Introduction into Market Design

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Content of Lectures

- Lecture 1. Introduction into Market and Mechanism Design
- Lecture 2. Revenue Management and Price Discrimination
- Lecture 3. Information Design and Price of Information

Topic 1. Introduction into Market Design

- how to design a market/game
- what are the constraints/what is feasible?
- two important insights:

1. revelation principle
2. revenue equivalence among auction formats

- a leading example: generalized second price

A Leading Example: Generalized Second Price

- an auction format tailored to its environment:
- sponsored search and display advertising
- Google's revenue in 2017 over $ 100B over 80% from GSP
- Other companies using GSP and its variations:
  - facebook
  - Bing - Microsoft
  - Amazon
  - Yahoo

History

unlike spectrum auctions and electricity auctions, which were designed essentially from scratch, sponsored search auctions evolved over time

- pre-internet advertising (think print media): volume pricing, person-to-person negotiations
- early Internet advertising (1994): per-impression pricing, person-to-person negotiations, no keyword targeting.
- Overture's (1997) generalized first-price auctions:
  – pay-per-click, for a particular keyword
  – completely automated, bids can be changed at any time
  – links are arranged in the descending order of bids
  – pay your own bid

YAHOO! SEARCH

Search Results

Results **1 - 10** of about **107,000,000** for **car insurance** - 0.05 sec. (About this page)

Also try: **car insurance** quotes, cheap **car insurance**, progressive **car insurance** More...

SPONSOR RESULTS

- GEICO **Car Insurance** - Official Site
  www.geico.com Get an instant quote and see how much you could save.
- Progressive **Car Insurance**: Official Site
  www.progressive.com Named #1 **insurance** website by Watchfire. Get a free direct quote now.
- Save on **Car Insurance** Instantly
  www.esurance.com Get a quote, compare quotes and buy your policy instantly online.
- State Farm **Car Insurance**: Official Site
  www.statefarm.com State Farm offers competitive and affordable rates for **car insurance**.

Car Insurance Quotes on Yahoo! Autos
Yahoo! Shortcut - About

1. State Farm **Insurance**
   Offering a full line of auto, home, life, health, and business **insurance** policies.
   Quick Links: Auto Rates - Get A Quote - Auto **Insurance** Promotions
   www.statefarm.com - More from this site - Save

2. Allstate Auto **Insurance**
   Offers auto **insurance** coverage options, **car insurance** quotes, and cost saving assistance.
   Category: Automotive **Insurance**
   auto-**insurance**.allstate.com - 22k - Cached - More from this site - Save

3. GEICO **Car Insurance**. Get an auto **insurance** quote and save today. Free online motorcycle quotes as well.
   GEICO auto **insurance**, online **car insurance** quote, motorcycle **insurance** quote, online **insurance** sales and service from a leading **insurance** company. See how much you could save with our rates. ... Buy Auto **Insurance** Make a Payment Auto Claims ... JavaScript capable browser is required to be able to select products Auto **Insurance** Motorcycle **Insurance** your quote ...
   Quick Links: Auto **Insurance** - Homeowners **Insurance** - Motorcycle **Insurance**
   geico.com - 32k - Cached - More from this site - Save

SPONSOR RESULTS

Car Insurance in CA, OH, GA, TN and FL
CA, OH, GA, TN, FL: Buy in 10 minutes - accidents and violations OK.
www.thegeneral.com

Allstate Auto **Insurance** - Official Site
Safe drivers save up to 20% with Allstate. Get a quote.
www.allstate.com

AIG Auto **Insurance** - Instant Quotes
Instant, online, accurate **car insurance** quotes direct from AIG...
www.aigauto.com

Compare Multiple Auto **Insurance** Quotes
NetQuote's fast free online rates. Comparison shop nationwide. Get...
www.netquote.com

Car Insurance at **Insurance**.com
Compare multiple instant, accurate **car insurance** rates at...
www.insurance.com

Car Insurance
Free side-by-side quotes from Safeco, Esurance, Hartford, Travelers...
www.answerfinancial.com

http://uv.bidtool.overture.com/d/search/tools/bidtool/index.jhtml?Keywords=car+insurance&mkt=us&lang=en_US&Partner=userbidtoc
overture bid tool

View Bids

Type in a search term and we'll show you the Max Bids and listings for that term.

car insurance

Search Cancel

1. **GEICO Car Insurance - Official Site**
   Get an instant quote and see how much you could save.
   www.geico.com
   (Advertiser's Max Bid: $10.49)

2. **Progressive Car Insurance: Official Site**
   Named #1 insurance website by Watchfire. Get a free direct quote now.
   www.progressive.com
   (Advertiser's Max Bid: $10.48)

3. **Save on Car Insurance Instantly**
   Get a quote, compare quotes and buy your policy instantly online. Print your insurance cards in minutes. Quote. Buy. Print. It's that easy to save on car insurance with Esurance.
   www.esurance.com
   (Advertiser's Max Bid: $9.15)

4. **State Farm Car Insurance: Official Site**
   State Farm offers competitive and affordable rates for car insurance. Deep discounts for safe drivers, good students and safety features in your car.
   www.statefarm.com
   (Advertiser's Max Bid: $9.14)

5. **Car Insurance in CA, OH, GA, TN and FL**
   CA, OH, GA, TN, FL: Buy in 10 minutes - accidents and violations OK.
   www.thegeneral.com
   (Advertiser's Max Bid: $8.99)

6. **Allstate Auto Insurance - Official Site**
   Safe drivers save up to 20% with Allstate. Get a quote.
   www.allstate.com
   (Advertiser's Max Bid: $8.50)

Done

Problem: Generalized First Price Auction

- Generalized First-Price Auction is unstable, because it generally does not have a pure strategy equilibrium, and bids can be adjusted dynamically.

Example.

- Two slots and three bidders.
- First slot gets 100 clicks per hour, second slot gets 70.
- Bidders 1, 2, and 3 have values per click of $10, $8, and $5, respectively.
- There is no pure strategy equilibrium in the one-shot version of the game. If bidders best respond to each other, they will want to revise their bids as often as possible.

History: Generalized Seconcd Price Auction

- Google's (2002) generalized second-price auction (GSP):
  - pay the bid of the next highest bidder
- Later adopted by Yahoo!/Overture and others.

Generalized Second-Price and Vickrey Auctions

"[Google's] unique auction model uses Nobel Prize-winning economic theory to eliminate [. . .] that feeling that you've paid too much."

— marketing materials at google.com

- With only one slot, GSP is identical to the standard second price auction (a.k.a. Vickrey, VCG).
- With multiple slots, the mechanisms are different
  - GSP charges bidder k the *bid* of bidder $k + 1$
  - VCG charges bidder k for his *externality*
- a misunderstanding...

Language for Market Design

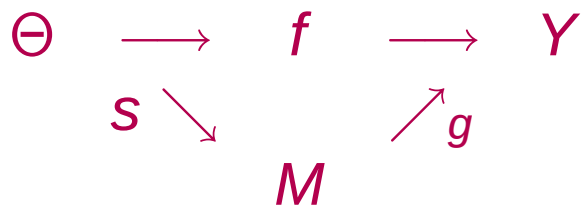
- agents $i = 1, ..., I$
- private information $\theta_i \in \Theta_i$
- common prior distribution F_i, $F = \times_{i=1}^{I} F_i$
- allocations $y = (x, t_1, \cdots, t_I)$
  - t_i : monetary transfer from agent i
  - x : non-monetary dimension
- utility $u_i(y, \theta)$

$$u_i(y, \theta) = v_i(x, \theta_i) - t_i$$

- leading examples: auction, public good

Mechanism Design

- social choice function f
- message m_i ,strategy: $s_i : \Theta_i \longrightarrow M_i$
- outcome function $g : M \longrightarrow Y$
- commuting diagram:



- market design: how to choose M and g?
- first-price, second-price, English, Dutch auctions all are instances of market/auction design

Constraints: Incentives and Participation

- agent have incentive to report thrutfully and wish to participate
- dominant strategy incentive compatibility:

$$u_i\left(f\left(\theta_i,\theta_{-i}\right),\theta_i\right) \geq u_i\left(f\left(\theta_i',\theta_{-i}\right),\theta_i\right), \quad \forall\theta_i,\forall\theta_i',\forall\theta_{-i}$$

- ex post individual rationality constraints:

$$v_i\left(x\left(\theta_i,\theta_{-i}\right),\theta_i\right) - t_i\left(\theta_i,\theta_{-i}\right) \geq v_0$$

- there are many ways to elicit information, many languages, many procedures, many queries
- is there a canonical, universal mechanism
- yes: revelation principle

Theorem

For every mechanism $\Gamma' = \langle M', g' \rangle$ *and every pure strategy DSE* σ' *of* Γ'*, there exists a direct mechanism* $\Gamma = \langle \Theta, f \rangle$ *and a pure strategy DSE* σ *of* $\langle \Theta, f \rangle$ *such that:*

1. $\sigma_i(\theta_i) = \theta_i$;
2. $g'(\sigma'(\theta)) = f(\theta)$.

- direct mechanism: $M = \Theta$
- truth-telling: $\theta = \theta'$

What Can We Achieve

- socially efficient objective, maximize social surplus, gains from trade
- later: consider variant, individual objectives
- Vickrey (1961), Clarke (1971), Groves (1973) (VCG)
- establish incentive compatibility in dominant strategies for efficient social choice function:

$$f^*(\theta) \in \arg\max_{x \in X} \sum_{i=1}^{I} v_i(x, \theta_i).$$

Efficient Mechanism Design

- agent i internalizes social objective by paying for report θ' :

$$t_i\left(\theta'\right) = -\sum_{j \neq i}^{I} v_j\left(f^*\left(\theta'\right), \theta'_j\right)$$

Theorem

Every efficient social choice function f can be truthfully implemented in a dominant strategy equilibrium by a VCG mechanism.

- leading example: second price auction
- highest bidder wins and pays second highest bid
- willingness to pay is announced truthfully!

Pivot Mechanism = Social Externality Cost

- transfer in pivot mechanism

$$t_i^P(\theta) = \sum_{j \neq i}^{I} v_j\left(f_{-i}^*(\theta), \theta_j\right) - \sum_{j \neq i}^{I} v_j\left(f^*(\theta), \theta_j\right)$$

- externality cost of i :
- i internalizes social objective as i pays her externality cost
- marginal contribution of i = utility of i - externality cost of i :

$$\sum_{j=1}^{I} v_j\left(f^*(\theta), \theta_j\right) - \sum_{j \neq i}^{I} v_j\left(f_{-i}^*(\theta), \theta_j\right)$$

- second price auction: highest bidder pays second highes bid

Return to GSP

- Two slots, three bidders. First slot gets 100 clicks per hour, second slot gets 70. Bidders 1, 2, and 3 have values per click of \$10, \$8, and \$5, respectively. If all advertisers bid truthfully, then bids are \$10, \$8, \$5.
- Under GSP, payments for slots one and two are \$8 and \$5 per click. Total payments of bidders one and two are \$800 and \$350, respectively.
- Under VCG, the second bidder's payment is still \$350. However, the payment of the first advertiser is now \$590: \$350 for the externality that he imposes on bidder 3 (by forcing him out of position 2) and \$240 for the externality that he imposes on bidder 2 (by moving him from position 1 to position 2 and thus causing him to lose $(100 - 70) = 30$ clicks per hour).

Truth-Telling is not a Dominant Strategy under GSP

- Per click values are \$10, \$8, and \$5
- CTR's are 100 and 70
- If everyone bids truthfully, bidder 1's payoff is

$$(\$10 - \$8) * 100 = \$200.$$

If instead bidder 1 bids \$6, his payoff is

$$(\$10 - \$5) * 70 = \$350 > \$200.$$

GSP and the Generalized English Auction

- $N \geq 2$ slots and $K = N + 1$ advertisers
- α_i is the expected number of clicks in position i
- s_k is the value per click to bidder k
- A clock shows the current price; continuously increases over time
- A bid is the price at the time of dropping out
- Payments are computed according to GSP rules
- Bidders' values are private information, drawn randomly from commonly known distributions

Strategy can be represented by

$p_k(i, h, s_k)$
s_k is the value per click of bidder k,
p_k is the price at which he drops out,
i is the number of bidders remaining (including bidder k), and
$h = (b_{i+1}, \ldots, b_{N+1})$ is the history of prices at which bidders $N+1, N, \ldots, i+1$ have dropped out.
If bidder k drops out after history h, he pays b_{i+1} (unless the history is empty, then set $b_{i+1} \equiv 0$).

Revenue Equivalence

Theorem. *In the unique perfect Bayesian equilibrium of the generalized English auction with strategies continuous in s_k, an advertiser with value s_k drops out at price*

$$p_k(i, h, s_k) = s_k - \frac{\alpha_i}{\alpha_{i-1}}(s_k - b_{i+1}).$$

In this equilibrium, each advertiser's resulting position and payoff are the same as in the dominant-strategy equilibrium of the game induced by VCG. This equilibrium is ex post: the strategy of each bidder is a best response to other bidders' strategies regardless of their realized values.

Conclusions

- market/mechanism design can reveal private information, willingness-to-pay for product
- important results: revelation principle, revenue equivalence
- aggregate many pieces of decentralized private information
- GSP looks similar to VCG, but is not the same:
- GSP is not dominant strategy solvable, and truth-telling is generally not an equilibrium;
- corresponding Generalized English Auction:
  - has a unique equilibrium and explicit analytic formulas for bid functions, which is very useful for empirical analysis;
  - is a robust mechanism—the equilibrium does not depend on distributions of types, beliefs, etc.

Application: Sponsored Search

- many positions, hence many items to be auctioned of
- value of position k for advertiser i:

$$\alpha_k v_i$$

- ranking of positions

$$\alpha_1 > .... > \alpha_k > ...\alpha_K > 0$$

- VCG mechanism ("internalizing externality)

$$p_k = \alpha_k v_{k+1} - \sum_{l=k+1}^{K} \alpha_l \left(v_l - v_{l+1}\right)$$

- special case: $\alpha_1 = ... = \alpha_K = \alpha$:

$$p_k = \alpha v_{K+1}, \text{ for all } k$$

A Misunderstanding: Generalized Second Price Auction

- Overture as predecessor of Yahoo
- k highest bidder has to pay bid of $k+1$ highest bidder
- by recursion:

$$b_{K+1} = \alpha_K v_{K+1} \Rightarrow p_K = \alpha_K v_{K+1}$$

- now what is bidder K willing to bid to get a higher rank

$$\alpha_{K-1} v_K - b_K = \alpha_K v_K - \alpha_K v_{K+1} \Rightarrow b_K = (\alpha_{K-1} - \alpha_K) v_K + \alpha_K v_{K+1}$$

- thus the price of bidder $K-1$ is:

$$p_{K-1} = \alpha_{K-1} v_K - \alpha_K (v_K - v_{K+1}),$$

and hence payoff equivalent to VCG mechanism

Countering the Winner's Curse: Auction Design in a Common Value Model

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Interdependence and Winner's Curse

- interdependence in values across bidders is frequent in auctions

$\rightarrow$ wildcatters bidding for an oil tract ...

$\rightarrow$ investment banks competing for shares in IPO's...

$\rightarrow$ lenders competing in syndicated loan-markets ...

