the number of units the auctioneer has for sale. Each bidder observes the price at which any other bidder drops out, and a bidder who drops out cannot reenter the auction.[13]

Each player's pure strategy specifies the price level up to which he will remain in the bidding, as a function of his private signal and of the price (if any) at which any other player quit previously. We look for perfect Bayesian equilibria, but we restrict attention to those equilibria in which symmetric bidders follow symmetric strategies. We also restrict attention to equilibria in which each bidder stays in the bidding just so long as he would be happy to find himself a winner, and stops bidding at that price at which he would be just indifferent were he to find himself a winner on the assumptions that any opponent(s) who drop out to make him a winner are of their lowest possible types assuming they have followed the equilibrium strategies prior to the current price.[14] Appendix B shows that this yields a unique (perfect Bayesian) equilibrium.

We write the actual ith-highest signal among all the potential bidders as $t_{(i)}$, and we write the highest and second-highest of two bidders as $t_{(1,2)}$ and $t_{(2,2)}$ respectively.

We write $h(t_i) \equiv f(t_i)/[1 - F(t_i)]$ for i's hazard rate, and we compress notation by writing h_i for $h(t_i)$. Bidder i's *marginal revenue*[15,16] is defined as

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[12] A classroom example involves three students, each of whom knows only the amount of money, t_i, in his or her own wallet. They compete for a prize equal to the combined contents of their wallets, $v = t_1 + t_2 + t_3$. See Klemperer (1998).

[13] Auction theorists call this a "Japanese auction." See Bikhchandani and Riley (1993) or the working paper version of our article (Bulow and Klemperer, 1997) for a formal description.

[14] These restrictions both avoid trivialities (although there are other equilibria, they do not seem very natural—see Appendix B) and greatly reduce the technical burden: Bikhchandani and Riley (1993) show how cumbersome and lengthy is a fully general analysis of even the completely symmetric version of our model. They too make assumptions to obtain a unique equilibrium (the same equilibrium as ours, though their model is a special case of ours).

[15] The marginal revenue of bidder i with signal t_i is exactly the marginal revenue extracted from the customer who is the same fraction of the way down the distribution of potential buyers of a monopolist whose demand is such that it has $q = 1 - F(t_i)$ customers who have values $\geq p = v_i(t_i)$ (i.e., there are $F(t_i)$ customers with values less than $v_i(t_i)$).

[16] Bulow and Roberts (1989) first showed how to interpret independent private-value auctions in terms of marginal revenues, and Bulow and Klemperer (1996) extended their interpretation to more general settings such as this one. This

$$MR_i = v_i - \frac{1}{h_i}\frac{\partial v_i}{\partial t_i}.$$

Note that since (we assumed) the α_i are all small, $MR_i \approx v - (1/h_i)$.

We write $E(t)$ and $E(v)$ for the (unconditional) expectations of t and v, and $E(t \mid t \geq z)$ for the expectation of the signal t conditional on its exceeding z, etc. Recall that if hazard rates are increasing (decreasing), the expected "time" $(t - z)$ to arrival of a signal, t, conditional on it not yet having arrived at time z, is decreasing (increasing) in z, that is,

Lemma 1. $E(t - z \mid t \geq z)$ is increasing (decreasing) in z if the hazard function h is decreasing (increasing).

Proof. See Appendix B.

3. Intuition

■ This section uses "marginal revenues" to develop the intuition behind our main results, recalling from Bulow and Klemperer (1996) that *the expected price from the auction equals the expected marginal revenue of the winning bidder(s).*[17]

□ **The symmetric case.** When bidders are symmetric, the bidder(s) with the highest signal(s) win(s) the unit(s).

Now in a pure private-value auction in which a bidder's value, v_i, just equals his own signal, t_i, i's marginal revenue $MR_i = v_i - 1/h_i = t_i - 1/h_i$ is increasing in t_i under weak assumptions that are satisfied by most standard distributions. Thus the standard single-unit auctions that assign a single unit to the high-value bidder thereby assign it to the bidder with the highest marginal revenue. Since in a two-unit auction one unit goes to the second-highest-value bidder who has a lower marginal revenue, it follows that revenue per unit is lower in the two-unit case.

However, in an (almost) pure common-value auction in which $v_i \approx v_j \approx v$, so $MR_i \approx v - 1/h_i$, the bidder with the highest signal has the highest marginal revenue only if hazard rates, h_i, are increasing in signals. This is a much more stringent condition. The intuition is that with private values, when a bidder has a higher signal it affects only his own value and marginal revenue. But with common values, when a bidder has a higher signal it also raises the other bidders' values and so raises the others' marginal revenues. So it takes a much stronger distributional condition to ensure that bidders with higher signals have higher marginal revenues.

If indeed bidders with higher signals do have higher marginal revenues in the common-value case, then the logic is the same as for the private-value case and revenue per unit is lower when selling two units than when selling one unit. But if instead bidders with higher signals have lower marginal revenues, as will be the case if hazard rates are decreasing in signals, a standard auction that sells to the bidder(s) with the highest signal(s) will sell to the *lowest*-marginal-revenue bidder(s). So in this case revenue per unit is higher when selling two units than when selling one unit.

□ **The asymmetric case.** The difference in the asymmetric case is that the bidder with the highest signal does not necessarily win a standard ascending auction.

In our case, in which it is common knowledge that bidder 1 is much the most likely to have the highest actual value, this bidder is much the most likely to win an ascending auction for a single unit. The reason is that the other bidders have greatly exaggerated "winner's curses" from beating bidder 1 and so must bid very cautiously, which also reduces 1's "winner's curse."

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interpretation allows the translation of results from monopoly theory into auction theory and so facilitates the analysis of auctions and the development of intuition about them.

[17] The result assumes that a bidder with the lowest possible signal never makes money; our asymmetric, multiunit, decreasing-hazard-rate case does give expected profits to a bidder with the lowest possible signal, but the required correction does not affect our results (see Section 5). Klemperer (1999, Appendix B) gives a simple exposition of the result.

But with increasing hazard rates, bidder 1 is *not* much more likely than bidders 2 and 3 to win an ascending auction for two units. The reason is that 2 and 3 compete against each other, symmetrically, for the second unit and do not face an abnormally large winner's curse in that competition. Because the prices of both units will be the same, the more aggressive bidding by 2 and 3 increases 1's winner's curse, which causes 1 to bid more cautiously and further reduces 2's and 3's winner's curses; we will show that when 1's value advantage is small it has only a small effect on the equilibrium.

So when three bidders compete for two units, and hazard rates are increasing, the bidders with the two highest signals who also have the two highest marginal revenues (almost) always win. But when three bidders compete for a single unit in the asymmetric case, bidder 1, who on average only has the average signal and average marginal revenue, (almost) always wins. So expected prices are higher when two units are sold.

The analysis of the decreasing-hazard-rate case is a little more complex (see Section 5), but in this case, too, the asymmetric case is the exact reverse of the symmetric case.

4. The symmetric case

■ To develop our results more formally, we begin with the symmetric case in which $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$ (but $\alpha \approx 0$).[18]

When three bidders compete for a single object, the lowest bidder, with signal $t_{(3)}$, quits when the price reaches the level where he would just be indifferent about winning if both his opponents dropped out, that is, at $(3+\alpha)t_{(3)}$. To see this, observe that conditional on winning at price $p' = (3+\alpha)t'$ (and assuming the kind of equilibrium behavior we are looking for), his inferred value of the unit equals $v' = (1+\alpha)t_{(3)} + 2t'$ and $v' > p' \Leftrightarrow t_{(3)} > t'$. So if he were to find himself a winner at any price higher than $(3+\alpha)t_{(3)}$, he would lose money; conversely, he would make money if he won at any lower price and so should not quit earlier. So the equilibrium is a separating one in which the other bidders can then infer (assuming equilibrium behavior) his actual signal, $t_{(3)}$. The next-lowest bidder then quits when the price reaches the point at which he would just be indifferent about winning were he the marginal winner, that is, were he tied for the highest signal, so he quits at $p = t_{(3)} + (2+\alpha)t_{(2)}$.[19] But $v = t_{(3)} + t_{(2)} + t_{(1)}$, so (since $\alpha \approx 0$) $p \approx v - (t_{(1)} - t_{(2)}) \Rightarrow E(p) \approx E(v) - E(t - t_{(2)} \mid t \geq t_{(2)})$.

Lemma 2. When three symmetric bidders compete for one object, the bidder with the highest signal wins and (for small α) the expected price $\approx E(v) - E(t - t_{(2)} \mid t \geq t_{(2)})$.

Proof. See Appendix B.

If instead, three bidders compete for two objects, the lowest quits in symmetric equilibrium at the price at which he would just be indifferent about winning were he the marginal winner, that is, were he tied with the second-highest signal. So the actual lowest-signal bidder with signal $t_{(3)}$ quits at the value to him if the second-highest-signal bidder has the same signal, ($t_{(3)}$), and the remaining signal equals its expected value given the two lowest signals are $t_{(3)}$, that is, $E(t \mid t \geq t_{(3)})$. So the lowest-signal bidder quits at $p = (1+\alpha)t_{(3)} + t_{(3)} + E(t \mid t \geq t_{(3)})$.[20] But $E(v) = E(t_{(3)}) + 2E(t \mid t \geq t_{(3)})$, so $E(p) \approx E(v) - E(t - t_{(3)} \mid t \geq t_{(3)})$.

Lemma 3. When three symmetric bidders compete for two objects, the bidders with the highest signals win and (for small α) the expected price $\approx E(v) - E(t - t_{(3)} \mid t \geq t_{(3)})$.

Proof. See Appendix B.

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[18] As $\alpha \longrightarrow 0$, the equilibrium of this case approaches the symmetric equilibrium of the pure common-values model.

[19] If he were to find himself a winner at any higher price he would lose money, since at price $p' = t_{(3)} + (2+\alpha)t'$ with $t' > t_{(2)}$, the inferred value of the unit equals $t_{(3)} + (1+\alpha)t_{(2)} + t'$ conditional on winning at price p', and conversely he would make money at any lower price and so should not quit before p.