- winning the object is informative about value estimate of competing bidders
- each bidder must carefully account for the interdependence in individual bidding behavior
- winner's curse: unconditional vs conditional expectation

Winner's Curse and Adverse Selection

- consider bidding for a natural resource, such as an oil tract
- richer samples suggest more oil reserves and induce higher bids
- winning means that the other samples' were relatively weak
- a winning bidder therefore faces adverse selection
- the expected value of the tract conditional on winning is less than the unconditional expectation

Winner's Curse and Auction Design

- winner's curse results in bid shading and lower revenues
- how can auction design attenuate the winner's curse...
- how can the resulting selection impact revenue:
  adverse, neutral or advantageous
- today: what is the revenue maximizing selling mechanism?
- prior literature has largely focused on private value

$\rightarrow$ thus a world without winner's curse and selection issues

Auction Design in A Common Value Model

- a pure common value model
- private signal gives partial information about common value
- key statistical feature:
  higher signals contain more information about common value than lower signals
- today:
  $\rightarrow$ highest signal is sufficient statistic of common value
  $\rightarrow$ lower signals carry no additional information

Revenue Maximizing Design

- characterize revenue maximizing mechanism
- maximal revenue is obtained by strikingly simple mechanism, stated at interim level (given signal of bidder i)

1. constant – signal independent – price
2. constant – signal independent – probability of getting object

- contrast with first, second, or ascending auction in an environment with private values

Revenue Maximizing Design: Posted Price

- optimal mechanism shares some features with posted price

1. constant – signal independent – price

- it coincides with posted price if

2. constant – signal independent – probability is $1/N$

- necessary and sufficient condition when optimal mechanism reduces exactly to posted price
- if posted price is an optimal mechanism it is inclusive: every bidder with every signal realization is willing to buy

Revenue Maximizing Design: Beyond Posted Price

- in general, aggregate assignment probability is < 1
- interim probability of getting object is constant and $< 1/N$
- ex post probability for i then depends on entire signal profile
- conditionally on allocating the object optimal mechanism:

1. favors bidders with lower signals
2. discriminates against bidder with highest signal

- "winner's blessing" rather than "winner's curse"
- advantageous rather than adverse selection

Contributions: Substantive

- setting where bidders with higher signals have more accurate information about common value;
- arises in market with intermediaries, and many other settings: auctions for resources, IPO's
- countervailing screening incentives, tension between selling to

1. bidder with higher expected value and
2. bidder with less private information

- optimal to screen "less" - with no screening in inclusive limit
- foundation for posted price mechanisms

Contributions: Methodological

- very few results extend characterization of optimal auctions beyond private value case
- we extend optimal auctions into interdependent values:

1. with private values, "local" incentive constraints are sufficient to pin down optimal mechanism
2. with interdependent values, "global" constraints matter, new arguments are required

Model

Common Value Model

- N bidders compete for a single object
- bidder i receives signal s_i :

$$s_i \in [\underline{s}, \overline{s}] \subset \mathbb{R}_+$$

  according to absolutely continuous distribution $F(s_i), f(s_i)$
- common value is the maximum of N independent signals:

$$v(s_1, \ldots, s_N) \triangleq \max\{s_1, \ldots, s_N\}$$

- "maximum signal model"
- signal distribution $F(s_i)$ induces value distribution $G_N(v)$:

$$G_N(v) = (F(s))^N$$

- common value is first-order statistic of N independent signals

Two Interpretations

- maximum signal model

$$v(s_1, \ldots, s_N) = \max\{s_1, \ldots, s_N\}$$

- two leading interpretations:

1. common value model with informational implications:
   - higher signal realizations contain more information about common value than lower signal realizations
   - specifically, conditional on highest signal, the other signals contain no additional information about the common value
   - drilling/sampling for mineral rights (Bulow and Klemperer (2002))

Two Interpretations

- maximum signal model

$$v(s_1, \ldots, s_N) = \max\{s_1, \ldots, s_N\}$$

- two leading interpretations:

2. private value model of intermediary (dealer) market

   - each intermediary bidder receives the signal (sample) about the downstream trading opportunities
   - final sale in downstream market is open to all intermediaries
   - IPO, syndicated loan-markets, inter-dealer markets (Viswanathan and Wang (2004))

Utility and Allocation

- bidder i is expected utility maximizer with quasilinear preferences, probability q_i of receiving object and transfers t_i:

$$u_i(s, q_i, t_i) = v(s)\, q_i - t_i$$

- feasibility of auction

$$q_i(s) \geq 0, \text{ with } \sum_{i=1}^{N} q_i(s) \leq 1$$

- *ex post* transfer $t_i(s)$ of bidder i, *interim* expected transfer:

$$t_i(s_i) = \int_{s_{-i} \in S^{N-1}} t_i(s_i, s_{-i})\, f_{-i}(s_{-i})\, ds_{-i},$$

where

$$f_{-i}(s_{-i}) = \prod_{j \neq i} f(s_j)$$

Incentive Compatibility

- bidder i surplus when reporting s_i' while observing s_i:

$$u_i\left(s_i, s_i'\right) \equiv \int_{s_{-i} \in S^{N-1}} q_i\left(s_i', s_{-i}\right) v\left(s_i, s_{-i}\right) f_{-i}\left(s_{-i}\right) ds_{-i} - t_i\left(s_i'\right)$$

- indirect utility given truthtelling is:

$$u_i(s_i) \equiv u_i\left(s_i, s_i\right)$$

- direct mechanism $\{q_i, t_i\}_{i=1}^N$ is *incentive compatible (IC)* if

$$u_i\left(s_i\right) \geq u_i\left(s_i, s_i'\right), \text{ for all } i \text{ and } s_i, s_i' \in S$$

- *... is individually rational (IR)* if $u_i\left(s_i\right) \geq 0$, for all i and $s_i \in S$

The Winner's Curse

Warm-Up: Second Price Auction

- second-price auction in maximum signal model:

$$b_i(s_i)$$

- bid of bidder i is based on his interim expectation:

$$\mathbb{E}[v(s_1, ..., s_N) \,|s_i]$$

- signal s_i is sharp lower bound on ex post (realized) value:

$$s_i \leq v(s_1, ..., s_N),$$

- signal s_i is lower bound for interim expectation of value:

$$s_i < \mathbb{E}[v(s_1, ..., s_N) \,|s_i]$$

Winner's Curse in Second Price Auction

- bidder with highest signal wins in second price auction
- equilibrium bid is given by:

$$b_i(s_i) = s_i$$

- bids as-if private value s_i, not common value $\max\{s_1, ..., s_N\}$
- conditional on winning, signal s_i turns into sharp upper bound:

$$v(s_1, ..., s_N) = \max\{s_1, ..., s_N\} \leq s_i$$

- this is the curse:

1. when bidding, s_i is sharp lower bound of expectation of value
2. when winning, s_i is sharp upper bound of expectation of value

Winner's Curse and Adverse Selection

- adverse selection:
  winner learns his signal is most favorable of all signals
- selection as winner is adverse information to winner
- magnitude of adverse selection is controlled by change in expectation from ex-interim to ex-post:

1. when bidding, s_i is sharp lower bound of expectation of value
2. when winning, s_i is sharp upper bound of expectation of value

- structure of information controls strength of winner's curse
- winner's curse lowers bids, thus lowers revenue of auctioneer
- maximal winner's curse is quantified by minimal revenue
  (in any given auction format)

Magnitude of Winner's Curse

Magnitude of the Curse

- can we quantify the winner's curse ?
- can we identify maximal winner's curse which generates minimal revenue?
- how does it relate to the structure of private information of bidders?
- making it operational:
  consider all possible information structures for a fixed distribution of values

Information and Winner's Curse

- fix a distribution of (common) values with N bidders:

$$G_N(v)$$

- ask how different common prior distribution of signals:

$$F(s\,|\,v)$$

  impact bidding and revenue for fixed distribution $G_N(v)$
- maximum signal model: an example of information structure, others are wallet game, afiliated mineral rights model, etc.

Revenue Minimum

- "Revenue Guarantee Equivalence" (AER forthcoming) finds:

1. equivalence: the maximum signal model attains the same revenue in all standard auctions: first-price, second-price, ascending auction, etc.
2. guarantee: the maximum signal model generates the lowest revenue across all information structures in every standard auction

- thus a sharp revenue guarantee can be established with the maximum signal model, and it is the same, hence revenue guarantee equivalence, across all standard auction formats
- revenue minimizing–winner's curse maximizing:

$$v\left(s_1, \ldots, s_N\right) = \max\left\{s_1, \ldots, s_N\right\}$$

A Visualization

- standard auction (with no reserve prices) with two bidders
- revenue and bidders surplus in all information structures

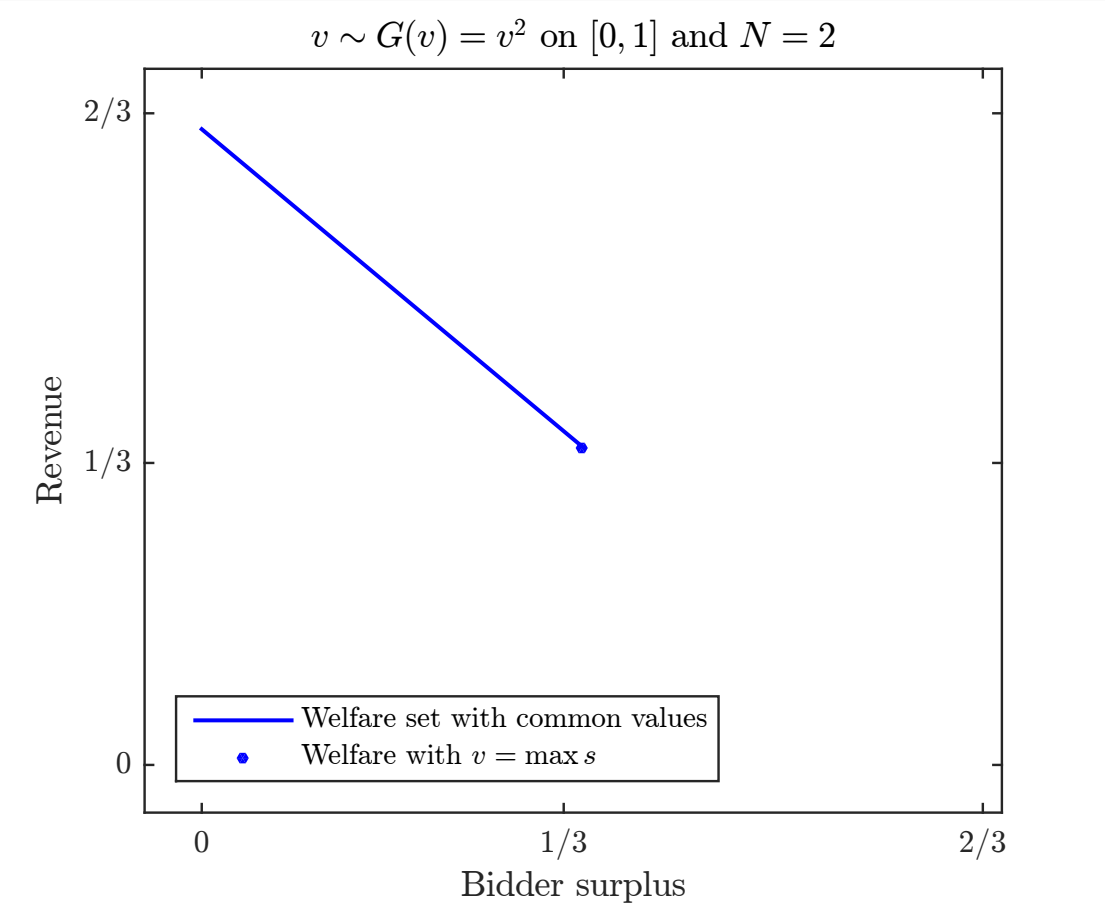

Figure 1: Revenue and Bidder Utility across All Information Structures

Structure of Incentive Constraints

- structure of incentive constraints in maximum signal model
- all upward deviations–relative to unique equilibrium bid–yield the equilibrium net utility
- all upward deviations are binding:

$$b' \in [b_i(s_i), b_i(\overline{s})], \qquad \forall s_i \in [\underline{s}, \overline{s}]$$

- global rather than local inventive constraints matter, everywhere!
- global constraints matter in all standard auction formats!

Upward Deviations

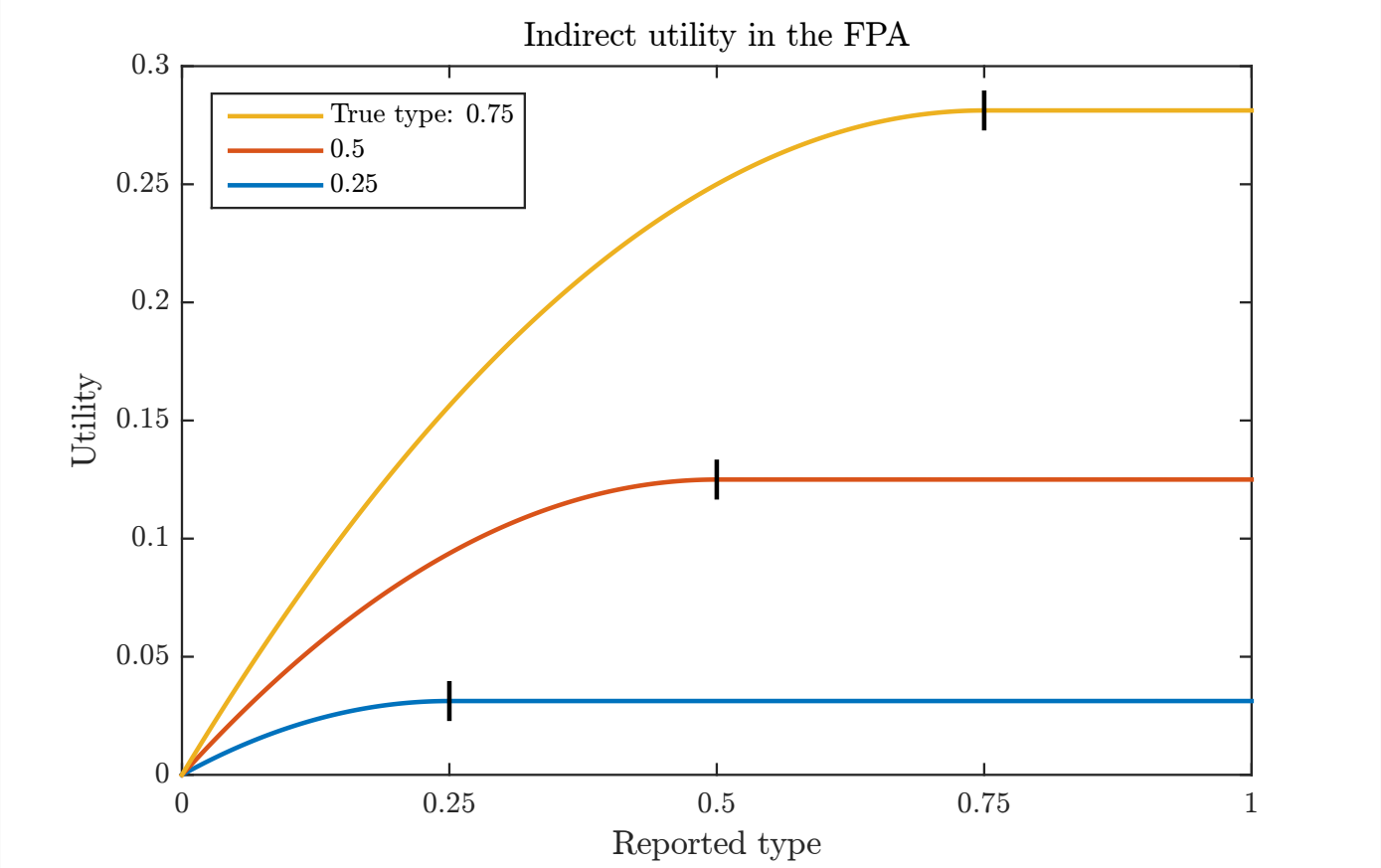

Figure 2: Uniform Upward Incentive Constraints and Winner's Curse

- counter the curse: find optimal auction

Counter the Curse

Adverse Selection and Winner's Curse

- assigning object to highest bidder conveys (too) much information to the winner
- adverse selection: winner learns that his signal was more favorable than all other signals
- winning bid is depressed by adverserial selection of winner
- what about neutral selection of winner?
- a neutral (symmetric) selection must be a random allocation among the bidders
- event of winning does not convey any additional information to the winner

Neutral Selection: Inclusive Posted Price

- a specific neutral selection
- every bidder receives the object with equal probability $1/N$
- every winning bidder is charged a posted price

$$p \triangleq \int_{s_{-i}} v\left(\underline{s}, s_{-i}\right) f_{-i}\left(s_{-i}\right) ds_{-i}$$

- even bidder with lowest signal, $s_i = \underline{s}$, is willing to buy at p,
- thus p is inclusive, does not exlude any signal s_i for any i

Revenue Improvement

- how does inclusive posted price fare?