[20] If either of the other bidders were to quit and leave him as a winner at any higher price, $p' = (2+\alpha)t' + E(t \mid t \geq t')$ with $t' > t_{(3)}$, he would expect to lose money, since he would then infer a unit's value to be $(1+\alpha)t_{(3)} + t' + E(t \mid t \geq t') < p'$, and conversely he would expect to profit from a victory at any lower price.

From Lemmas 2 and 3, selling two units rather than one lowers (raises) the per-unit price obtained if $E(t - t_{(3)} \mid t \geq t_{(3)})$ is greater than (less than) $E(t - t_{(2)} \mid t \geq t_{(2)})$. But for any realization of $t_{(3)}$ we know from Lemma 1 that this is true when hazard rates are increasing (decreasing). So increasing hazard rates imply that the expected price is greater with one unit, and decreasing hazard rates imply that the expected price is greater with two units.

Proposition 1. With three symmetric bidders, the expected price per unit is higher when one unit is sold than when two units are sold if hazard rates, h_i, are increasing in the signals, t_i. The expected price per unit is lower when one unit is sold than when two units are sold if hazard rates are decreasing.

The difference between the prices in the one-unit and two-unit auctions, that is, the difference between $E(t-t_{(2)} \mid t \geq t_{(2)})$ and $E(t-t_{(3)} \mid t \geq t_{(3)})$, is just the difference between the information rents of the winning bidders. Since the seller's revenue equals the social surplus less the information rent of the bidders, and the social surplus is (almost) independent of the allocation in (almost) common-value auctions, a higher information rent implies a lower price.

It is now easy to find examples in which selling more units raises the expected price. (See Example 1 below.)

Section 3 provided some intuition for our result; the key is that the expected price equals the expected marginal revenue of the winner(s), that is, of the highest-signal bidder(s) in the symmetric case. So the expected price is decreasing in the number of winners only if lower-signal bidders have lower marginal revenues. In a private-value model, the condition for lower-signal bidders to have lower marginal revenues is just that a bidder's marginal revenue is downward sloping, that is, that a monopoly firm with demand $q = 1 - F(p)$ has marginal revenue downward sloping in its *own* output.[21,22] But with common values, when a bidder has a higher signal it also raises the other bidders' values and so raises the others' marginal revenues; the corresponding condition in the common-value case is that the same firm's marginal revenue is steeper than its demand curve.[23]

Equivalently, the firm's marginal revenue must be downward sloping in an *opponent's* output; this is exactly the condition required to guarantee that firm outputs in a quantity-setting oligopoly are "strategic substitutes," in the terminology of Bulow, Geanakoplos, and Klemperer (1985a). And the assumption of strategic substitutes, while commonly made, and *perhaps* more plausible than the converse assumption of strategic complements, is not a reasonable general assumption.[24] Indeed, among the most commonly used demand curves, linear demand ($p = A - Bq \Leftrightarrow q = (A-p)/B$) yields strategic substitutes, constant elasticity demand ($p = Aq^{1/\eta} \Leftrightarrow q = (p/A)^{\eta}$, $\eta < -1$) yields strategic complements, and logarithmic demand ($p = A - (1/\lambda)\log q \Leftrightarrow q = e^{-\lambda(p-A)}$, i.e., quantity is exponential in price) yields strategic independence (neither strategic substitutes nor strategic complements) for a monopolist facing a small new entrant.

Corresponding exactly to the oligopoly cases, we have the following example.

Example 1. With uniformly distributed signals, $F(t) = (t - \underline{t})/(\bar{t} - \underline{t})$, expected price is decreasing

[21] The demand curve $q = 1 - F(p)$ is just the conventional demand curve that would be created by a very large number of buyers with values $v_i(t_i)$ when the t_i are drawn independently from the distribution $F(t_i)$. (Buyers are atomistic with total mass 1.) For more discussion of the analogy between a bidder with signal distributed as $F(t_i)$ and a market with demand curve $1 - F(p)$, see Bulow and Klemperer (1996).

[22] Note that the expected marginal revenue of the second-highest bidder is always below the expected marginal revenue of the highest bidder in a private-value model, because the second-highest value both equals the expected marginal revenue of the highest bidder and exceeds the actual marginal revenue of the second-highest bidder.

[23] If $MR_j = v - 1/h_j(t_j)$, then the slope of the "demand curve" is $\partial v/\partial t_j$, whereas the slope of marginal revenue is $\partial v/\partial t_j + (\partial h_j/\partial t_j)/(h_j(t_j))^2$. The latter is greater (less) than the former if h is increasing (decreasing). In the common-value auction this determines whether an increase in j's signal increases his marginal revenue relative to i's. In oligopoly it determines whether an increase in j's output increases i's marginal revenue and thus i's output.

[24] See Bulow, Geanakoplos, and Klemperer (1985b) for an example in which a monopolist facing a new entrant views products as strategic complements in the terminology they introduced.

in supply.[25] With constant-elasticity distributed signals, $F(t) = 1 - (t/\underline{t})^{\eta}$, expected price is increasing in supply.[26] With exponentially distributed signals, $F(t) = 1 - e^{-\lambda(t-\underline{t})}$, expected price is independent of supply.

So, just as in oligopoly, it is an empirical matter whether firms' outputs are strategic substitutes or strategic complements, so in symmetric pure common-value auctions it must be an empirical matter whether price is increasing or decreasing in supply.

The next section, however, will show that even small asymmetries can make the relationship between supply and price even less predictable.

5. The asymmetric case

■ This section will show that when the result that greater supply lowers expected price holds for the perfectly symmetric case, it can fail when there are small asymmetries between the bidders. In particular it fails if the item(s) for sale are arbitrarily close to pure common values but one bidder, say bidder 1, almost certainly has a (slightly) higher actual value than the others. We assume $\alpha_1 > \alpha_2 = \alpha_3 = \alpha > 0$, but $\alpha_1 \approx 0$ and $(\alpha/\alpha_1) \approx 0$. We begin by analyzing bidding behavior in more detail.

Lemma 4. When three bidders compete for one object in the asymmetric case (i.e., small α_i and small α/α_1), the advantaged bidder (almost always) wins and the expected price $\approx E(v) - E(t - \underline{t})$.

Proof. See Appendix B.

The logic is straightforward. Bidder i quits where he would be just indifferent about finding himself a winner, so his marginal type $\underline{t}_i$ quits at price $p = (1+\alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k$, where $\underline{t}_j$ and $\underline{t}_k$ are his expectations of j's and k's signals conditional on his winning at this price. That is, $\underline{t}_j$ is the marginal type of bidder j who is just quitting if any type of j is currently quitting,[27] and similarly for $\underline{t}_k$. Likewise, type $\underline{t}_j$ of j is in fact just quitting if and only if $p = \underline{t}_i + (1+\alpha_j)\underline{t}_j + \underline{t}_k$. So types $\underline{t}_i$ and $\underline{t}_j$ quit simultaneously if and only if $(1+\alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k = \underline{t}_i + (1+\alpha_j)\underline{t}_j + \underline{t}_k \Leftrightarrow \alpha_i\underline{t}_i = \alpha_j\underline{t}_j$, and $\underline{t}_i$ quits before (after) $\underline{t}_j$ if and only if $\alpha_i\underline{t}_i < (>)\alpha_j\underline{t}_j$. So since $\alpha_1\underline{t} > \alpha t_2$ and $\alpha_1\underline{t} > \alpha t_3$ for almost all actual signals t_2 and t_3 of bidders 2 and 3 for sufficiently large α_1/α, bidder 1 is almost always the winner. If, for example, in fact $\alpha_1\underline{t} > \alpha t_2 > \alpha t_3$, then bidder 3 quits first at $(1+\alpha)t_3 + t_3 + \underline{t}$ (since at this price he knows $t_2 \geq t_3$ so the current lowest types of bidders 2 and 1 that could remain are $\underline{t}_2 = t_3$ and $\underline{t}_1 = \underline{t}$), and bidder 2 quits next at $p = (1+\alpha)t_2 + t_3 + \underline{t} \approx t_2 + t_3 + \underline{t} = v - (t_1 - \underline{t})$.

The intuition is that because bidder 1 (almost always) values the asset the most, bidders 2 and 3 face enormous winner's curses if bidder 1 ever exits, and they must therefore assume $t_1 \approx \underline{t}$ whenever he bids. So they quit at $\approx t_2 + t_3 + \underline{t}$, and bidder 1 almost always wins.

However, with three bidders competing for two units and increasing hazard rates, bidder 1's advantage is almost eliminated and he wins only when he has one of the two highest signals.

Lemma 5. When three bidders compete for two objects, in the asymmetric case (i.e., small α_i and small α/α_1), (i) if hazard-rates, h_i, are increasing in signals, the bidders with the highest signals (almost always) win and the expected price $\approx E(v) - E(t - t_{(3)} \mid t \geq t_{(3)})$, and (ii) if hazard-rates are decreasing, the advantaged bidder and the disadvantaged bidder with the higher signal win and the expected price $\approx E(v) - E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$.

Proof. See Appendix B.

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[25] For example, if $F(t) = t/4$ for $4 \geq t \geq 0$ (which corresponds probabilistically to a linear demand curve), the expected values of the three signals would be 1, 2, and 3. The expected price in a three-for-one auction would be $1 + 2 + 2 = 5$, and the expected price in a three-for-two auction would be $1 + 1 + (2+3)/2 = 4.5$.

[26] For example, if $F(t) = 1 - t^{-2}$ for $t \geq 1$ (which corresponds probabilistically to a demand curve $q \equiv 1 - F(p) = p^{-2}$, that is, constant elasticity of -2), the expected values of the three signals would be 1.2, 1.6, and 3.2. So the expected price in a three-for-one auction would be $1.2 + 1.6 + 1.6 = 4.4$, and the expected price with three-for-two would be $1.2 + 1.2 + (1.6 + 3.2)/2 = 4.8$.

[27] If j has already quit, then $\underline{t}_j$ is j's inferred signal, and if j has not quit but no type of j is quitting, then $\underline{t}_j$ is j's lowest possible signal consistent with equilibrium.