Proposition (Bulow & Klemperer 2002)

The inclusive posted price yields a (weakly) higher revenue than the absolute second price auction.

- notable features of inclusive posted price

1. random allocation–rather than deterministic allocation
2. constant allocation in signal – rather than increasing in signal
3. no selection on either signal or value, thus no screening

Neutral Selection and Exclusion

- exclusion–not selling the object when the value is low–may increase the revenue
- in private value environments it famously does: Myerson (1981)
- can neutral selection be maintained with exclusion?

Exclusive Posted Price

- uniform exclusion at a threshold r:

$$q_i(s) = \begin{cases} \frac{1}{N} & \text{if } \max s \geq r; \\ 0 & \text{otherwise.} \end{cases}$$

- supported by a pair of prices:

1. an unconditional price:

$$p_u \triangleq r,$$

2. a conditional price:

$$p_c \triangleq \frac{\int_r^{\overline{s}} \max\{s_{-i}\}\, dF_{-i}(s)}{1 - F^{N-1}(r)} > r = p_u,$$

$\Leftrightarrow$ right censored first order statistic of $N-1$ samples

Exclusive Posted Price Mechanism

- object is sold if and only if at least one bidder is willing to make an unconditional purchase at $p_u = r$.
- then *all* bidders get object with probability $1/N$ at price p_c
- with one exception... if only one bidder ask for unconditional purchase, then this bidder gets object at $p_u < p_c$

Proposition (Posted Price Pair)

The posted price pair (p_c, p_u) yields a (weakly) higher revenue than any other inclusive or exclusive posted price.

Implications of Exclusive Posted Price

- uniform screening among bidders with respect to highest signal
- uniform exclusion among bidders'
- winning at generates winner's blessing:

$$\mathbb{E}[v(s_1, ..., s_N) \,|s_i] < \mathbb{E}[v(s_1, ..., s_N) \,|s_i, x_i > 0]$$

- two-tiered pricing similar to syndicated loan arrangement: one for lead lender, and one for all syndicate lenders
- turned from adverse to neutral selection
- now turn from neutral to to advantageous selection!

Advantageous Selection

- there is a fixed reserve price r and a random reserve price $x > r$
- if bidder i reports highest signal $s_i > r$, then:

1. he receives priority status,
2. he is offered object at price:

$$p \triangleq max\{x, s_{-i}\}$$

- otherwise, other bidders receive object with probability

$$1/(N-1),$$

- if at least one bidder has declared priority status and pay price:

$$p \triangleq \max\{r, s_{-i}\} = v(s_1, ..., s_N).$$

Random Reserve Price

- reserve price r^* is smallest solution to:

$$x - \int_{y=x}^{\bar{s}} \frac{1 - F(y)}{F(y)} dy = 0$$

- distribution of random reserve price is:

$$H^*(x) = \frac{1}{N}(1 - \frac{F^N(r)}{F^N(x)})$$

Theorem (Random Reserve Price)

The random reserve price mechanism (r^*, H^*) *is a revenue maximizing mechanism.*

- interim probability of receiving object is constant in signal s_i
- interim transfer is constant in signal s_i
- advantageous selection
- all downward incentive constraints are binding!

A Visualization

- with random reserve price, each bidder is indifferent between his equilibrium bid and any lower bid

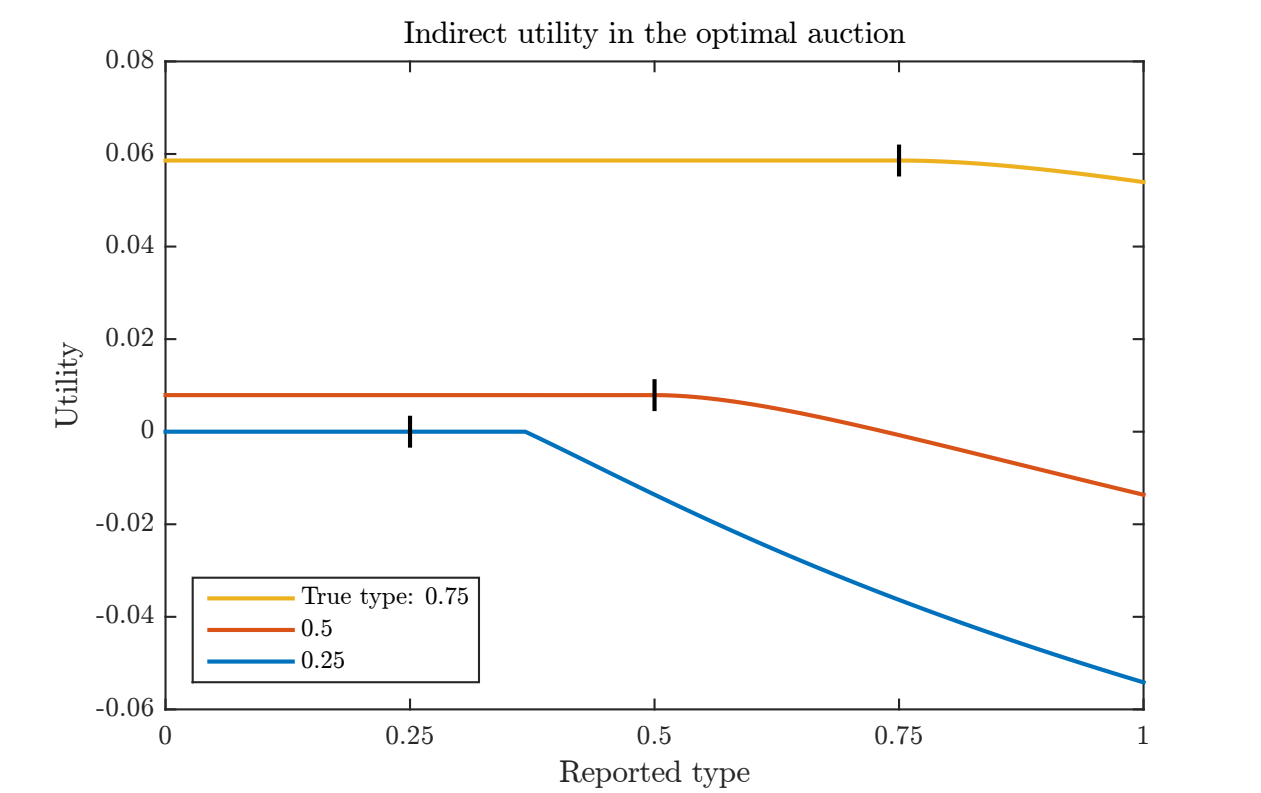

Figure 3: Uniform Downward Incentive Constraints

A Study in Contrasts

- optimal vs standard mechanisms
- exactly flip the orientation of the constraints, and more...

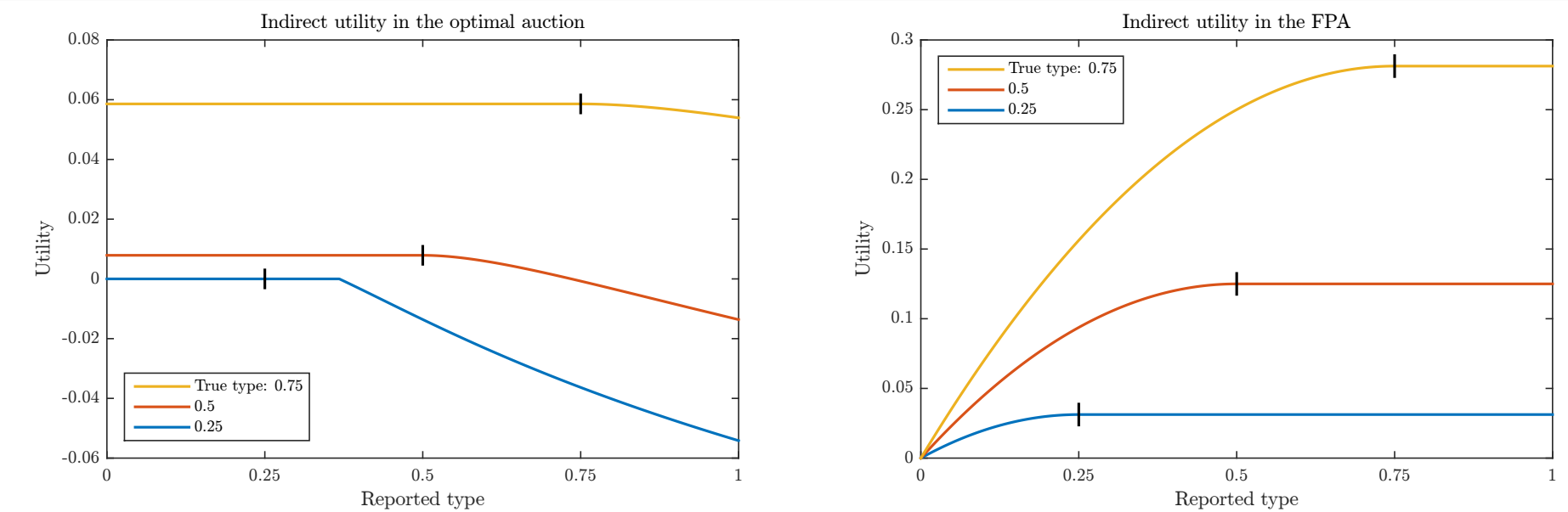

Figure 4: Uniform Downward vs Upward Incentive Constraints

Bounds on Bidder Surplus and Revenue

A New Problem

- how to establish the optimality of the mechanism?
- evidently, the local constraints are binding, but many others, non-local constraints are binding as well
- thus, we need to consider local as well global constraints
- but which ones?
- analyze a relaxed problem which consists of local and small class of global constraints
- use these constraints to derive:

1. an upper bound on seller revenue
2. a lower bound on bidder utility

A Relaxed Problem

- consider a smaller–one-dimensional–family of constraints:
- instead of reporting signal s_i, report a random signal

$$s_i' < s_i,$$

drawn from truncated prior on support $[\underline{s}, s_i]$:

$$F\left(s_i'\right) / F\left(s_i\right)$$

- *misreporting a redrawn lower signal*

A Lower Bound on Bidder Utility

- what are the gains from *misreporting a redrawn lower signal?*
- equilibrium surplus of a bidder with type x is
  –from envelope condition of local constraints:

$$u_i(s_i) = \int_{x=\underline{s}}^{s_i} \widehat{q}_i(x)\, dx$$

- surplus from misreporting the redrawn lower signal

$$\frac{1}{F(s_i)} \int_{x=\underline{s}}^{s_i} u_i(s_i, x)\, f(x)\, dx$$

- gains vary depending on realized misreport
  average gains across all misreports are easy to compute

Average Gains from Misreporting

- misreport is redrawn from prior, bidder i is equally likely to fall anywhere in distribution of signals, unconditional on misreport, ex-ante likelihood that i receives object and x is highest signals

$$q_i(x)\, g_N(x)$$

- if highest report is less than s_i, surplus that bidder i obtains from being allocated object is s_i rather than x, so $s_i - x$ is difference between deviator and truthtelling surplus:

$$\frac{1}{F(s_i)} \int_{x=\underline{s}}^{s_i} \left[(s_i - x)\, q_i(x)\, g_N(x) + u_i(x)\, f(x) \right] dx$$

- thus the incentive constraint requires:

$$u_i(s_i) \geq \frac{1}{F(s_i)} \int_{x=\underline{s}}^{s_i} \left[(s_i - x)\, q_i(x)\, g_N(x) + u_i(x)\, f(x) \right] dx$$

Lower Bound As Equality

- lower bound of bidder's surplus through small class of deviations:

$$u_i(s_i) \geq \frac{1}{F(s_i)} \int_{x=\underline{s}}^{s_i} \left[(s_i - x) q_i(x) g_N(x) + u_i(x) f(x) \right] dx$$

- inequality hold as sum across all i :

$$u(s) \geq \frac{1}{F(s)} \int_{x=\underline{s}}^{s} \left[(s - x) q(x) g_N(x) + u(x) f(x) \right] dx$$

- lowest solution $\underline{u}(s)$ exists and solves inequality as equality
- monotonic operator on increasing functions has unique smallest fixed point by Knaster-Tarski fixed point
- can be integrated by parts as

$$\underline{U} = \int_{x \in S} \underline{u}(s) f(s) ds = \int_s \left(\int_{x=s}^{\overline{s}} \frac{1 - F(x)}{F(x)} dx \right) q(s) g_N(s) ds$$

A Generalized Virtual Utility Formula

- with the lower bound on bidder surplus:

$$\underline{U} = \int_{x \in S} \underline{u}(s) f(s) ds = \int_{s} \left(\int_{x=s}^{\overline{s}} \frac{1 - F(x)}{F(x)} dx \right) q(s) g_N(s) ds$$

- we obtain our final formula for revenue, which is

$$\overline{R} = TS - \underline{U} = \int_{v} \psi(v) q(v) g_N(v) dv$$

where

$$\psi(v) = v - \int_{x=v}^{\overline{s}} \frac{1 - F(x)}{F(x)} dx,$$

- compare to virtual utility in private value environments:

$$\pi(x) = x - \frac{1 - F(x)}{f(x)}$$

Upper Bound on Revenue

- generalized virtual utility:

$$\psi(x) = x - \int_{y=x}^{\overline{s}} \frac{1 - F(y)}{F(y)} dy,$$

Theorem (Revenue Upper Bound)

In any auction in which the probability of allocation is given by q, bidder surplus is bounded below by $\underline{U}$ and expected revenue is bounded above by $\overline{R}$.

- bound is valid for any allocation policy $q(v)$

Corollary (Random Reserve Price)

The random reserve price mechanism attains the revenue upper bound.

Posted Price As Optimal Mechanism

Posted Prices

- consider mechanisms where object is always allocated
- pure common values – allocation is therefore socially efficient

Theorem (Revenue Optimality among Efficient Mechanisms)

Among all mechanisms that allocate the object with probability one, revenue is maximized by setting a posted price of

$$p = \int_{v=\underline{s}}^{\overline{s}} v g_{N-1}(v)\, dv,$$

i.e., expected value of object conditional on having lowest signal $\underline{s}$.

- posted price is inclusive: all types purchase at p
- all bidders equally likely to receive object: $q_i(v) = 1/N, \forall i, v$.
- optimal selling mechanism is attained with constant interim transfer $t = t_i(s_i)$ and probability $q = q_i(s_i)$

Optimality of Posted Price

- next, optimality of posted price among all – possibly inefficient – mechanisms

Corollary (Revenue Optimality of Posted Prices)

A posted price mechanism is optimal if and only if

$$\psi(\underline{s}) = \underline{s} - \int_{\underline{s}}^{\overline{s}} \frac{1 - F(x)}{F(x)} dx \geq 0.$$

If a posted price p is optimal, then it is fully inclusive.

The Power of Optimal Auctions

Auctions vs Optimal Mechanism

- Bulow and Klemperer (1996) establish the limited power of optimal mechanisms as opposed to standard auction formats
- revenue of optimal auction with N bidders is strictly dominated by standard absolute auction with $N + 1$ bidders
- current common value environment is an instance of general interdependent value setting – with one exception
- virtual utility function—or marginal revenue function—is not monotone due to maximum operator in common value model

A Closer Look at the Virtual Utility

- non-monotonicity leads to an optimal mechanism with features distinct from standard first or second price auction.
- it elicits information from bidder with highest signal but minimizes probability of assigning him the object subject to incentive constraint
- *virtual utility* of each bidder, $\pi_i(s_i, s_{-i})$:

$$\pi_i(s_i, s_{-i}) = \begin{cases} \max_j\{s_j\}, & \text{if} \quad s_i \leq \max\{s_{-i}\}; \\ \max\{s_j\} - \frac{1-F_i(s_i)}{f_i(s_i)}, & \text{if} \quad s_i > \max\{s_{-i}\}. \end{cases}$$

- downward discontinuity in virtual utility indicates why seller wishes to minimize the probability of assigning the object to the bidder with the high signal

Revenue Comparison

- virtual utility of bidder i fails monotonicity assumption even when hazard rate of distribution function is increasing everywhere
- BK (1996) require monotonicity of virtual utility when establishing their main result that an absolute English auction with $N + 1$ bidders is more profitable than any optimal mechanism with N bidders
- revenue ranking does not extend to current auction environment
- compare revenue from optimal auction with N bidders to absolute, English or second-price, auction with $N + K$ bidders
- absolute as there is no reserve price imposed

Reversal in Revenue Comparison

Theorem (Revenue Comparison)

For every $N \geq 1$ and every $K \geq 1$, the revenue from an absolute second-price auction with $N + K$ bidders is strictly dominated by the revenue of an optimal auction with N bidders.

- comparison of *second order statistic* of $N + K$ i.i.d. signals and *first order statistic* of $N + K - 1$ i.i.d. signals
- second order statistic of $N + K$ signals is revenue of absolute second-price auction with $N + K$ bidders.
- by earlier Theorem, optimal mechanism (weakly) exceeds revenue from a posted price set equal to the maximum of $N + K - 1$ signals.

Revenue Comparison: Continued

- but pure common value of the object is not affected by number of bidders, it is as if the remaining K signals are simply not disclosed, but the N participating bidders still form the expectation over the $N + K - 1$ signals.
- now, if instead of $N + K$ bidders, the optimal auction only has N bidders, then it is as if only N independent and identical distributed signals are revealed to the N bidders
- thus an attainable revenue for the seller is to offer the object at random to a bidder at a posted price set equal to the maximum of $N + K - 1$ signals

Conclusion

- characterized novel revenue maximizing auctions for a class of common value models
- common value models with qualitative feature that values are more sensitive to private information of bidders with more optimistic beliefs
- second interpretation as auction with intermediary/resale market
- countering the winner's curse
- optimal auctions discriminate in favor of less optimistic bidders since they obtain less information rents from being allocated the object

The Limits of Price Discrimination

Dirk Bergemann

November 2020

Introduction: A classic economic issue ...

- a classic issue in the analysis of monpoly is the impact of discriminatory pricing on consumer and producer surplus
- a monopolist engages in third degree price discrimination if he uses additional information - beyond the aggregate distribution - about consumer characteristics to offer different prices to different segments

...information and segmentation...

- additional information leads to segmentation of the population
- different segments are offered different prices
- with additional information about the valuations of the consumers seller can match / tailor prices to consumer characteristics
- what are then the possible (consumer surplus, producer surplus) pairs (for some information)?
- in other words, what are possible welfare outcomes from *third degree price discrimination*?

.. and a modern issue

- if market segmentations are exogenous (location, time, age), then only specific segmentations may be of interest,
- but, increasingly, data intermediaries collect and distribute information, and in consequence segmentations become increasingly endogeneous, choice variables
- for example, if data is collected directly by the seller, then as much information about valuations as possible might be collected, consumer surplus is extracted
- by contrast, if data is collected by an intermediary, to increase consumer surplus, or for some broader business model, then the choice of segmentation becomes an instrument of design
- implications for privacy regulations, data collection, data sharing, etc....

A Classical Economic Problem: A First Pass

- Fix a demand curve
- Interpret the demand curve as representing single unit demand of a continuum of consumers
- If a monopolist producer is selling the good, what is producer surplus (monopoly profits) and consumer surplus (area under demand curve = sum of surplus of buyers)?

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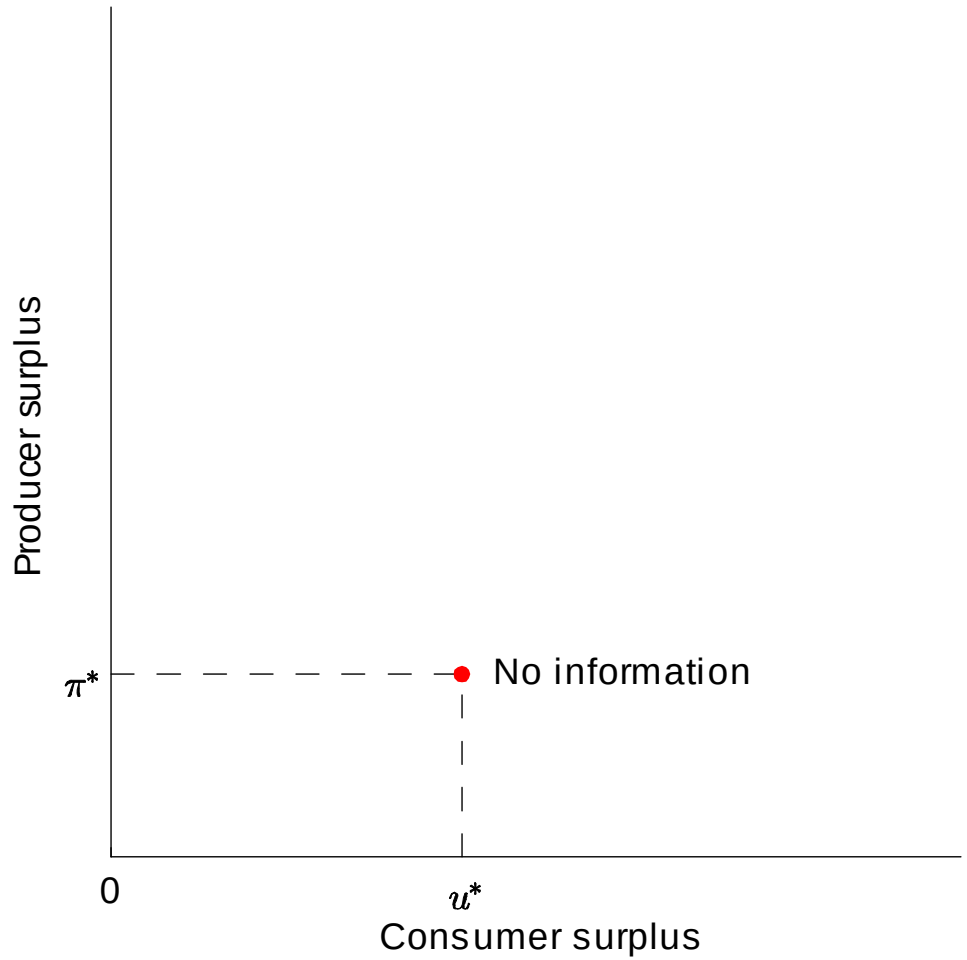
- If a monopolist producer is selling the good, what is producer surplus (monopoly profits) and consumer surplus (area under demand curve = sum of surplus of buyers)?
- If the seller cannot discriminate between consumers, he must charge uniform monopoly price

The Uniform Price Monopoly

- Write u^* for the resulting consumer surplus and π^* for the producer surplus ("uniform monopoly profits")



Perfect Price Discrimination

- But what if the producer could observe each consumer's valuation perfectly?
- Pigou (1920) called this "first degree price discrimination"
- In this case, consumer gets zero surplus and producer fully extracts efficient surplus $w^* > \pi^* + u^*$

First Degree Price Discrimination

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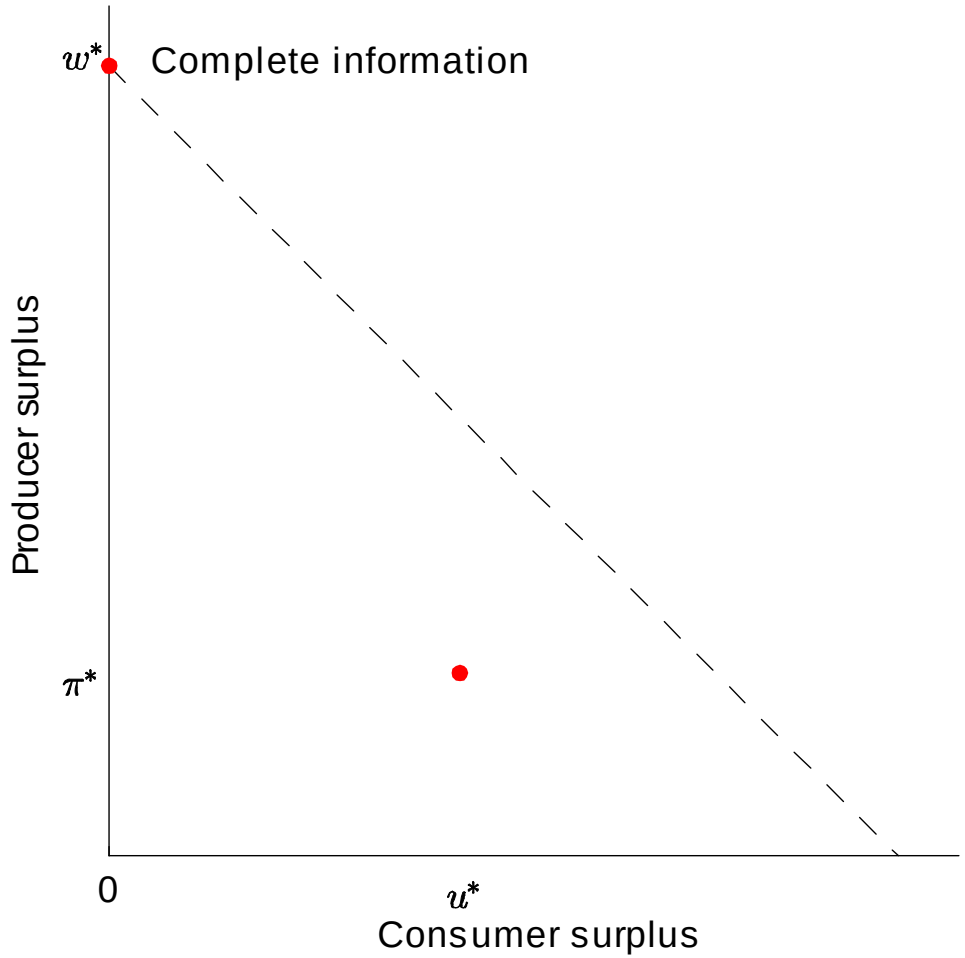

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- Pigou (1920) called this "third degree price discrimination"
- What can happen?
- A large literature (starting with Pigou (1920)) asks what happens to consumer surplus, producer surplus and thus total surplus if we segment the market in particular ways

The Limits of Price Discrimination

- Our main question:
  - What could happen to consumer surplus, producer surplus and thus total surplus for all possible ways of segmenting the market?

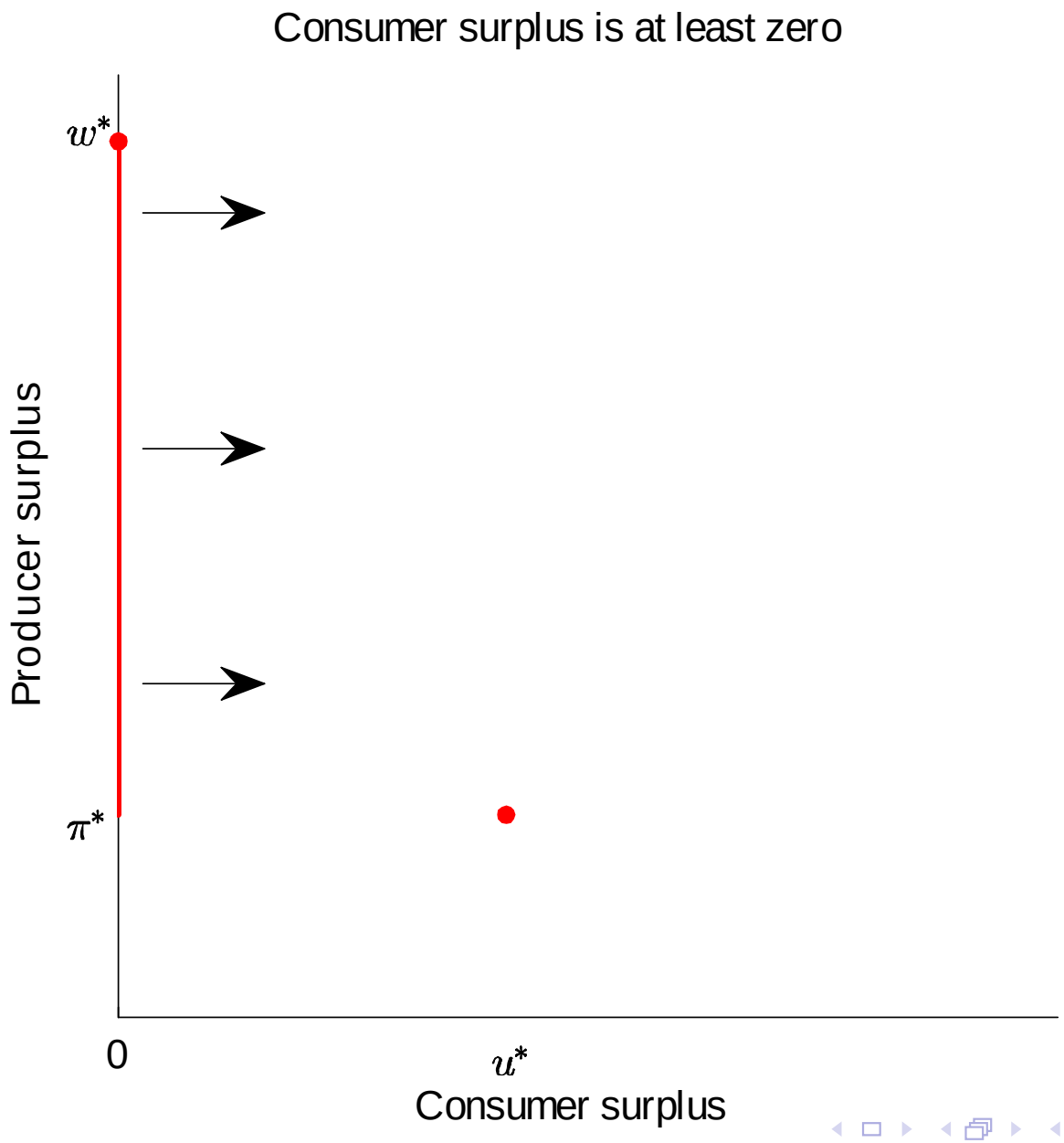
The Limits of Price Discrimination

- Our main question:
  - What could happen to consumer surplus, producer surplus and thus total surplus for all possible ways of segmenting the market?
- Our main result
  - A complete characterization of all (consumer surplus, producer surplus) pairs that can arise...