To understand Lemma 5, again begin by observing that bidder i quits where he would be just indifferent about finding himself a winner. If $\underline{t}_i$, $\underline{t}_j$, and $\underline{t}_k$ are the lowest possible signals of bidders i, j, and k assuming equilibrium behavior up to the current price, type $\underline{t}_i$ of bidder i has expected value $(1+\alpha_i)\underline{t}_i+\underline{t}_j+E(t_k \mid t_k \geq \underline{t}_k)$ if j quits now, and expected value $(1+\alpha_i)\underline{t}_i+E(t_j \mid t_j \geq \underline{t}_j)+\underline{t}_k$ if k quits now. So type $\underline{t}_i$ quits at $p = (1+\alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k + x_j\text{Prob}(k \text{ quits now} \mid j \text{ or } k \text{ quits now}) + x_k\text{Prob}(j \text{ quits now} \mid j \text{ or } k \text{ quits now})$ in which $x_j \equiv E(t_j - \underline{t}_j \mid t_j \geq \underline{t}_j)$ and x_i and x_k are defined similarly. Since $\alpha_2 = \alpha_3 = \alpha < \alpha_1$, some types of bidders 2 and 3 quit (symmetrically) before any type of bidder 1 quits. Now note that for small-enough α and α_1, the differences between $\alpha_1\underline{t}_1$ and $\alpha\underline{t}_2(=\alpha\underline{t}_3)$ are very small relative to differences between x_1 and $x_2(=x_3)$ (except if hazard rates are constant). So if hazard rates are increasing, so x_i is decreasing in $\underline{t}_i$, then if $\underline{t}_1$ were to fall much behind $\underline{t}_2(=\underline{t}_3)$, then x_2 would become small relative to x_1 and $\underline{t}_1$ would wish to quit at a lower price than $\underline{t}_2$. So types of bidder 1 would have to quit until $\underline{t}_1$ roughly caught up to the value of $\underline{t}_2(=\underline{t}_3)$. Therefore increasing hazard rates require $\underline{t}_1 \approx \underline{t}_2 = \underline{t}_3$. So bidder i quits at (approximately) $(1+\alpha_i)t_i + t_i + E(t_k \mid t_k \geq t_j = t_i)$, just as in symmetric equilibrium with symmetric bidders, and the bidder with the lowest signal, $t_{(3)}$, (approximately) quits first. So the expected price $\approx E(t_{(3)} + t_{(3)} + E(t \mid t \geq t_{(3)})) \approx E(v) - E(t - t_{(3)} \mid t \geq t_{(3)})$, and we have part (i) of Lemma 5.

In words, a bidder exits when the current price equals (P), the private value component of the bidder's signal, *plus* (C), the common value to the bidder if all others exited immediately, *plus* (X), the expected value in excess of (C) conditional on the auction ending immediately. (C) is the same for everyone. With only one unit available for sale (X) is necessarily zero, so the bidder with the highest (P) always wins. But with more than one unit available, (X) swamps (P) if hazard rates are increasing and the private values are small, so the bidders with the highest signals win.

The intuition is that even if bidder 1 had a large advantage, bidders 2 and 3 would compete against each other for the second unit and would not face an abnormally large winner's curse in that competition. So bidders 2 and 3 bid more aggressively than when there is just one unit for sale, which forces bidder 1 to pay more and may cause bidder 1 to exit if his signal is low enough, which further reduces the other bidders' winner's curses.

So bidder 1's position is greatly weakened by the sale of the second unit in the "normal" increasing-hazard-rates case. When just one unit is for sale, bidder 1 always wins it. But when there are two units for sale, he wins barely more often than his opponents do.

Notice that although this turns out to be good for revenue, there is a (small) efficiency cost in that bidder 1 (almost always) has the highest value for a unit.

On the other hand, if hazard rates are decreasing, x_i is increasing in $\underline{t}_i$, so once $\underline{t}_1$ falls behind $\underline{t}_2(=\underline{t}_3)$, then x_2 becomes large relative to x_1 so $\underline{t}_1$ wishes to quit at a still higher price relative to $\underline{t}_2$, so (since some types of bidders 2 and 3 start quitting first) no type of bidder 1 ever quits. As the auction proceeds and more types of bidders 2 and 3 quit, $x_2 \equiv E(t_2 - \underline{t}_2)$ and $x_3 \equiv E(t_3 - \underline{t}_3)$ increase while $x_1 \equiv E(t_1 - \underline{t}_1)$ remains unchanged, and thus even the lowest type of bidder 1 expects a larger and larger surplus conditional on winning; the higher the bidding goes, the more *under*priced bidder 1 thinks the object is. Since bidders 2 and 3 are symmetric, the bidder with the lower of their two signals loses, so, writing this signal as $t_{(2,2)}$, he quits at price $p = E(t_1) + (1+\alpha)t_{(2,2)} + t_{(2,2)}$. But $E(v) = E(t_1 + t_{(1,2)} + t_{(2,2)})$, so $E(p) \approx E(v) - E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$, giving us part (ii) of Lemma 5.

Lemmas 4 and 5 yield:

Proposition 2. In the asymmetric case, the expected price per unit is *lower* when one unit is sold than when two units are sold if hazard rates, h_i, are increasing in the signals, t_i. The expected price per unit is *higher* when one unit is sold than when two units are sold if hazard rates are decreasing.

Proof. Since $\underline{t}$ is the lowest possible value of t, we know from Lemma 1 that $E(t - \underline{t})$ is greater than (less than) $E(t - t_{(3)} \mid t \geq t_{(3)})$ and $E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$ if hazard rates are increasing (decreasing). The proposition then follows directly from Lemmas 4 and 5. *Q.E.D.*

As with the symmetric case, marginal revenues help us understand these results better (see Section 3): When a single unit is sold it always goes to bidder 1, so the expected price equals the expected marginal revenue of a randomly drawn signal.

When two units are sold, and hazard rates are increasing, the winners are the bidders with the two highest signals, and increasing hazard rates imply that these bidders have the highest marginal revenues. So two units sell at a higher per-unit price, on average, than one unit.

When hazard rates are decreasing, on the other hand, and two units are sold, they are won by bidder 1, whose expected marginal revenue is that of a randomly drawn signal, and by the other bidder who has the higher of the other signals and the *lower* of the other marginal revenues. So the expected marginal revenue of a winner, and therefore expected price, is lower when two units are sold than when just one unit is sold.

There is a further reason why with decreasing hazard-rates two units yield a lower per-unit price than a single unit. Recall that the expected revenue from an auction equals the expected marginal revenue of the winner only when any bidder with the lowest feasible signal receives no expected surplus (just as the Revenue Equivalence Theorem applies only to auctions where bidders with the worst possible signals make no money). If the assumption fails, expected revenue is reduced by the sum of the expected profits of the bidders conditional on their having their lowest possible signals. In most standard auctions the assumption holds trivially, but our decreasing-hazard-rate multiunit asymmetric auction is an exception in which even the lowest type of bidder 1 always wins and makes positive expected surplus: this type makes zero expected surplus at the lowest feasible sale price, and an ever-increasing expected surplus at higher sale prices. So the two-unit decreasing-hazard-rate auction is even less profitable relative to any of our other auctions that do all satisfy the standard assumption.

In short, selling more units leads to higher prices when bidders are asymmetric and hazard rates are increasing, and when bidders are symmetric and hazard rates are decreasing.

6. Rationing and initial public offerings

■ Selling two half-units yields the same per-unit prices as selling two whole units in our model. So rationing each bidder to buy only a half-unit yields a higher expected price than selling a single unit in those cases in which increasing supply raises price. The intuition is that creating additional winners reduces the winner's curse that any of them faces and so elicits more aggressive bidding behavior. By the same logic, the seller can do better still in the decreasing-hazard-rate case by simply offering each buyer one-third of a unit at a fixed price or, alternatively, by choosing the winner randomly among those prepared to pay the fixed price.

Proposition 3. Rationing to all three bidders at the fixed price $\underline{t} + 2E(t) = E(v) - E(t - \underline{t})$ is the optimal way to sell the good when hazard rates are decreasing.[28]

Proof. See Appendix B.

The result that rationing among all bidders is more profitable than raising the price to clear the market requires decreasing hazard rates.[29] The more general point, however, is that the difference between the expected revenues from choosing a price that guarantees an immediate sellout and from searching for the best possible price may be small; because searching for a high price may reveal some negative information (about where low bidders quit), it can lead to either a higher or a lower price than the pooling equilibrium that rationing induces. If the seller is risk averse, it may prefer the sure price that rationing guarantees.

In many finance and oil-lease models, signals are assumed to be distributed log-normally, so hazard rates are first increasing and then decreasing. In these cases, with symmetric bidders, the

[28] We are restricting ourselves to mechanisms that always yield a sale. Sometimes a seller can do better in expectation by having minimum prices that may lead to no sale.

[29] Rationing is *strictly* more profitable than raising the price to clear the market in the symmetric case, and/or when two units are available. In the asymmetric case, rationing a single unit to all three bidders is as profitable in expectation as the standard ascending auction (independent of hazard rates). (See Appendix B.)

seller does best to gradually raise price to eliminate the buyers with the lowest signals but then ration when a high-enough price is reached; Example 2 illustrates. This fits closely with practice in IPOs where a range of prices may be explored, but the final price is often fixed at a point where excess demand is most likely.[30] More mildly stated, it may be surprisingly inexpensive for sellers to choose a rationing price in IPOs, making the practice sustainable even if it is not necessarily optimal.

Example 2. With signals distributed according to the standard log-normal distribution (in which $\ln(t)$ is normally distributed with mean 0 and standard deviation 1), symmetric bidders, and one unit, it is optimal to run an ascending auction that separates all bidders with signals less than $t^* \approx .24$ and then ration among bidders with signals above t^*, if there are any. Thus if no bidder has a signal below t^*, the price will rise continuously to $3t^*$, at which point the seller will announce a rationing price of $t^* + 2y^*$ in which $y^* \equiv E(t \mid t \geq t^*) \approx 1.77$ (and all three bidders will then accept this price). If $t_{(3)} < t^*$, the lowest-signal bidder will quit at $3t_{(3)}$, and if no further bidder quits before price $t_{(3)} + 2t^*$, the good will be rationed at price $t_{(3)} + t^* + y^*$ (which both remaining bidders will then accept). Expected price is 3.414; by contrast, a pure ascending auction (or a sealed-bid auction) would yield 3.06.