Three Payoffs Bounds

1. Voluntary Participation: Consumer Surplus is at least zero

Payoff Bounds: Voluntary Participation



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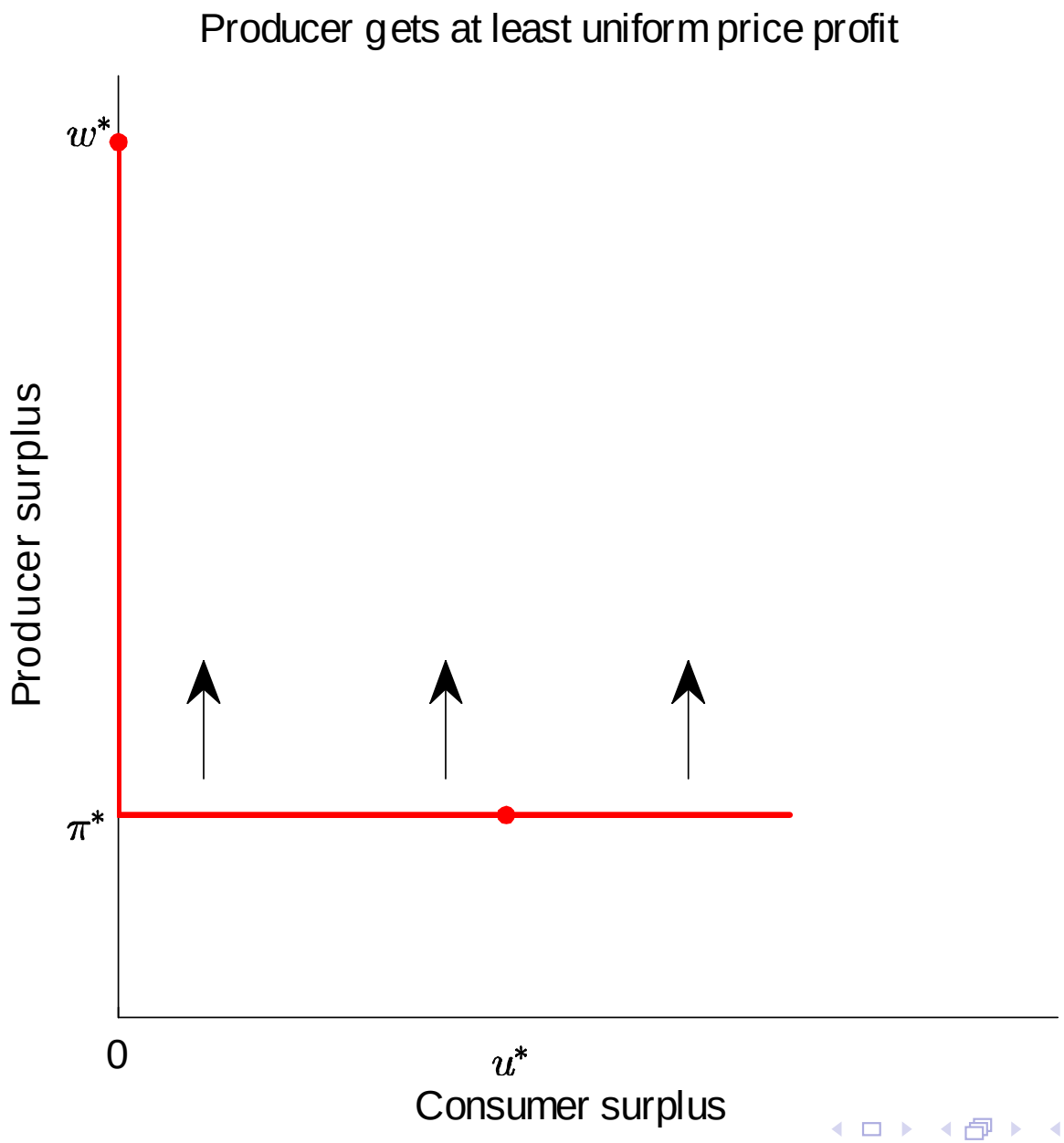
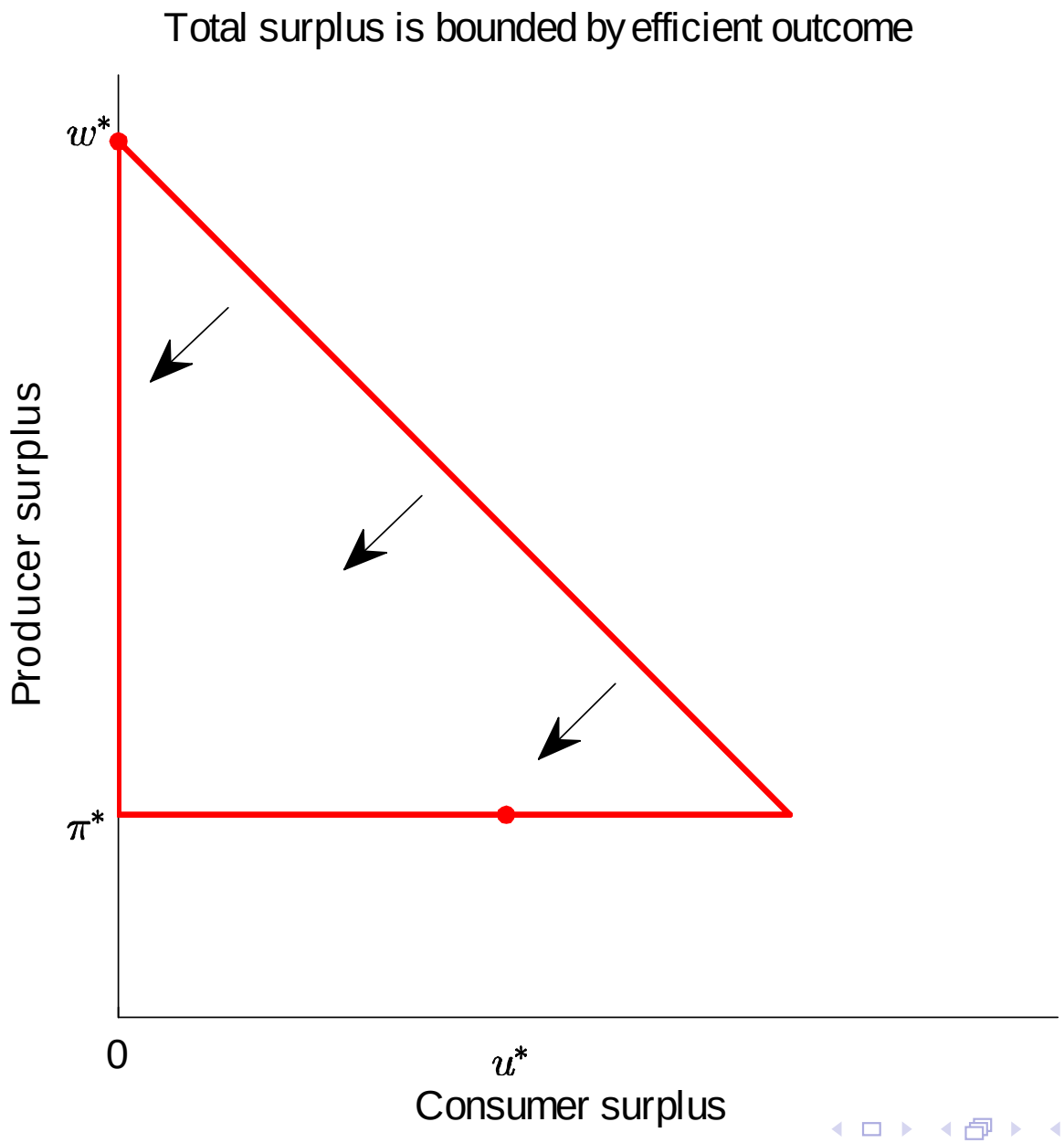
Payoff Bounds: Nonnegative Value of Information



Three Payoff Bounds

1. Voluntary Participation: Consumer Surplus is at least zero
2. Non-negative Value of Information: Producer Surplus bounded below by uniform monopoly profits π^*
3. Social Surplus: The sum of Consumer Surplus and Producer Surplus cannot exceed the total gains from trade

Payoff Bounds: Social Surplus



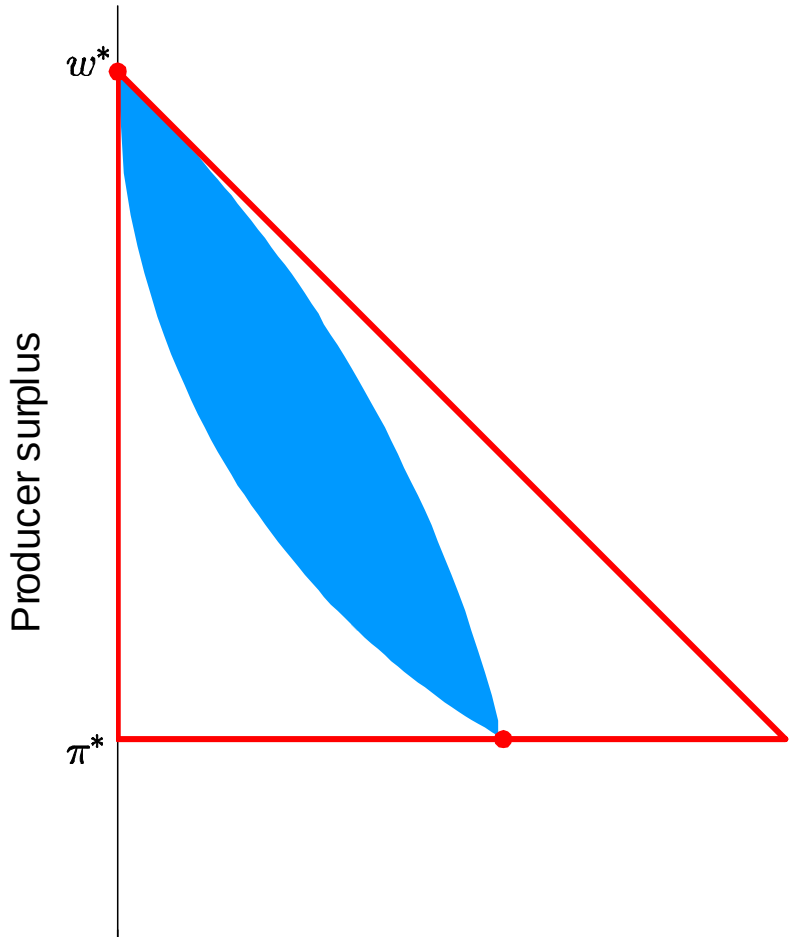
Beyond Payoff Bounds

1. Includes point of uniform price monopoly, (u^*, π^*),
2. Includes point of perfect price discrimination, $(0, w^*)$
3. Segmentation supports convex combinations

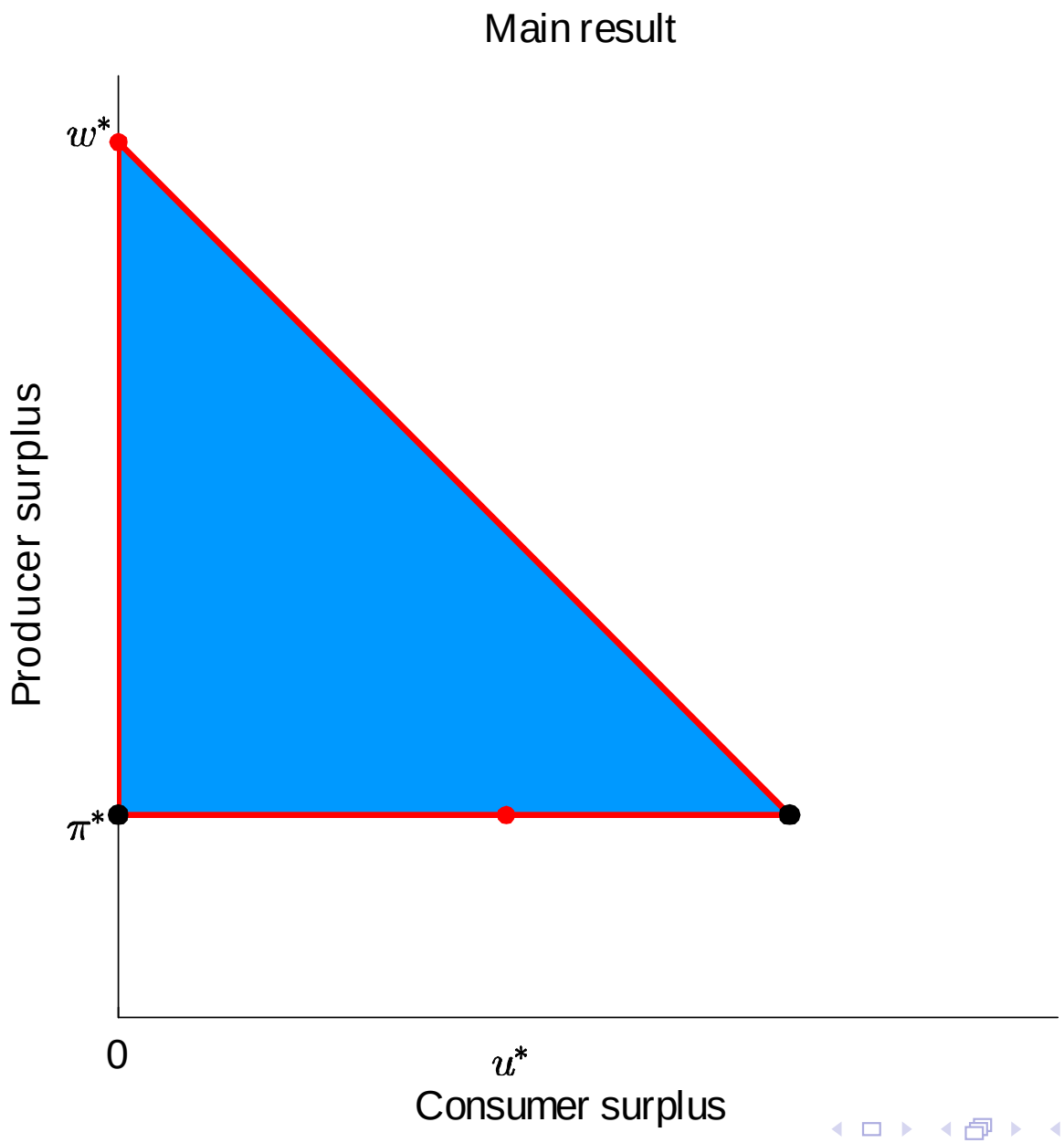
Payoff Bounds and Convexity

1. Includes point of uniform price monopoly, (u^*, π^*),
2. Includes point of perfect price discrimination, $(0, w^*)$
3. Segmentation supports convex combinations

What is the feasible surplus set?