With two units, it is again optimal to start with an ascending auction but then to ration at a price of $t^* + 2y^*$ if all three bidders have signals above t^*. Expected price is 3.411, just below the expected price in the optimal one-unit mechanism.[31]

Similarly, restaurateurs, theater owners, and football teams may raise expected revenue by choosing prices that assure excess demand—these examples may closely fit our model, which assumes that all customers demand the same number of units.

7. Restricting the number of bidders

■ It is also evident that when increasing supply raises price, so can restricting demand.

Again the intuition is that reducing the number of bidders reduces each bidder's winner's curse. Since in a private-value ascending auction bidders follow the same strategy regardless of the number of bidders (they bid up to their true value), it should be no surprise that with common values each bidder bids more aggressively when there are fewer of them.[32,33] This effect can dominate the effect of the winner having a lower signal, on average, when there are fewer bidders.

As before, it is quickest to see the results using marginal revenues, though we will offer proofs using more traditional methods.

When n symmetric bidders compete for one unit, the expected price equals the expected marginal revenue of the winner, which equals the expected marginal revenue of the bidder with the highest signal among the n bidders.[34] So if hazard rates are decreasing, that is, the bidders with the higher signals have the lower marginal revenues, then the expected price is decreasing in

[30] This is true even when the final IPO price is set above the initially specified range. For other theories of rationing, see DeGraba (1995), DeGraba and Mohammed (1999), Denicolò and Garella (1999), Gilbert and Klemperer (2000), and the references they cite.

[31] Because of the decreasing hazard rates for higher signals, the expected price in a pure ascending auction for two units (3.31) exceeds the expected price in a pure ascending auction for one unit.

[32] In the symmetric case, with two bidders, i bids up to $2t_i + E(t)$. With three bidders, i bids up to $2t_i + t_{(3)}$.

[33] Kagel, Levin, and Harstad (1995) noted that bidders bid more aggressively when there are fewer of them in sealed second-price auctions, and Matthews (1984) argues that this also typically applies in first-price auctions. Our result applies equally to the asymmetric case.

[34] Any bidder's actual marginal revenue is a function of all the other active bidders' signals, so it depends on n. But with independent signals a bidder's marginal revenue, $\widetilde{MR}_i(t_i, t_j)$, when i and j are active, equals his expected marginal revenue conditional on t_i and t_j, $E_{t_k}\{MR_i(t_i, t_j, t_k)\}$, when an additional bidder k is active. So with two bidders and decreasing hazard rates, expected profits are $E \min(\widetilde{MR}_1(t_1, t_2), \widetilde{MR}_2(t_1, t_2)) = E \min(E_{t_3}\{MR_1(t_1, t_2, t_3)\}, E_{t_3}\{MR_2(t_1, t_2, t_3)\}) \geq E \min(MR_1(t_1, t_2, t_3), MR_2(t_1, t_2, t_3)) > E \min(MR_1(t_1, t_2, t_3), MR_2(t_1, t_2, t_3), MR_3(t_1, t_2, t_3))$ = expected profits when all three bidders are present.

n. On the other hand, when bidders are asymmetric and all three bidders are present, bidder 1 is the winner and, in expectation, has the marginal revenue of a randomly selected bidder. But when only two bidders are selected, the winner will be the bidder with the higher of their two signals when bidders 2 and 3 are selected (and the winner will be bidder 1 otherwise). So if marginal revenues are higher (lower) for the higher-signal bidders, the expected price will be higher (lower) when the number of bidders is arbitrarily restricted to two. That is, the results for the asymmetric case are again opposite to those for the symmetric case.

Proposition 4. In the symmetric case, the expected price when one unit is sold is lower when only two bidders are allowed to participate than when all three compete if hazard rates, h_i, are increasing in the signals. The price is higher when only two bidders are allowed to participate than when all three compete if hazard rates are decreasing.

In the asymmetric case the opposite results apply.

Proof. See Appendix B.

In sum, restricting the number of bidders allowed to participate is likely to be a profitable strategy when bidders are asymmetric if hazard rates are increasing, or when bidders are completely symmetric (in what is publicly known about them) but hazard rates are decreasing.[35] Our model can thus explain strategies such as, for example, a merger-target opening negotiations with only a limited number of potential acquirers.

Of course these results contrast with our earlier work, Bulow and Klemperer (1996), which emphasized conditions under which restricting bidding is not merely undesirable for the seller, but is even a bad idea for a seller who can gain additional negotiating power by limiting participation. The point of this section is that while the conditions specified in our earlier work are very natural for private-value auctions with symmetric bidders, they are less compelling for symmetric common-value or almost-common-value auctions, and perhaps even unnatural for asymmetric almost-common-value auctions.

8. Sealed-bid auctions

■ How are our results affected if the other of the two most common auction forms, that is, a sealed-bid or first-price auction, is used?[36]

The answer is "hardly at all" when bidders are symmetric, since the highest-signal bidder(s) win(s) in any standard auction, so it follows from the Revenue Equivalence Theorem that expected revenues are the same in any standard auction.[37] So Lemmas 2 and 3 apply in expectation, and Proposition 1 applies exactly as before.

But the outcome of a sealed-bid auction, in stark contrast to that of an ascending auction, is, it is believed, almost unaffected by small asymmetries between the bidders.[38] Assuming this is true, it then follows easily that in the asymmetric case the expected revenue is higher for one unit and is the same for two units from a sealed-bid auction than from an ascending auction if bidders' hazard rates are increasing, and is lower for one unit and higher for two units from a sealed-bid auction than from an ascending auction if hazard rates are decreasing (see Appendix A). In the symmetric case, the two auction types are always equally profitable.

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[35] We are assuming that the participants are chosen randomly when their numbers are restricted. Restricting numbers by requiring bidders to pay an entry fee would be very unprofitable, since it would select precisely those bidders (higher signals in the symmetric case, and advantaged bidders in the asymmetric case) that the seller wishes to exclude.

[36] In a sealed-bid or first-price auction for two units, bidders simultaneously and independently submit bids. The winners are the two high bidders, and each pays his actual bid.

[37] The Revenue Equivalence Theorem is due to Myerson (1981) and Riley and Samuelson (1981). For a simple exposition, see Appendix A of Klemperer (1999).

[38] To our knowledge there is no general theorem proving this, although Avery and Kagel (1997) demonstrate the results for a model that is almost a special case of ours, and Bulow, Huang, and Klemperer (1999) prove the result in a related context.

In sum, without detailed information about the distribution of bidders' signals, it is very hard to make any predictions about which of sealed-bid and ascending auctions are more profitable.[39]

9. Extensions, and the "Maximum Game"

■ **More bidders and more units.** Extending our results to $n > 3$ bidders and k units is trivial for the symmetric case. As before, whether more or fewer units raises expected price depends on whether hazard rates are decreasing or increasing. Appendix A gives more detail and shows that with decreasing hazard rates a larger k may yield a significantly higher expected price, even when n is very large. It also seems straightforward that the asymmetric case generalizes to more than two disadvantaged bidders in the obvious way.[40]

□ **General value functions.** Both for tractability and to make a clear contrast with the pure private-value case, $v_i = t_i$, we restricted analysis to value functions that are the sum of bidders' signals. However, our "perverse" results in no way depend on this.

More generally, if i's (almost pure common) value is $v_i(t_1, t_2, t_3)$, then i's marginal revenue, MR_i, equals $v_i - (\partial v_i/\partial t_i) \cdot (1/h_i)$, so in the symmetric case, for example, fewer bidders or more units raise prices if $(\partial v_i/\partial t_i) \cdot (1/h_i)$ is increasing in t_i, which can clearly be satisfied *either* because hazard rates h_i are (sufficiently) decreasing *or* because $\partial v_i/\partial t_i$ is (sufficiently) increasing.

Most of our results hold quite generally if we substitute the condition "the bidders with the higher signals have the higher MRs (i.e., $t_i > t_j \Leftrightarrow MR_i > MR_j$)" for the condition "hazard rates are increasing in the signals," and substitute the condition "the bidders with the higher signals have the lower MRs" for the condition "hazard rates are decreasing," throughout.[41]

□ **The Maximum Game.** Example 3, which we call the Maximum Game, provides a good illustration of how a different choice of value function than the one we used can make it easy to obtain extreme "perverse" results.

Example 3 (the Maximum Game). Assume bidders' common value equals the *maximum* of the signals, $v_i = v = \max\{t_i\}$. Then with n bidders competing for one unit in an ascending auction, every bidder bids up to his signal, and the price will equal the actual second-highest of the n signals. But if one bidder is randomly chosen for a take-it-or-leave-it offer, he will be willing to pay at least the expected highest of the $n - 1$ other signals, which is greater in expectation.

More generally, with k units, and if only a random $m \geq k$ of the n bidders are allowed to compete, the expected price is, as we show below, $(1 - m/nk)E(t_{(1)}) + (m/nk)E(t_{(2)})$. So the more units, and the fewer bidders, the higher the expected price. Likewise, a rationing mechanism that selects the top ℓ bidders and sells each of them $1/\ell$ of the available supply yields an expected price of $(1 - 1/\ell)E(t_{(1)}) + (1/\ell)E(t_{(2)})$; the *best* of these mechanisms sets $\ell = n$, that is, it rations the available supply to all n bidders at the highest price (the expected highest of $n - 1$ signals) at which potential buyers will want to buy, while the *worst* of these mechanisms sets $\ell = 1$, that is, it is a pure ascending auction.

The results of the Maximum Game are best understood using marginal revenues. For all bidders but the one with the highest signal, $\partial v_i/\partial t_i = 0$, so marginal revenue for these bidders simply equals the value of the highest bidder, which in expectation is $E(t_{(1)})$. Furthermore, recall that the expected revenue in any auction equals the expected marginal revenue of the winning bidder. If a single-unit Maximum Game were conducted as an ascending auction, the bidder with the highest signal would win and pay the second-highest signal. So the expected revenue in such

39 Klemperer (1998) builds on the first draft of this article, to discuss how auctions of PCS licenses and auctions of companies can be designed to capture the benefits that first-price auctions offer. See also Klemperer (2002a, forthcoming).

40 The results seem less clear-cut if there are multiple units for sale and multiple advantaged bidders.

41 We conjecture that this applies to Lemma 5 and hence to Propositions 2 and 4, at least for value functions of the form $v_i(t_1, t_2, t_3) = g(t_1, t_2, t_3) + \alpha t_i$ as well as (more obviously, and for more general almost pure common value functions) to Proposition 1.

an auction is $E(t_{(2)})$, and this must therefore be the expected marginal revenue of the bidder with the highest signal.[42] So the bidder with the highest signal has the lowest marginal revenue.

Since the expected price is the expected marginal revenue of the winners, and since the highest-signal bidder is always a winner whenever he is allowed to participate and so on average wins a fraction (m/nk) of the units while other bidders win the remaining units, the expected sale price is $(1 - m/nk)E(t_{(1)}) + (m/nk)E(t_{(2)})$.[43] Clearly, it is optimal to minimize the fraction of units that will be sold to the highest-signal bidder—so rationing, restricting participation, and increasing supply all raise the average sale price.

The Maximum Game may be a more natural model than our additive model if, for example, oil or other mineral rights are being sold. (Conditional on one bidder's test finding substantial oil, the results of other tests that find little or no oil may be irrelevant.)

Several symmetric common-value models whose results are driven by the Maximum Game's logic have been analyzed by other authors. In Matthews (1984), the signals are distributed as $F(t) = (t/v)^{\theta}$ $(\theta > 0, 0 \leq t \leq v)$, so the maximum signal is a sufficient statistic for the true value, v. Of course, bidders' signals are affiliated in this model, which complicates matters, and Matthews examines sealed-bid auctions, but he finds that expected price is often decreasing in the number of bidders, just as in our Maximum Game.

Harstad and Bordley (1996) examine an example in which bidders have diffuse priors about the true value v, and their signals are drawn from a uniform distribution on $[v - \theta, v + \theta]$ $(\theta > 0)$, so the expected value of v conditional on all n signals is the average of the highest and lowest signal—and independent of all the other signals. Because the lowest-signal bidder, as well as the highest-signal bidder, therefore has a marginal revenue below the actual value ($\partial v_i/\partial t_i = 1/2 > 0$ for these bidders), selling to the lowest-signal bidder is inferior to selling to any of the other bidders with signals below the highest. And once we eliminate the lowest-signal bidder (e.g., by raising the price until the point at which he quits the auction), the remaining problem is exactly a Maximum Game. So, as we would expect, Harstad and Bordley find that profitable selling mechanisms are those that avoid selling to the highest- and lowest-signal bidders. Rationing equally to all except the lowest-signal bidder is the most profitable mechanism they examine. Again, bidders' signals are affiliated, but again the results are driven by the "Maximum Game" structure that is embedded within the model.[44]

Two other articles subsequent to ours also have the Maximum Game underlying them. Levin (2001) emphasizes a model in which a weighted average of the two highest signals is a sufficient statistic for the true value, while in Parlour and Rajan (2001) all the signals are drawn from a distribution centered on the true value. Since their very rich model neither restricts this distribution to be uniform nor restricts priors about the true value to be diffuse, the highest and lowest signals are no longer sufficient statistics for the true value. However, exactly as we would expect, they find that as the priors become more diffuse and the distribution of the signals conditional on the value becomes more uniform (so their model more closely approximates the Maximum Game), rationing performs relatively better.[45]

□ **Affiliation.** Finally, we noted that bidders' signals are affiliated in several other authors' models. We conjecture that affiliation makes rationing less likely to be optimal and makes our other "perverse" results harder to obtain. Excluding bidders or rationing or adding more winners means less information is shared in the auction. This tends to mean lower prices with affiliation because of the conservatism that affiliation creates in marginal bidders; see Milgrom and Weber

[42] See Bulow and Klemperer (1996) and Appendix B of Klemperer (1999) for more details.

[43] The results can, of course, be confirmed directly by noting that the price is set by the actual $(k+1)$st-highest-signal participant who quits at his expectation of the maximum signal among the $(k + 1)$ remaining bidders (including himself) and the $(n - m)$ excluded bidders, conditional on the kth-highest-signal remaining bidder having the same signal as he does (since he would be indifferent about finding himself a winner at this price).

[44] Of course, in this and other models with affiliated signals, there exist selling mechanisms, albeit implausible ones, that extract bidders' entire surplus (Crémer and McLean, 1985).

[45] Campbell and Levin (2001) have very recently developed additional results about the Maximum Game.

(1982, 2000). So the standard intuition that more bidders and fewer winners will lead to higher prices will be more likely to hold if signals are affiliated. For example, when rationing to all bidders in the Maximum Game, the seller must offer each bidder a take-it-or-leave-it price equal to the expected highest of $n-1$ signals contingent on the nth signal being the minimum possible. With affiliation this could easily be below the unconditional expected second-highest of n signals, the expected price in an ascending auction with a single winner.

10. Conclusion

■ Economists' intuition has been developed from the partial equilibrium analysis of fully informed buyers and sellers. These agents know the value they place on assets. So in "private-value" auctions, more buyers raise prices, more quantity implies a lower price, and if demand exceeds supply it always makes sense for a seller to try to raise price.

We have shown that this intuition does not carry over to "common-value" settings such as financial markets where buyers have different assessments of assets that would be valued similarly by all if they shared their information.

With symmetric agents, the standard results only occur with a rather strong distributional assumption, equivalent to what is needed for strategic substitutes in Cournot competition. When this assumption fails, setting a price that guarantees excess demand and rationing, as in IPOs, may be more profitable than finding the price that clears the market. Furthermore, restricting entry to an auction may increase expected revenues.

With asymmetric agents the standard results fail under exactly the conditions for which they hold under symmetry. This may explain why, in the FCC's initial PCS auction, prices seemed to be lower in some regions where a single license was sold, than in markets where two licenses were available.

Appendix A

■ This Appendix discusses extensions of the model to more bidders and more units, and to sealed-bid auctions.

□ **More bidders and more units.** Generalizing from Lemmas 2 and 3, the expected price in a symmetric common-value auction in which the true value to the bidders, v, is the sum of the signals of n bidders, and the top $k < n$ bidders win, is $E(p_k) = E(v) - E(t - t_{(k+1)} \mid t \geq t_{(k+1)})$. Example A1 (which extends Example 1) illustrates that, as for just three bidders, whether more or fewer units raises expected price depends on whether hazard rates are decreasing or increasing.

Example A1. With n symmetric bidders competing for k units and uniformly distributed signals, $F(t) = (t - \underline{t})/(\bar{t} - \underline{t})$, we have $E(p_k) = E(v) - [\bar{t} - \underline{t}][(k+1)/2(n+1)]$. With exponentially distributed signals, $F(t) = 1 - e^{-\lambda(t-\underline{t})}$, we have $E(p_k) = E(v) - 1/\lambda$. With constant-elasticity distributed signals, $F(t) = 1 - (t/\underline{t})^{\eta}$, we have $E(p_k) = E(v) + [1/(\eta+1)]E(t_{(k+1)}) = E(v) + [1/(\eta+1)]\underline{t}\prod_{j=k+1}^{n} j\eta/(j\eta+1)$. In all three cases the ratio of buyer surplus (i.e., expected value minus expected price) with one winner to k winners is independent of n.

The limit of Example A1 as n becomes large can be looked at in different ways: In the uniform case, for any k the winner's expected surplus goes to zero as n becomes large. In the constant-hazard-rate case, the winner's expected surplus is independent of n but does therefore become small both relative to the expectation of v (which is proportional to n) and to the standard deviation in v, which rises with the square root of n. In the constant-elasticity case, the winner's expected surplus rises as n increases but becomes small relative to the expected value of v. Relative to the standard deviation in v, however, the expected surplus becomes larger (smaller) with n depending on whether the absolute value of η is smaller (larger) than two.

The conclusion, then, is that for the "normal" case of increasing or constant hazard rates, k becomes irrelevant as n grows because buyer surplus becomes relatively small—and hence expected price approximates value—under all mechanisms. But in the "abnormal" case of decreasing hazard rates, a larger k may still yield a significantly lower buyer surplus, and hence also a significantly higher expected price, even when n becomes large.

□ **Comparison of sealed-bid and ascending auctions.** In the asymmetric case, when one unit is sold, the ascending auction yields $\approx \underline{t}+2E(t) = E(v) - E(t - \underline{t})$ in expectation (Lemma 4). The sealed-bid auction yields $\approx E(v) - E(t - t_{(2)} \mid t \geq t_{(2)})$ in expectation—we assume the conjecture in Section 8 that the expected revenue from the sealed-bid auction is almost unaffected by the small asymmetries between the bidders and so is almost Revenue Equivalent to the situation in Lemma 2. Applying Lemma 1, the ascending auction then yields less (more) expected revenue when hazard rates are increasing (decreasing).

When two units are sold in a sealed-bid auction, expected price $\approx E(v) - E(t - t_{(3)} \mid t \geq t_{(3)})$ in expectation, assuming approximate Revenue Equivalence to the situation in Lemma 3. The ascending auction yields the same in expectation if hazard rates are increasing, but $\approx E(v) - E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$ in expectation if hazard rates are decreasing (Lemma 5). But with decreasing hazard rates, $E(t - t_{(2,2)} \mid t \geq t_{(2,2)}) > E(t - t_{(3)} \mid t \geq t_{(3)})$, since for any random three signals $t_{(2,2)} \geq t_{(3)}$. Therefore expected revenue is less in the ascending auction when there are two units and decreasing hazard rates.

(Thinking about marginal revenues is the quickest way to see the result for the one-unit case, since the sealed-bid and ascending auctions yield the expected marginal revenue of the highest-signal bidder and the average bidder, respectively. For the two-unit, decreasing-hazard-rate case, however, marginal revenue calculations are trickier, since this is the special case in which the ascending auction gives positive expected surplus to the lowest type of bidder 1 (see Section 5), so the expected revenue from the ascending auction is the sum of the expected marginal revenues of the winning bidders minus this expected surplus.)