Main Result: Payoff Bounds are Sharp



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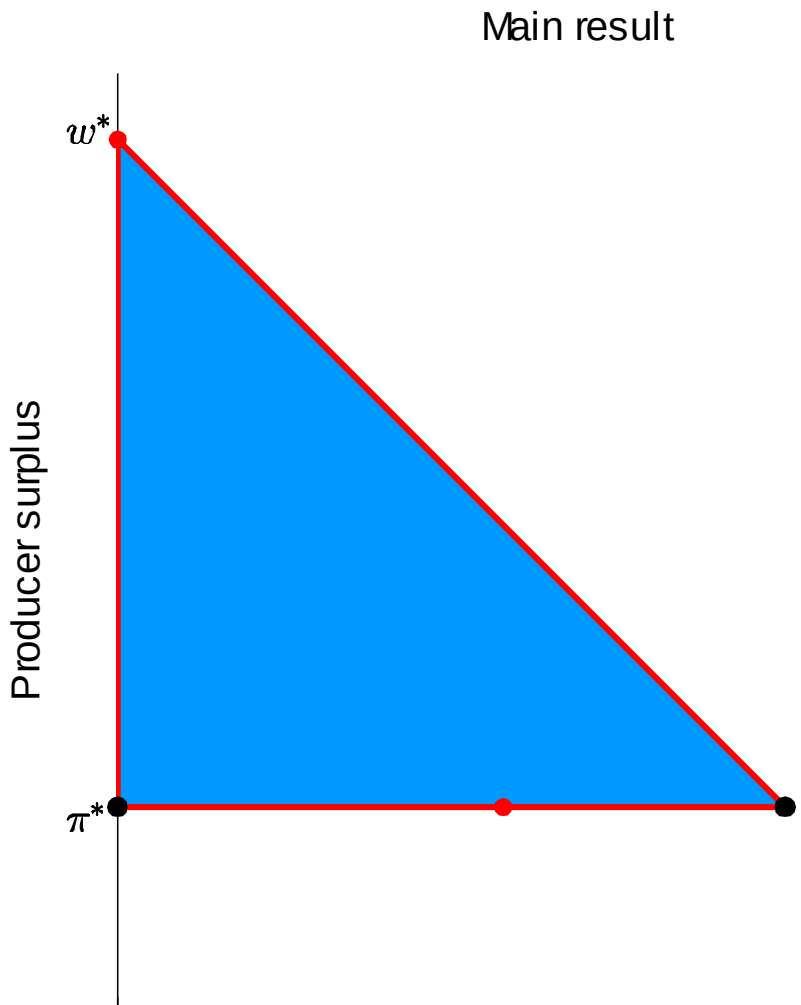
1. a *consumer surplus maximizing segmentation* where
    1. the producer earns uniform monopoly profits,
    2. the allocation is efficient,
    3. and the consumers attain the difference between efficient surplus and uniform monopoly profit.
2. a *social surplus minimizing segmentation* where
    1. the producer earns uniform monopoly profits,
    2. the consumers get zero surplus,
    3. and so the allocation is very inefficient.

The Surplus Triangle

- convex combination of any pair of achievable payoffs as binary segmentation between constituent markets
- it suffices to obtain the vertices of the surplus triangle



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- A solution concept, "Bayes correlated equilibrium," characterizes what could happen in (Bayes Nash) equilibrium for all information structures
- Advantages:
  - do not have to solve for all information structures separately
  - nice linear programming characterization

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3. Linear Normal Symmetric
    1. Stylised applications within continuum player, linear best response, normally distributed games with common values (aggregate uncertainty) ("Robust Predictions in Incomplete Information Games", Econometrica 2013)
    2. "Information and Volatility" (with Tibor Heumann): economy of interacting agents, agents are subject to idiosyncratic and aggregate shocks, how do shocks translate into individual, aggregate volatility, how does the translation depend on the information structure?
    3. "Market Power and Information" (with Tibor Heumann): adding endogeneous prices as supply function equilibrium

Model

- continuum of consumers
- finite set of valuations:

$$0 < v_1 < v_2 < ... < v_k < ... < v_K$$

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  where x_k is the proportion of consumers with valuation v_k
- set of possible markets X is the K-dimensional simplex,

$$X \triangleq \left\{ x \in \mathbb{R}_+^K \,\middle|\, \sum_{k=1}^{K} x_k = 1 \right\}.$$

- the price v_i is *optimal* for a given market x if and only if

$$v_i \sum_{j \geq i} x_j \;\geq\; v_k \sum_{j \geq k} x_j, \quad \forall k$$

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- write X_i for the set of markets where price v_i is optimal,

$$X_i \triangleq \left\{ x \in X \;\middle|\; v_i \sum_{j \geq i} x_j \geq v_k \sum_{j \geq k} x_j, \;\; \forall k \right\}.$$

- the price v_i is *optimal* for a given market x if and only if

$$v_i \sum_{j \geq i} x_j \;\geq\; v_k \sum_{j \geq k} x_j, \quad \forall k$$

- write X_i for the set of markets where price v_i is optimal,

$$X_i \triangleq \left\{ x \in X \;\middle|\; v_i \sum_{j \geq i} x_j \geq v_k \sum_{j \geq k} x_j, \;\forall k \right\}.$$

- each X_i is a convex polytope in the probability simplex

Aggregate Market

- there is an "aggregate market" x^*:

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Aggregate Market

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- define the uniform monopoly price for aggregate market x^*:

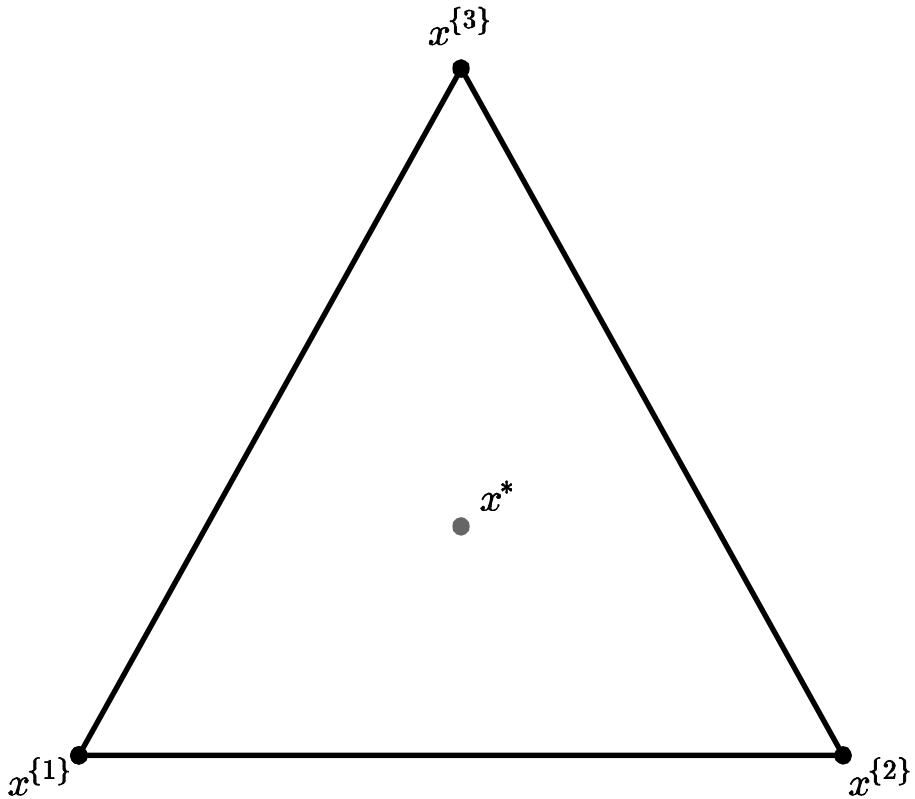
$$p^* = v_{i^*}$$

such that:

$$v_{i^*} \sum_{j \geq i^*} x_j^* \;\geq\; v_k \sum_{j \geq k} x_j^*, \quad \forall k$$

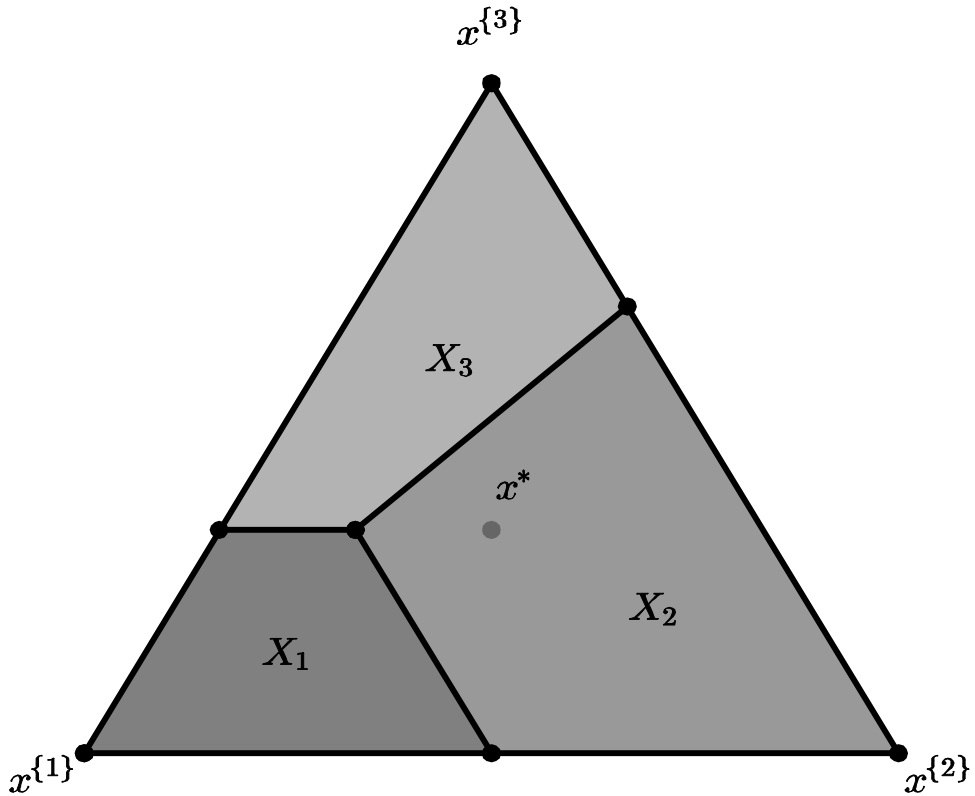
A Visual Representation: Aggregate Market

- given aggregate market x^* as point in probability simplex
- here $x^* = (1/3, 1/3, 1/3)$ uniform across $v \in \{1, 2, 3\}$



A Visual Representation: Optimal Prices and Partition

- composition of aggregate market $x^* = (x_1^*, ..., x_k^*, ..., x_K^*)$ determines optimal monopoly price: $p^* = 2$



Segmentation of Aggregate Market

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- a *pricing strategy* for segmentation σ specifies a price in each market in the support of σ,

$$\phi : \mathbf{supp}(\sigma) \to \Delta\{v_1, ..., v_K\},$$

Segmentation as Splitting

- consider the uniform market with three values
- a segmentation of the uniform aggregate market into three market segments:

	$v = 1$	$v = 2$	$v = 3$	weight
market 1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
market 2	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$
market 3	0	1	0	$\frac{1}{6}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Joint Distribution

- the segments of the aggregate market form a joint distribution over market segmentations and valuations

	$v = 1$	$v = 2$	$v = 3$
market 1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{9}$
market 2	0	$\frac{1}{18}$	$\frac{1}{9}$
market 3	0	$\frac{1}{6}$	0

Signals Generating this Segmentation

- additional information (signals) can generate the segmentation
- likelihood function

$$\lambda : V \to \Delta(S)$$

- in the uniform example

λ	$v = 1$	$v = 2$	$v = 3$
signal 1	1	$\frac{1}{3}$	$\frac{2}{3}$
signal 2	0	$\frac{1}{6}$	$\frac{1}{3}$
signal 3	0	$\frac{1}{2}$	0

Segmentation into "Extremal Markets"

- this segmentation was special

	$v = 1$	$v = 2$	$v = 3$	weight
$\{1, 2, 3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
$\{2, 3\}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$
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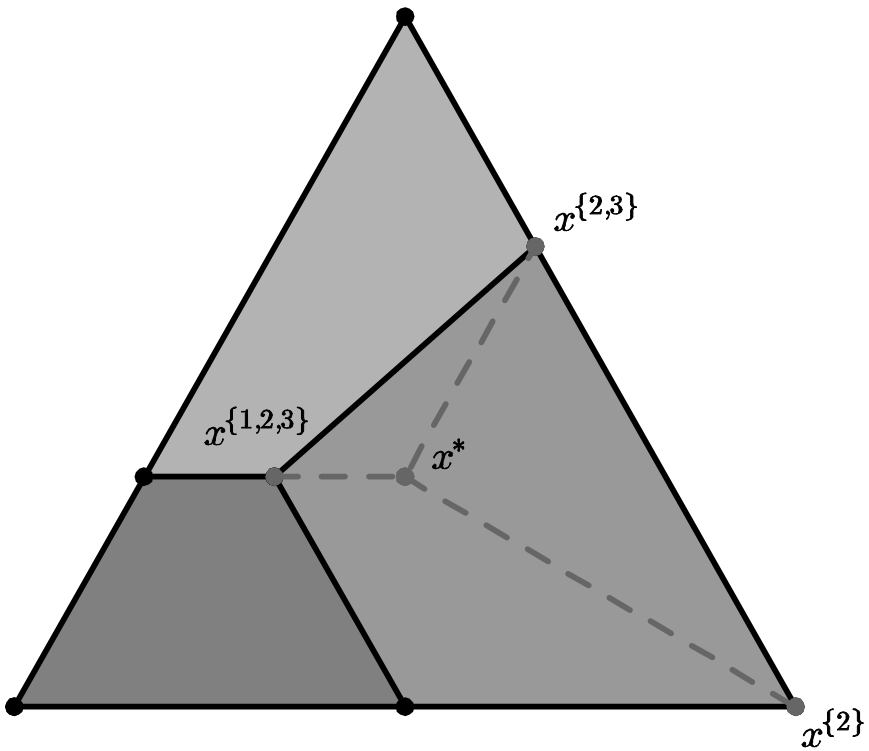
- price 2 is optimal in all markets
- in fact, seller is always indifferent between all prices in the support of every market segment, "unit price elasticity"

- extremal segment x^S: seller is indifferent between all prices in the support of S



- *an* optimal policy: always charge lowest price in the support of every segment:

	$v = 1$	$v = 2$	$v = 3$	price	weight
$\{1, 2, 3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	1	$\frac{2}{3}$
$\{2, 3\}$	0	$\frac{1}{3}$	$\frac{2}{3}$	2	$\frac{1}{6}$
$\{2\}$	0	1	0	2	$\frac{1}{6}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		1

- another optimal policy: always charge highest price in each segment:

	$v = 1$	$v = 2$	$v = 3$	price	weight
$\{1, 2, 3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	3	$\frac{2}{3}$
$\{2, 3\}$	0	$\frac{1}{3}$	$\frac{2}{3}$	3	$\frac{1}{6}$
$\{2\}$	0	1	0	2	$\frac{1}{6}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		1

Extremal Market: Definition

- for any support set $S \subseteq \{1, ..., K\} \neq \varnothing$, define market x^S :

$$x^S = \left(...., x_k^S, ... \right) \in X,$$

with the properties that:

1. no consumer has valuations outside the set $\{v_i\}_{i \in S}$;
2. the monopolist is indifferent between every price in $\{v_i\}_{i \in S}$.