Appendix B

■ Proofs of Propositions 3–4 and Lemmas 1–5 follow.

Proof of Proposition 3. For any allocation mechanism, let $P_i(t_i)$ be the probability that i will receive the object, in equilibrium, and let $S_i(t_i)$ be the equilibrium expected surplus to bidder i. Incentive compatibility requires that $S_i(t_i+dt_i) \geq S_i(t_i)+(1+\alpha_i)dt_i P_i(t_i)$ (since v_i is $(1+\alpha_i)dt_i$ higher for type t_i+dt_i than for type t_i, independent of the other bidders' signals). Likewise $S_i(t_i) \geq S_i(t_i+dt_i) - (1+\alpha_i)dt_i P_i(t_i+dt_i)$, so $(1+\alpha_i)P_i(t_i+dt_i) \geq [S_i(t_i+dt_i) - S_i(t_i)]/dt_i \geq (1+\alpha_i)P_i(t_i)$.

So $P_i(t_i)$ must be a (weakly) increasing function.

Also $S_i(t_i)$ has derivative $(1+\alpha_i)P_i(t_i)$, so

$$S_i(t_i) = S_i(\underline{t}) + \int_{\underline{t}}^{t_i} (1+\alpha_i)P_i(t)dt,$$

and

$$\begin{aligned} E_{t_i}\Big((S_i(t_i)) &= S_i(\underline{t}) + \int_{\underline{t}}^{\infty}\int_{\underline{t}}^{t_i} (1+\alpha_i)P_i(t)dtf(t_i)dt_i \\ &= S_i(\underline{t}) + \int_{\underline{t}}^{\infty} (1-F(t_i))(1+\alpha_i)P_i(t_i)dt_i \qquad \text{(integrating by parts)} \\ &= S_i(\underline{t}) + E_{t_i}\frac{1}{h_i(t_i)}(1+\alpha_i)P_i(t_i))\Big). \end{aligned}$$

Now expected seller profits are the expected value of the good to the winning bidder less the expected surplus of the bidders. Since the value of the good to the winner is the same for all mechanisms (ignoring the α_i), profits are maximized by minimizing the bidder's surplus. But $(1/h_i(t_i))$ is increasing when hazard rates are decreasing, so since $P_i(t_i)$ is required to be increasing, $\sum_{i=1}^{3} E_{t_i}[(1/h_i(t_i))(1+\alpha_i)P_i(t_i)]$ is minimized (ignoring the α_i) among all schemes that always sell (i.e., have $\sum_{i=1}^{3} P_i(t_i) = 1$) by choosing $P_i(t_i) =$ constant, for all i. And selling at price $\underline{t} + 2E(t)$ yields $S_i(\underline{t}) = 0$ (individual rationality implies $S_i(t_i) \geq 0$ for all t_i). So rationing equally to all three bidders at price $\underline{t} + 2E(t)$ maximizes the seller's expected profits. *Q.E.D.*

(The expected profit from rationing is $E(\underline{t} + 2E(t)) = E(v) - E(t - \underline{t})$, which exceeds the expected profits from a standard auction to clear the market of either one unit (apply Lemma 1 to Lemmas 2 and 4) or two units (apply Lemma 1 to Lemmas 3 and 5). In the asymmetric, two-unit, decreasing-hazard-rate case, rationing yields the same profits as the ascending auction.)

Proof of Proposition 4. In the symmetric case, with three bidders the expected price $\approx v - E(t - t_{(2)} \mid t \geq t_{(2)})$, by Lemma 2. When only two bidders, say i and j, are permitted to participate, the loser, say bidder i, quits at the point at which he would just be indifferent about winning conditional on being tied with bidder j, that is, $2t_i + E(t_k)$, so the expected price $\approx E(v) - E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$. Since $t_{(2)}$ is the second-highest of three signals while $t_{(2,2)}$ is the lower of two signals, then for any set of three signals, $t_{(2)} \geq t_{(2,2)}$. So the symmetric case of the proposition follows from Lemma 1.

In the asymmetric case, with three bidders the price $\approx \underline{t} + t_2 + t_3$ and the expected price $\approx E(v) - E(t - \underline{t})$—see Lemma 4. Likewise, if bidder 2 is excluded, the price $\approx \underline{t} + t_3 + E(t_2)$ and the expected price $\approx E(v) - E(t - \underline{t})$. Similarly, if bidder 3 is excluded, the expected price $\approx E(v) - E(t - \underline{t})$. But if bidder 1 is excluded and bidder i loses, then the price $\approx 2t_i + E(t_1)$, so the expected price $\approx E(v) - E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$ (as for the symmetric case). But $E(t - \underline{t}) > (<) E(t - t_{(2,2)} \mid t \geq t_{(2,2)})$ if hazard rates are increasing (decreasing) by Lemma 1, so the asymmetric case of the proposition follows. *Q.E.D.*

Proof of Lemma 1. $E(t-z \mid t \geq z) = [1/(1-F(z))]\int_z^{\infty} tf(t)dt - z = \int_z^{\infty}(1-F(t))/(1-F(z))dt$ (integrating by parts with $u = t, du = dt, v = -(1-F(t)), dv = f(t)dt$). But $1 - F(t) = e^{-\int_{\underline{t}}^{t} h(x)dx}$ (the probability that a signal exceeds t equals 1 discounted by all the hazard rates between $\underline{t}$ and t). So $E(t - z \mid t \geq z) = \int_z^{\infty} e^{-\int_z^t h(x)dx}dt = \int_{\underline{t}}^{\infty} e^{-\int_{\underline{t}}^{t} h(x+z-\underline{t})dx}dt$. This final formula is clearly increasing (decreasing) in z if h is a decreasing (increasing) function. *Q.E.D.*

Proof of Lemmas 2–5. At a given price p and for a given history (i.e., the first quitter's quit price if there has been a quit), we write $\underline{t}_i$ for the lowest (or infimum), i.e., marginal, type of bidder i remaining in equilibrium, or we write $\underline{t}_i$ for bidder i's expected signal if he has already exited, and write $w_i = (1+\alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k$. Write $x_i = E(t_i - \underline{t}_i \mid t_i \geq \underline{t}_i)$. (Thus $\underline{t}_i$, w_i, and x_i are all functions of p and the history, but we will not usually write this dependence explicitly.) It will be convenient to write $\underline{x} = E(t_i - \underline{t})$.

□ **Analysis of the one-unit auction.** We are looking for an equilibrium in which i stays in the bidding if and only if $p < w_i$. Now $\alpha_i \underline{t}_i \geq \alpha_j \underline{t}_j \Rightarrow w_i \geq w_j \Rightarrow$ type $\underline{t}_i$ of i cannot quit if type $\underline{t}_j$ of j remains in the bidding. So types t_i of i and t_j of j quit simultaneously if and only if $\alpha_i t_i = \alpha_j t_j$. So if $\alpha_1 \geq \alpha_2 = \alpha_3 = \alpha$, bidders $i = 2, 3$ quit according to $\underline{t}_2 = \underline{t}_3$, with t_i quitting at price $p = \underline{t} + (1+\alpha)t_i + t_i$ for $p < (1+\alpha_1)\underline{t} + (\alpha_1/\alpha)\underline{t} + (\alpha_1/\alpha)\underline{t}$, and if bidder i quits in this range, then type t_j of the other of these two bidders quits at price $p = \underline{t} + \underline{t}_i + (1+\alpha)t_j$ for $p < (1+\alpha_1)\underline{t} + \underline{t}_i + (\alpha_1/\alpha)\underline{t}$, and beyond this price bidder j and bidder 1 both quit according to $\alpha_1 \underline{t}_1 = \alpha \underline{t}_j$ and $p = (1+\alpha_1)\underline{t}_1 + \underline{t}_i + (\alpha_1/\alpha)\underline{t}_1 = \underline{t}_1 + \underline{t}_i + (1+\alpha)\underline{t}_j$ $(j = 2, 3;\ j \neq i)$. (Bidders 1 and j infer i's actual signal $\underline{t}_i$ from the price at which he quit.)

No type of bidder 1 quits until $p = (1+\alpha_1)\underline{t}_1 + \underline{t}_2 + \underline{t}_3$. If neither of the other bidders quits before this price (so then $p = (1+\alpha_1)\underline{t} + (\alpha_1/\alpha)\underline{t} + (\alpha_1/\alpha)\underline{t}$ and $\alpha_1 \underline{t} = \alpha \underline{t}_2 = \alpha \underline{t}_3$), then all three bidders quit according to $\underline{t}_2 = \underline{t}_3 = (\alpha_1/\alpha)\underline{t}_1$ (and $p = w_1 = w_2 = w_3$) thereafter, and after one bidder has quit the remaining bidders ℓ and m quit according to $\alpha_\ell \underline{t}_\ell = \alpha_m \underline{t}_m$ and $p = w_\ell = w_m$. It is straightforward that this is a (perfect Bayesian) equilibrium and is unique under our assumptions.

Thus if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, the final price is $(1+\alpha)t_{(2)} + t_{(2)} + t_{(3)} \approx 2t_{(2)} + t_{(3)}$ for small α. If $\alpha_1 > \alpha_2 = \alpha_3 = \alpha$, then as $\left(\frac{\alpha}{\alpha_1}\right) \to 0$ the probability of bidder 1 winning approaches one, so the final price is (almost always) $\underline{t} + t_2 + t_3 + \alpha \max(t_2, t_3) \approx \underline{t} + t_2 + t_3$. This proves Lemmas 2 and 4.

□ **Analysis of the two-unit auction.** We look for an equilibrium in which a bidder quits when he would be just indifferent were he to find himself a winner. In such an equilibrium let $H_i(p)$ be the hazard rate with which i quits at price p, that is,[46]

$$H_i(p) = \frac{\underline{t}_i'(p) f(\underline{t}_i(p))}{1 - F(\underline{t}_i(p))}.$$

So type t_i of i quits when

$$p = (1+\alpha_i)t_i + \underline{t}_j + \underline{t}_k + x_j\left(\frac{H_k}{H_j + H_k}\right) + x_k\left(\frac{H_j}{H_j + H_k}\right),$$

that is, the price equals i's expected value conditional on winning, since, in this case, with probability $[H_j/(H_j + H_k)]$ it is j who has quit so $t_j = \underline{t}_j$ and $E(t_k) = \underline{t}_k + x_k$.

We begin with the asymmetric case. Let $\alpha_1 > \alpha_2 = \alpha_3 = \alpha$.

□ **Increasing hazard rates.** Begin with the standard case in which the hazard rate, $f(t_i)/(1 - F(t_i))$, is increasing in t_i, so x_i is decreasing in $\underline{t}_i$. No one quits until $p = (3+\alpha)\underline{t} + \underline{x}$, at which price the lowest types of bidders 2 and 3 quit. Since 2 and 3 behave symmetrically, $\underline{t}_2 = \underline{t}_3$, and $p = \underline{t} + (2+\alpha)\underline{t}_2 + \underline{x}$ until $\alpha_1 \underline{t} + x_2 = \alpha \underline{t}_2 + \underline{x}$, at which price, say $\underline{p}$, bidder 1 is also just indifferent about finding himself a winner. For types of all three bidders to be quitting simultaneously, we require

$$\begin{aligned}\alpha_1 \underline{t}_1 + x_2\left(\frac{H_3}{H_2 + H_3}\right) + x_3\left(\frac{H_2}{H_2 + H_3}\right) &= \alpha \underline{t}_2 + x_1\left(\frac{H_3}{H_1 + H_3}\right) + x_3\left(\frac{H_1}{H_1 + H_3}\right)\\ &= \alpha \underline{t}_3 + x_1\left(\frac{H_2}{H_1 + H_2}\right) + x_2\left(\frac{H_1}{H_1 + H_2}\right),\end{aligned}$$

so using the symmetry of bidders 2 and 3 yields

$$\begin{aligned}&\alpha_1 \underline{t}_1 + x_2 = \alpha \underline{t}_2 + x_1\left(\frac{H_2}{H_1 + H_2}\right) + x_2\left(\frac{H_1}{H_1 + H_2}\right)\\ &\Rightarrow \frac{\alpha_1 \underline{t}_1 - \alpha \underline{t}_2}{x_1 - x_2} = \frac{H_2}{H_1 + H_2} = \frac{1}{\frac{d\underline{t}_1}{d\underline{t}_2}\frac{h_1}{h_2} + 1} \Rightarrow \frac{d\underline{t}_1}{d\underline{t}_2} = \frac{h_2}{h_1}\left[\left(\frac{x_1 - x_2}{\alpha_1 \underline{t}_1 - \alpha \underline{t}_2}\right) - 1\right],\end{aligned}$$

in which $h_i(\underline{t}_i) \equiv f(\underline{t}_i)/(1 - F(\underline{t}_i))$ (i.e., as defined in Section 2). Since $h_i(\underline{t}_i)$ is finite everywhere, this yields $\underline{t}_1$ as a continuous upward-sloping function of $\underline{t}_2$, that is, $\infty > d\underline{t}_1/d\underline{t}_2 \geq 0$ everywhere, and $\underline{t}_2 > \underline{t}_1$, $x_1 > x_2$, and $\alpha_1 \underline{t}_1 > \alpha \underline{t}_2$

46 More precisely, the equilibrium functions $\underline{t}_i(p)$ are defined by the equilibrium hazard rates $H_i(p)$, since otherwise $\underline{t}_i'(\cdot)$ might not be defined by $\underline{t}_i(\cdot)$. The condition that a bidder quits when he is just indifferent about winning ensures that $H_i(p)$ is finite, that is, $\underline{t}_i(p)$ is single-valued and continuous.

everywhere. (As $(\alpha_1\underline{t}_1 - \alpha\underline{t}_2) \to 0$, $d\underline{t}_1/d\underline{t}_2 \to \infty \Rightarrow \alpha_1\underline{t}_1 > \alpha\underline{t}_2$ everywhere. As $(x_1 - x_2) - (\alpha_1\underline{t}_1 - \alpha\underline{t}_2) \to 0$, $d\underline{t}_1/d\underline{t}_2 \to 0$ so also $dx_1/d\underline{t}_2 \to 0$ while $dx_2/d\underline{t}_2 < 0, \Rightarrow (x_1 - x_2) - (\alpha_1\underline{t}_1 - \alpha\underline{t}_2) > 0$ everywhere when $\underline{t}_1 > \underline{t}$. So also $\infty > d\underline{t}_1/d\underline{t}_2 > 0$ when $\underline{t}_1 > \underline{t}$.)

Does the (unique) solution to this differential equation define a (perfect Bayesian) Nash equilibrium? To see that it does, first note that having $\underline{t}_1$ as a function of $\underline{t}_2$ (uniquely) defines $\underline{t}_1(p)$ and $\underline{t}_2(p)$ using $p = (1 + \alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2$ (since $\underline{t}_1$ and $\underline{t}_2 + x_2$ and hence p are all continuous and upward-sloping functions of $\underline{t}_2$). Now assume bidders 2 and 3 bid according to $\underline{t}_2(p)$ (and $\underline{t}_3(p) = \underline{t}_2(p)$). Then type t_1 of bidder 1's profits from finding himself a winner at price p are $p - ((1 + \alpha_1)t_1 + 2\underline{t}_2 + x_2) = (1 + \alpha_1)(\underline{t}_1 - t_1)$, and we have shown $\underline{t}_1$ is continuous and increasing in p for $p \geq \underline{p}$ (i.e., where $\underline{t}_1 > \underline{t}$), so type t_1's uniquely optimal strategy is to quit at $t_1 = \underline{t}_1$. Similarly, assume bidders 1 and 3 bid according to $\underline{t}_1(p)$ and $\underline{t}_2(p)$, respectively. Then type t_2 of bidder 2's profits from finding himself a winner at price p are

$$p - \left((1 + \alpha)t_2 + \underline{t}_1 + \underline{t}_2 + x_1\left(\frac{H_2}{H_1 + H_2}\right) + x_2\left(\frac{H_1}{H_1 + H_2}\right)\right) = (1 + \alpha)(\underline{t}_2 - t_2),$$

and $\underline{t}_2$ is continuous and increasing in p (for $p \geq \underline{p}$ from our analysis of the differential equation, and for $p \in ((3 + \alpha)\underline{t} + \underline{x}, \underline{p})$ from our earlier argument). So type t_2 quits at $t_2 = \underline{t}_2$. Thus our equations define a (perfect Bayesian) Nash equilibrium. (If there is an upper bound, $\bar{t}$, on t_i, then above the price where $\underline{t}_2 = \underline{t}_3 = \bar{t}$, we can continue to define 1's strategy according to $p = (1 + \alpha)\underline{t}_1 + \bar{t} + \bar{t} + 0$.)

Finally, it is easy to check that there are no other candidate equilibria in the increasing-hazard-rate case. At any price after $\underline{p}$ (at which price

$$p - (\underline{t}_1 + \underline{t}_2 + \underline{t}_3) = \alpha_1\underline{t}_1 + x_2 = \alpha\underline{t}_2 + x_1\left(\frac{H_2}{H_1 + H_2}\right) + x_2\left(\frac{H_1}{H_1 + H_2}\right)$$

is first satisfied), it yields a straightforward contradiction for there to be no types of 1 quitting, or no types of 2 and 3 quitting, or no types of any of 1, 2, and 3 quitting, as the price rises.[47]

So the equilibrium we have found is unique under our assumptions. Finally, note from the differential equation that $x_1 - x_2$ is of order $\alpha_1\underline{t}_1 - \alpha\underline{t}_2$. ($h_2/h_1 > 1$, so $d\underline{t}_1/d\underline{t}_2 > ((x_1 - x_2)/(\alpha_1\underline{t}_1 - \alpha\underline{t}_2)) - 1$, so $x_1 - x_2$ cannot become much larger than $\alpha_1\underline{t}_1 - \alpha\underline{t}_2$ without $d\underline{t}_1/d\underline{t}_2$ becoming large and so reducing $x_1 - x_2$.) So as $\alpha_1 \to 0$, $x_1 \to x_2$ and so $\underline{t}_2 \to \underline{t}_1$ along the equilibrium path.[48] So the winners are almost always the bidders with the higher signals, and the price is almost always set by the bidder with the lowest signal, $t_{(3)}$, who quits at $\approx (1 + \alpha)t_{(3)} + t_{(3)} + E(t \mid t \geq t_{(3)})$.

□ **Decreasing hazard rates.** As in the increasing-hazard-rate case, no one quits until $p = (3 + \alpha)\underline{t} + \underline{x}$, at which price the lowest types of bidders 2 and 3 start quitting symmetrically according to $\underline{t}_2 = \underline{t}_3$ and $p = \underline{t} + (2 + \alpha)\underline{t}_2 + \underline{x}$. Now with decreasing hazard rates, as $\underline{t}_2$ increases so does x_2, so if, as we assume, α is small, $(1 + \alpha_1)\underline{t} + 2\underline{t}_2 + x_2 > \underline{t} + (2 + \alpha)\underline{t}_2 + \underline{x} = p$ for all $\underline{t}_2$. That is, for bidder 1 to never quit while bidders 2 and 3 quit symmetrically satisfies the first-order conditions for equilibrium everywhere. It is straightforward that this also defines a (perfect Bayesian) Nash equilibrium: if bidders 2 and 3 bid according to $\underline{t}_2(p)$, no types of player 1 ever wish to quit. If no type of bidder 1 ever quits, while bidder 3 bids according to $\underline{t}_2(p)$, then the expected profits of type t_2 of bidder 2 if he finds himself a winner at price p are $p - ((1 + \alpha)t_2 + \underline{t} + \underline{t}_2 + \underline{x}) = (1 + \alpha)(\underline{t}_2 - t_2)$, which is continuous and increasing in p, so t_2 optimally quits at $t_2 = \underline{t}_2$.