- for every S, this uniquely defines a market

$$x^S = \left(...., x_k^S, ... \right) \in X$$

- writing $\underline{S}$ for the smallest element of S, the unique distribution is:

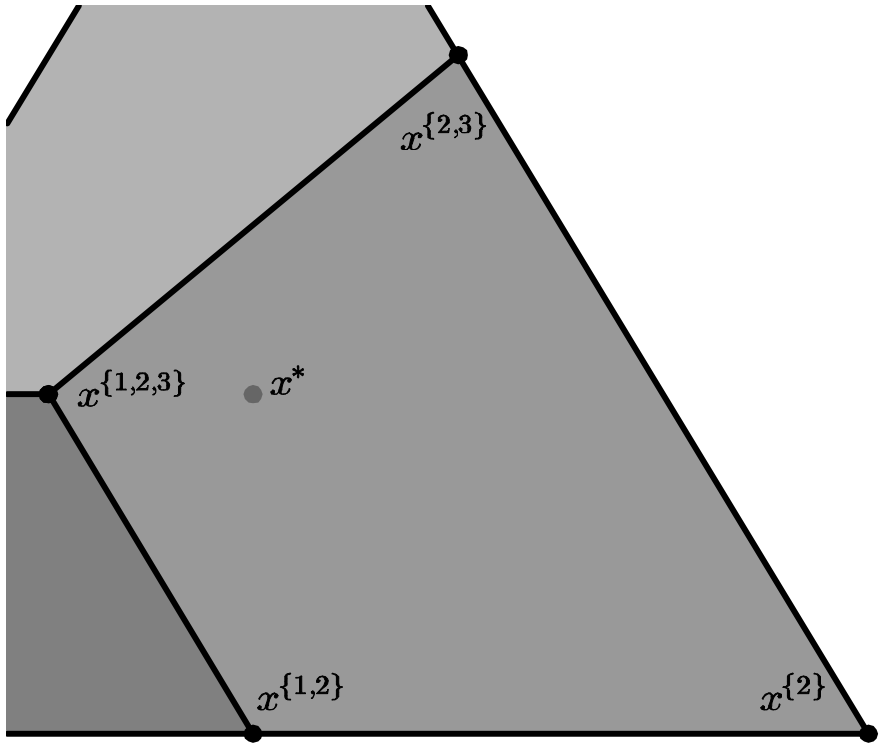
$$x_k^S \triangleq \begin{cases} \frac{v_{\underline{S}}}{v_k} - \sum_{k'>k} x_{k'} & \text{if} \quad k \in S \\ 0, & \text{if} \quad k \notin S. \end{cases}$$

  (a discrete version of the Pareto distribution)

- for any S, market x^S is referred to as *extremal market*

Geometry of Extremal Markets

- extremal markets



- set of markets X_{i^*} where uniform monopoly price $p^* = v_{i^*}$ is optimal:

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- pick any $x \in X$ where price v_{i^*} is optimal (i.e., $x \in X_{i^*}$) but there exists k such that valuation v_k arises with strictly positive probability (so $x_k > 0$) but is not an optimal price

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- so x is not an extreme point of X_{i^*}

Remainder of Proof of Main Result

- Split x^* into *any* extremal segmentation
- There is a pricing rule for that one segmentation that attains *any* point on the bottom of the triangle, i.e., producer surplus π^* anything between 0 and $w^* - \pi^*$.
- The rest of the triangle attained by convexity

Pricing Rules

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- So minimum pricing rule maximizes consumer surplus (bottom right corner of triangle)
- So maximum pricing rule minimizes total surplus (bottom left corner of triangle)

Theorem (Minimum and Maximum Pricing)

1. *In every extremal segmentation, minimum and maximum pricing strategies are optimal;*
2. *producer surplus is π^* under every optimal pricing strategy;*
3. *consumer surplus is zero under maximum pricing strategy;*
4. *consumer surplus is $w^* - \pi^*$ under minimumpricing strategy.*

A Simple "Direct" Construction

We first report a simple direct construction of a consumer surplus maximizing segmentation (bottom right hand corner):

1. first split:
    1. We first create a market which contains all consumers with the lowest valuation v_1 and a constant proportion q_1 of valuations greater than or equal to v_2
    2. Choose q_1 so that the monopolist is indifferent between charging price v_1 and the uniform monopoly price v_{i*}
    3. Note that v_{i*} continues to be an optimal price in the residual market
2. Iterate this process

A Simple "Direct" Construction

We first report a simple direct construction of a consumer surplus maximizing segmentation (bottom right hand corner):

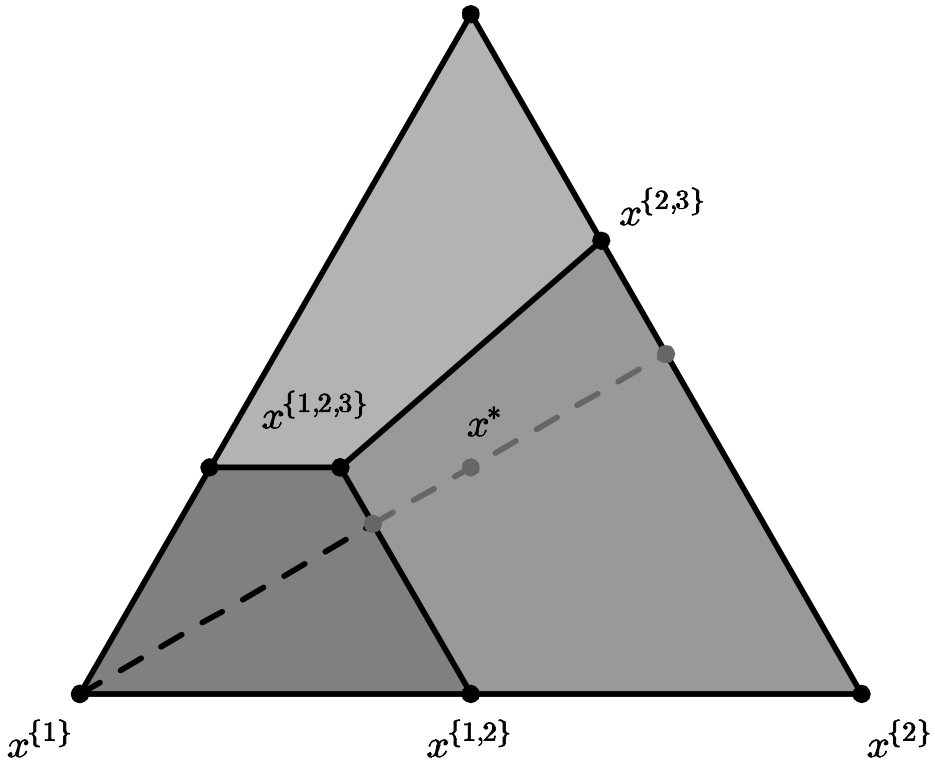
1. first split:
2. Iterate this process
3. thus at round k,
    1. first create a market which contains all consumers with the lowest remaining valuation v_k and a constant proportion q_k of valuations greater than or equal to v_{k+1}
    2. Choose q_k so that the monopolist is indifferent between charging price v_k and the uniform monopoly price v_{i*} in the new segment
    3. Note that v_{i*} continues to be an optimal price in the residual market

A Simple "Direct" Construction

In our three value example, we get:

	$v = 1$	$v = 2$	$v = 3$	price	weight
first segment	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1	$\frac{2}{3}$
second segment	0	$\frac{1}{2}$	$\frac{1}{2}$	2	$\frac{1}{3}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		1

A Simple "Direct" Construction



Advice for the Consumer Protection Agency?

- Allow producers to offer discounts (i.e., prices lower the uniform monopoly price)
- Put enough high valuation consumers into discounted segments so that the uniform monopoly price remains optimal

A Dual Purpose Segementation: Greedy Algorithm

1. Put as many consumers as possible into extremal market $X^{\{1,2,\ldots,K\}}$
2. Generically, we will run out of consumers with some valuation, say, v_k
3. Put as many consumers as possible into residual extremal market $X^{\{1,2,\ldots,K\}/\{k\}}$
4. Etc....

- In our three value example, we get first:

	$v = 1$	$v = 2$	$v = 3$	weight
$\{1, 2, 3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
$\{2, 3\}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

- Then we get

	$v = 1$	$v = 2$	$v = 3$	weight
market 1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
market 2	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$
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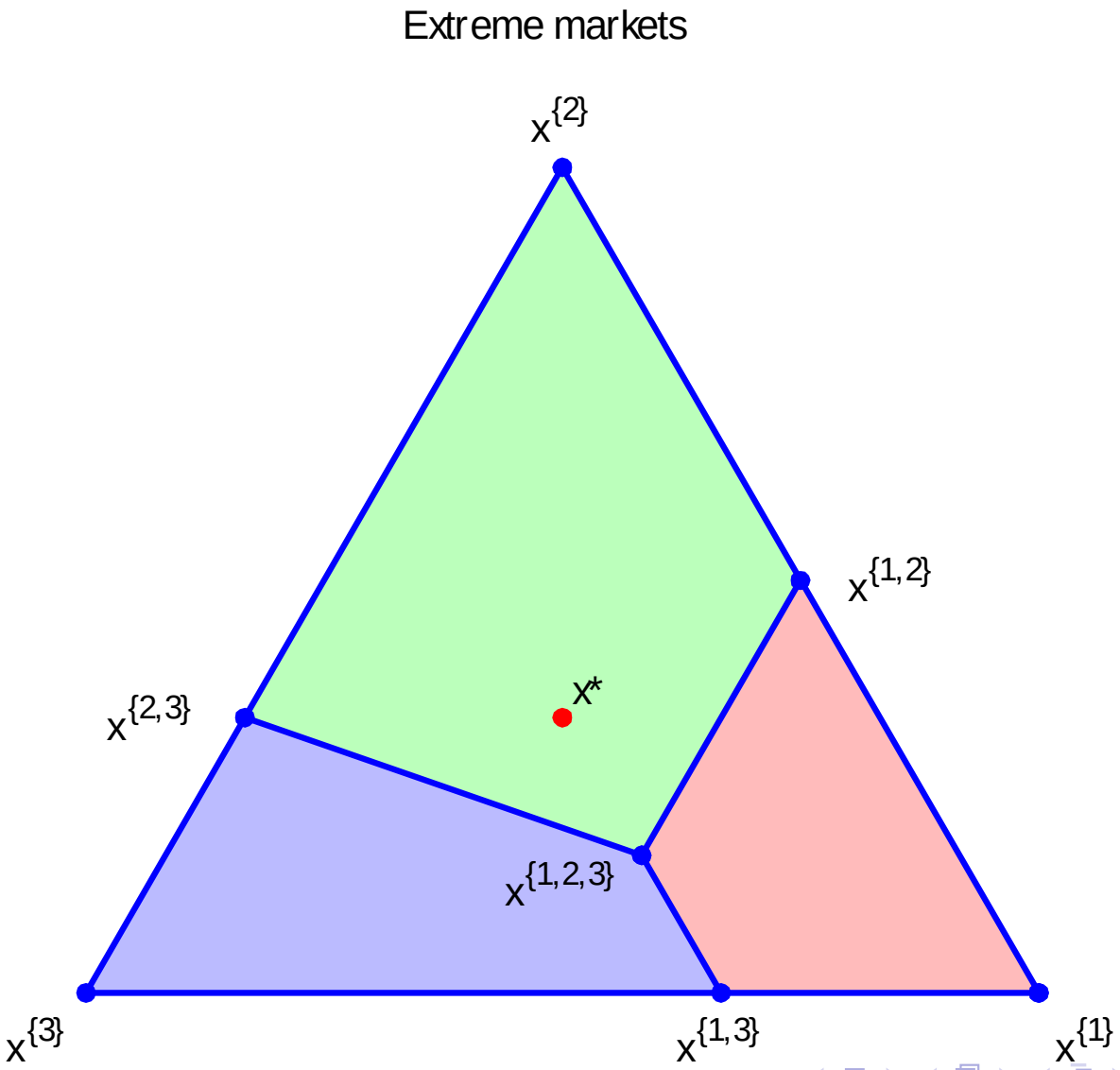
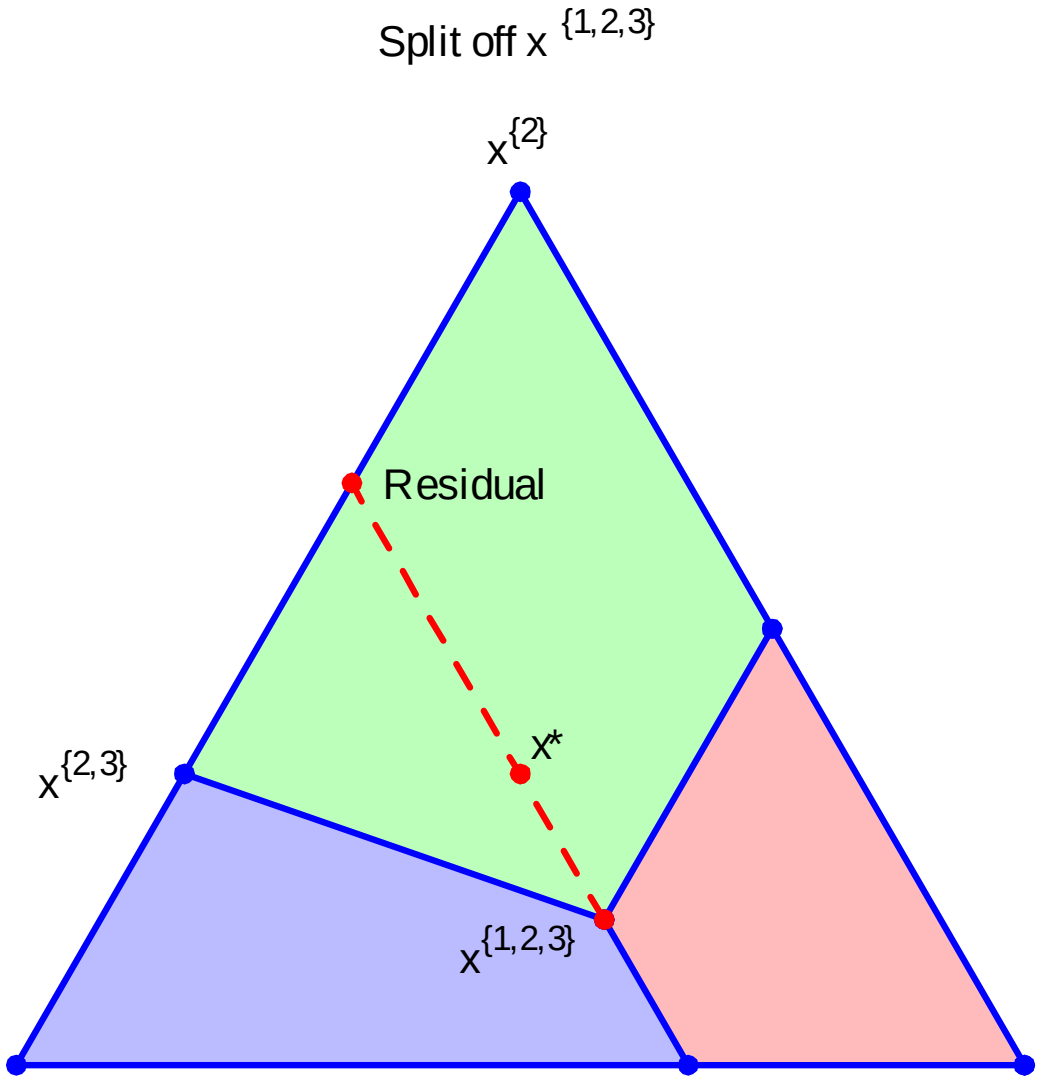
A Visual Proof: Extremal Markets