Are there any other equilibria in the decreasing-hazard-rate case? Clearly, as the price rises with $p = \underline{t} + (2 + \alpha)\underline{t}_2 + \underline{x}$ there is no point at which some types of 1 start quitting. (Their expected values from being a winner always exceed the price.) However, we need to consider the possibility that at some price at or above $(3 + \alpha)\underline{t} + \underline{x}$, no types of any players are quitting. This would require beliefs that conditional on the out-of-equilibrium event that player 2 does find himself a winner, he believes that player 3 quit with probability $\leq \lambda$, where $p = \underline{t} + (2 + \alpha)\underline{t}_2 + \lambda\underline{x} + (1 - \lambda)x_2$.

Note that as p rises, λ falls, since $x_2 > \underline{x}$. So no types of players 2 and 3 can ever start quitting again unless types of player 1 also do, since if the marginal types of 2 and 3 (but not 1) quit, their expected value conditional on being a winner is $\underline{t} + (2 + \alpha)\underline{t}_2 + \underline{x} - p = (1 - \lambda)(\underline{x} - x_2) < 0$, so an atom of types of 2 and 3 wishes to quit, so (almost) all of these types lose money conditional on winning, which is a contradiction.

Now one possibility is $\alpha_1\underline{t} > \alpha\underline{t}_2$, so no types of player 1 would ever quit, since their expected values from being a winner exceed $(1 + \alpha_1)\underline{t} + 2\underline{t}_2 + x_2 > \underline{t} + (2 + \alpha)\underline{t}_2 + \lambda\underline{x} + (1 - \lambda)x_2 = p, \forall\lambda \in [0, 1]$. In this case we have a contradiction at the price that yields $\lambda = 0$ (this price is reached with positive probability—as are all prices—since the hazard rate is decreasing): The price cannot rise above this price without at least the marginal types of 2 wishing to quit, so an atom

[47] If 2 and 3 alone stop quitting, their marginal types would earn $(\underline{t}_1 + \underline{t}_2 + \underline{t}_3) + \alpha_2\underline{t}_2 + x_2 - p$, i.e., strictly lose money in expectation, if they found themselves winners; if 1 alone stops quitting, the marginal types of 2 and 3 would earn $(\underline{t}_1 + \underline{t}_2 + \underline{t}_3) + \alpha_2\underline{t}_2 + x_1 - p$ by winning and so would also stop quitting; if all stopped quitting, 1 would earn $(\underline{t}_1 + \underline{t}_2 + \underline{t}_3) + \alpha_1\underline{t}_1 + x_2 - p$, so 1 would instead continue to quit.

[48] More precisely, $\forall\varepsilon, \forall K, \exists\,\delta$ s.t. $\{\alpha_1 < \delta \Rightarrow \mid \underline{t}_2 - \underline{t}_1 \mid < \varepsilon \quad \forall \quad \underline{t}_2 < K\}$. To see this, let $\min_{0 \leq t_i \leq K}\{-x_i'(t_i)\} = \phi > 0$ (this minimum exists because $-x_i'(t_i) = 1 - x_i(t_i)(h_i(t_i))$ and $f_i(t_i)$ and hence $-x_i'(t_i)$ is continuous, and $\phi > 0$ because $x_i(t_i) < 1/h_i(t_i)$ follows from the increasing hazard rate). So $x_1 - x_2 > \phi(\underline{t}_2 - \underline{t}_1)$. So if $\delta < \phi\varepsilon/4K$, then $(\underline{t}_2 - \underline{t}_1) > \varepsilon/2 \Rightarrow d\underline{t}_1/d\underline{t}_2 > [\phi(\varepsilon/2)/(\phi\varepsilon/4)] - 1 = 1 \Rightarrow \underline{t}_2 - \underline{t}_1$ can never increase above $\varepsilon/2$.

of types 2 and 3 wishes to quit (as above), so (almost) all of these lose money conditional on winning, which is a contradiction.

Another possibility is $\alpha_1 \underline{t} = \alpha \underline{t}_2$. In this case the marginal types of 1 also wish to quit at $\lambda = 0$. But the price cannot then rise higher without any types of 2 and 3 quitting, since 1's marginal condition would imply $p = (1 + \alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2 \geq \underline{t}_1 + (2 + \alpha)\underline{t}_2 + x_2$, which implies that types of 2 and 3 must quit, but for any (actual) relative probability $\lambda = H_3/(H_1 + H_3)$ with which player 2 believes that another player who quits is player 3, player 2's expected value from winning is $\underline{t}_1 + (2 + \alpha)\underline{t}_2 + \lambda x_1 + (1 - \lambda)x_2 < p$, so an atom of types of 2 and 3 must quit, which is a contradiction, as before.

Finally, we may have $\alpha_1 \underline{t} < \alpha \underline{t}_2$ at the price at which types of 2 and 3 stop quitting. In this case types of player 1 start quitting at price $p = (1 + \alpha_1)\underline{t} + 2\underline{t}_2 + x_2 = \underline{t} + (2 + \alpha)\underline{t}_2 + \lambda \underline{x} + (1 - \lambda)x_2$ for some $\lambda \in (0, 1)$. At this price the marginal types of players 2 and 3 must also start quitting at hazard rates such that $\lambda = H_2/(H_1 + H_2)\ (= H_3/(H_1 + H_3))$. (If not, the marginal types of players 2 and 3 would either be strictly losing or strictly making money (in expectation) conditional on winning. Both are contradictions, the latter because the types just below the current marginal types of 2 and 3 would not have been willing to quit earlier where their first-order conditions were satisfied.)

Now where types of 1 start quitting, we have $\underline{t}_1 = \underline{t} < (\alpha/\alpha_1)\underline{t}_2$ and $x_2 > x_1$, so when α_1 is small we require λ small, that is, H_1/H_2 large, hence $d\underline{t}_1/d\underline{t}_2$ is large. So $\alpha_1 \underline{t}_1 - \alpha \underline{t}_2 \longrightarrow 0$ and $\alpha_1 \underline{t}_1 = \alpha \underline{t}_2$ is achieved for finite $\underline{t}_2$. (Until this point $\underline{t}_1$ and $\underline{t}_2$ must just be following the differential equation determined by $p = (1 + \alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2 = \underline{t}_1 + (2 + \alpha)\underline{t}_2 + [H_2/(H_1 + H_2)]x_1 + [H_1/(H_1 + H_2)]x_2$, that is, the same differential equation as in the increasing-hazard-rate case.) But at $\alpha_1 \underline{t}_1 = \alpha \underline{t}_2$, and hence $H_2/(H_1 + H_2) = 0$, we have the same contradiction that we had with $\alpha_1 \underline{t} = \alpha \underline{t}_2$ and $\lambda = 0$. (Any finite rate of quitting of player 2 would imply that all types close to $\underline{t}_2$ strictly wished to quit, which is a contradiction, but if no types of player 2 quit as the price, and hence $\underline{t}_1$, rises, then we will have $p = (1 + \alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2 > \underline{t}_1 + (2 + \alpha)\underline{t}_2 + x_2$, which is also a contradiction.)

So the equilibrium we found, in which player 1 never quits while players 2 and 3 quit symmetrically according to $\underline{t}_2 = \underline{t}_3$ and $p = \underline{t} + (2 + \alpha)\underline{t}_2 + \underline{x}$, is the unique (perfect Bayesian) Nash equilibrium satisfying our assumptions, and the final price is $\underline{t} + \underline{x} + (2 + \alpha)\min(t_2, t_3) = E(t) + (2 + \alpha)\min(t_2, t_3) \approx E(t) + 2\min(t_2, t_3)$, in which t_2 and t_3 are the actual signals of bidders 2 and 3.

□ **The symmetric case.** When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, it is straightforward that it is a (perfect Bayesian) equilibrium for bidders to quit according to $\underline{t}_1 = \underline{t}_2 = \underline{t}_3$ and $p = (3 + \alpha)\underline{t}_1 + x_1$, and that this is the unique equilibrium satisfying our assumptions. In this case the final price is $(3 + \alpha)t_{(3)} + x_{(3)} \approx 2t_{(3)} + E(t \mid t \geq t_{(3)})$. Thus we have proved Lemmas 3 and 5.

□ **Other equilibria.** Note that there are other equilibria of the ascending auction that do not satisfy the additional assumptions we imposed in Section 2. In particular:

(i) When three bidders compete for a single unit, and player 2 or 3 receives a signal sufficiently low ($\alpha_2 t_2 < \alpha_1 \underline{t}$ or $\alpha_3 t_3 < \alpha_1 \underline{t}$) that he knows he will surely lose to player 1, different equilibria can be constructed by making different assumptions about how far he bids up the price in such a case (however, assumptions different from ours would not importantly affect our results).

(ii) When three bidders compete for a single unit, the chance of both opponents quitting at any point is of second order, so equilibria can be constructed in which a player may bid beyond the point where he would wish to win but then drop out immediately when another player does. (Allowing equilibria of these kinds would not affect any of our results, assuming symmetric players behave symmetrically; note also that these equilibria would not survive if bids increased in discrete jumps so that there was generally positive probability of both opponents quitting simultaneously.)

(iii) Even when just two players i and j compete for a single unit, it is an equilibrium for i to quit immediately while j never quits. (With unbounded supports of the signals, this is a perfect Bayesian Nash equilibrium supported by j believing that if he were to observe the out-of-equilibrium behavior that i stays in to price p, then i's signal is at least p; such equilibria can be ruled out by having a largest possible signal, or by insisting each player bids up at least as far as his minimum possible value given his own information.) Obviously these kinds of equilibria also arise when three bidders compete for either one or two units.

(iv) When three bidders compete for two units, it seems possible to construct equilibria in which symmetric bidders behave asymmetrically; beyond a certain price just one bidder is quitting, so there is no restriction on this bidder's beliefs about who has quit conditional on the out-of-equilibrium event that he finds himself a winner, and the careful choice of beliefs may support an equilibrium.

(v) When three players compete for two units and hazard rates are decreasing, equilibria can be constructed in which the first-order conditions fail because a player initially expects to lose money conditional on winning, but he expects to make up these losses (in expectation) if the bidding continues for a while. These equilibria also seem particularly unnatural because they require symmetric players to behave asymmetrically.

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