- extremal markets $x^{\{\dots\}}$



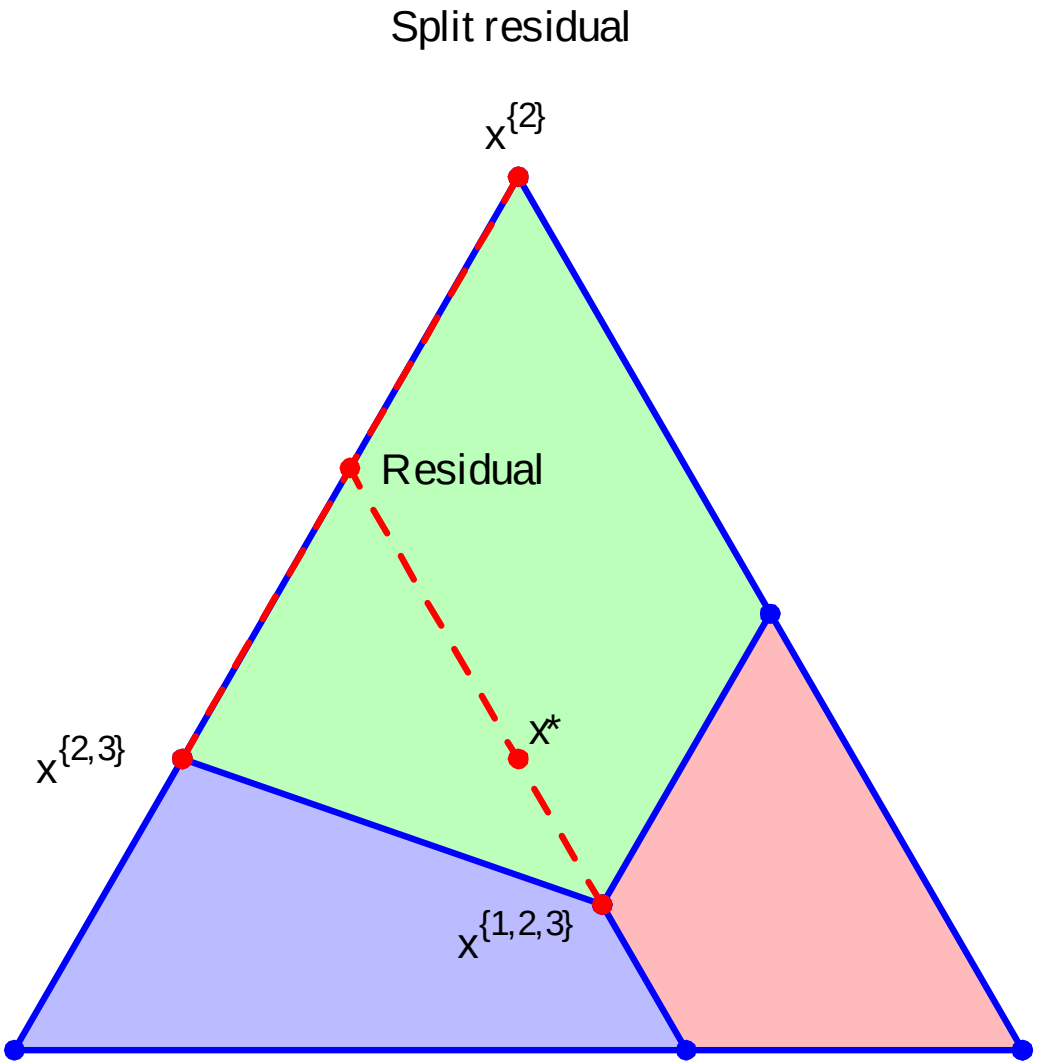
A Visual Proof: Splitting into Extremal Markets

- splitting the aggregate market x^* into extremal markets $x^{\{\dots\}}$



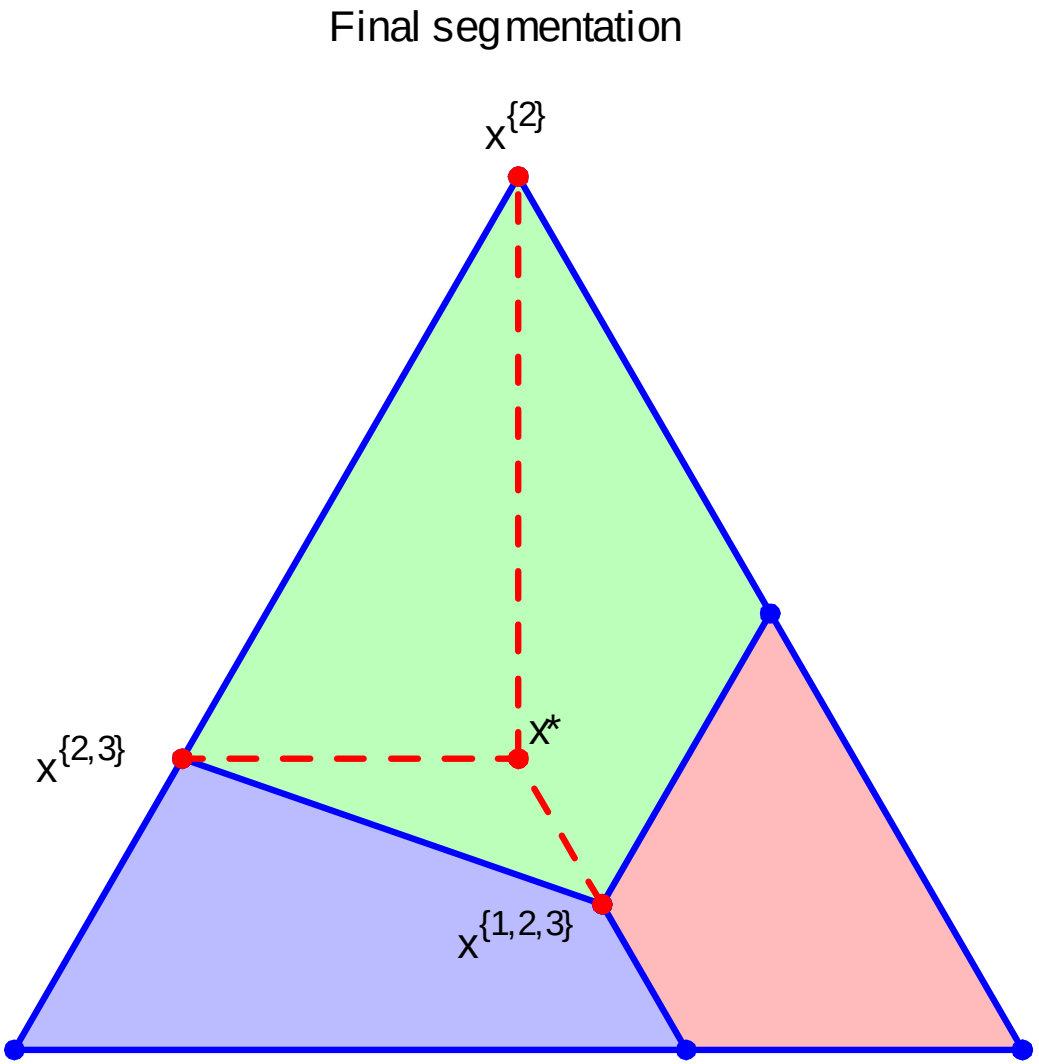
A Visual Proof: Splitting and Greedy Algorithm

- splitting greedily: maximal weight on the maximal market



A Visual Proof: Extremal Market Segmentation

- splitting the aggregate market x^* into extremal market segments all including $p^* = 2$



- minimal and maximal pricing rule maintained π^*
- first degree price discrimination resulted in third vertex

Theorem (Surplus Triangle)

There exists a segmentation and optimal pricing rule with consumer surplus u and producer surplus π if and only if (u, π) satisfy $u \geq 0$, $\pi \geq \pi^*$ and $\pi + u \leq w^*$

- convexity of information structures allows to establish the entire surplus triangle

Continuous Demand Case

- All results extend
- Main result can be proved by a routine continuity argument
- Constructions use same economics, different math (differential equations)
- Segments may have mass points

Third Degree Price Discrimination

- classic topic:
  - Pigou (1920) *Economics of Welfare*
  - Robinson (1933) *The Economics of Imperfect Competition*
- middle period: e.g.,
  - Schmalensee (1981)
  - Varian (1985)
  - Nahata et al (1990)
- latest word:
  - Aguirre, Cowan and Vickers (AER 2010)
  - Cowan (2012)

Existing Results: Welfare, Output and Prices

- examine welfare, output and prices
- focus on two segments
- price rises in one segment and drops in the other if segment profits are strictly concave and continuous: see Nahata et al (1990))
- Pigou:
  - welfare effect = output effect + misallocation effect
  - two linear demand curves, output stays the same, producer surplus strictly increases, total surplus declines (through misallocation), and so consumer surplus must strictly decrease
- Robinson: less curvature of demand $(-\frac{p \cdot q''}{q'})$ in "strong" market means smaller output loss in strong market and higher welfare

Our Results (across all segmentations)

- Welfare:
  - Main result: consistent with bounds, anything goes
  - Non first order sufficient conditions for increasing and decreasing total surplus (and can map entirely into consumer surplus)
- Output:
  - Maximum output is efficient output
  - Minimum output is given by *conditionally efficient* allocation generating uniform monopoly profits as total surplus (note: different argument)
- Prices:
  - all prices fall in consumer surplus maximizing segmentation
  - all prices rise in total surplus minimizing segmentation
  - prices might always rise or always fall *whatever* the initial demand function (this is sometimes - as in example - consistent with weakly concave profits, but not always)

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- no complete characterization
- we study what drives our results by seeing what happens as we move towards general screening problems by adding a little non-linearity
- corresponds to Pigou's "second degree price discrimination", i.e., charging different prices for different quantities / qualities

Re-interpret our Setting and adding small concavity

- Our main setting: Consumer type v consuming quantity $q \in \{0, 1\}$ gets utility $v \cdot q$

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- Equivalently, we are adding small convexity to cost, i.e., increasing marginal cost
- Note that efficient allocation for all types is 1

Three Types and Three Output Levels

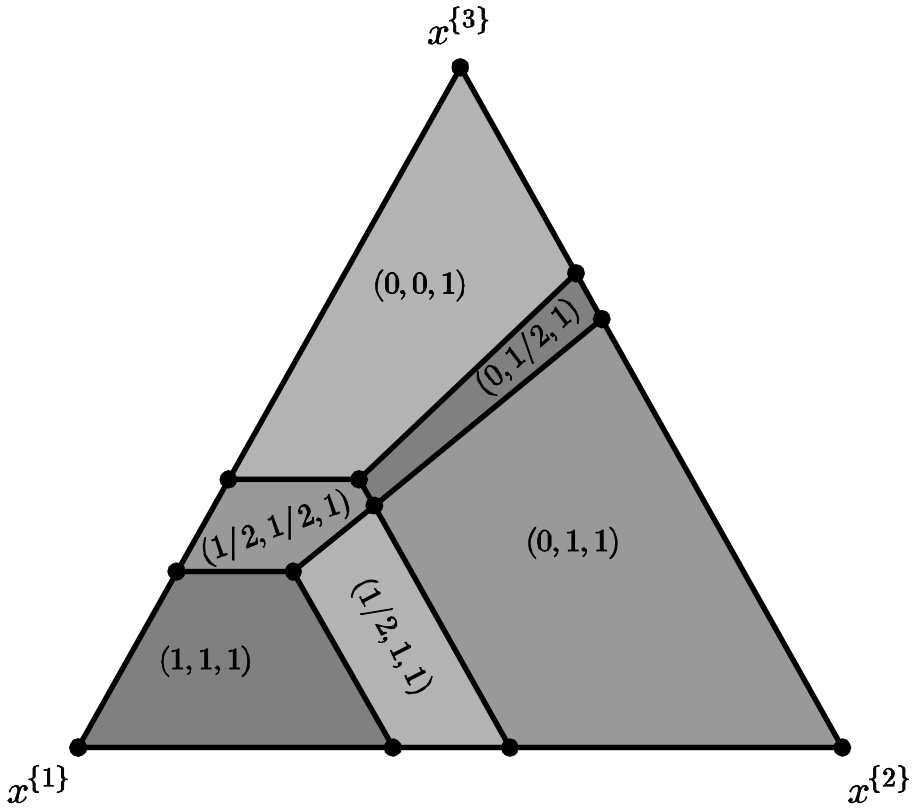
- Suppose $v \in \{1, 2, 3\}$; $q \in \left\{0, \frac{1}{2}, 1\right\}$
- Always efficient to have allocation of 1
- Note that in this case, utilities are given by

	0	$\frac{1}{2}$	1
1	0	$\frac{1}{2} + \varepsilon$	1
2	0	$1 + \varepsilon$	2
3	0	$\frac{3}{2} + \varepsilon$	3

- contract $q = (q_1, q_2, q_3)$ specifies output level for each type
- six contracts which are monotonic and efficient at the top:
  - $(0, 0, 1), \left(0, \frac{1}{2}, 1\right), (0, 1, 1), \left(\frac{1}{2}, \frac{1}{2}, 1\right), \left(\frac{1}{2}, 1, 1\right)$ and $(1, 1, 1)$
- Now we can look at analogous simplex picture
- Illustrates geometric structure in the general case

Picture

- richer partition of probability simplex

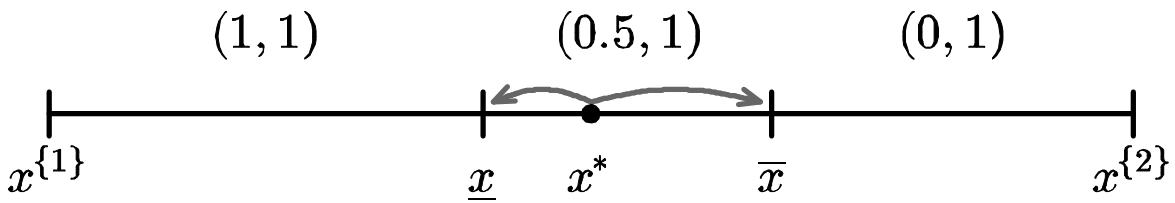


- additional allocations beyond binary appear as optimal

Two Types and Three Output Levels

- Now restrict attention to $v \in \{1, 2\}$
- probability simplex becomes unit interval
- denote by x probabilit of low valuation:

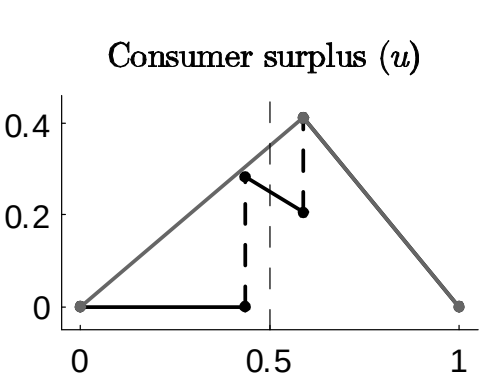
$$x \triangleq \Pr(v = 1)$$

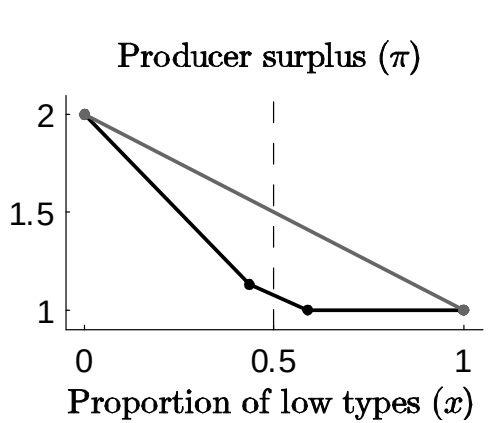


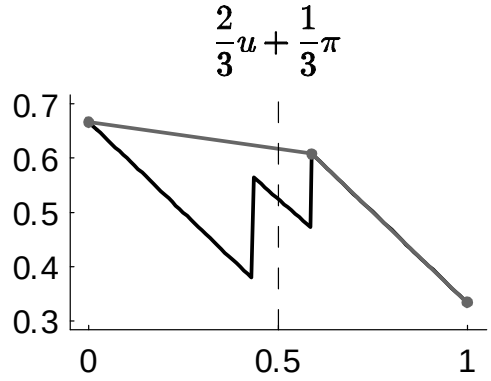
- extremal markets are $\underline{x}$ and $\overline{x}$

Surplus and Concavified Surplus

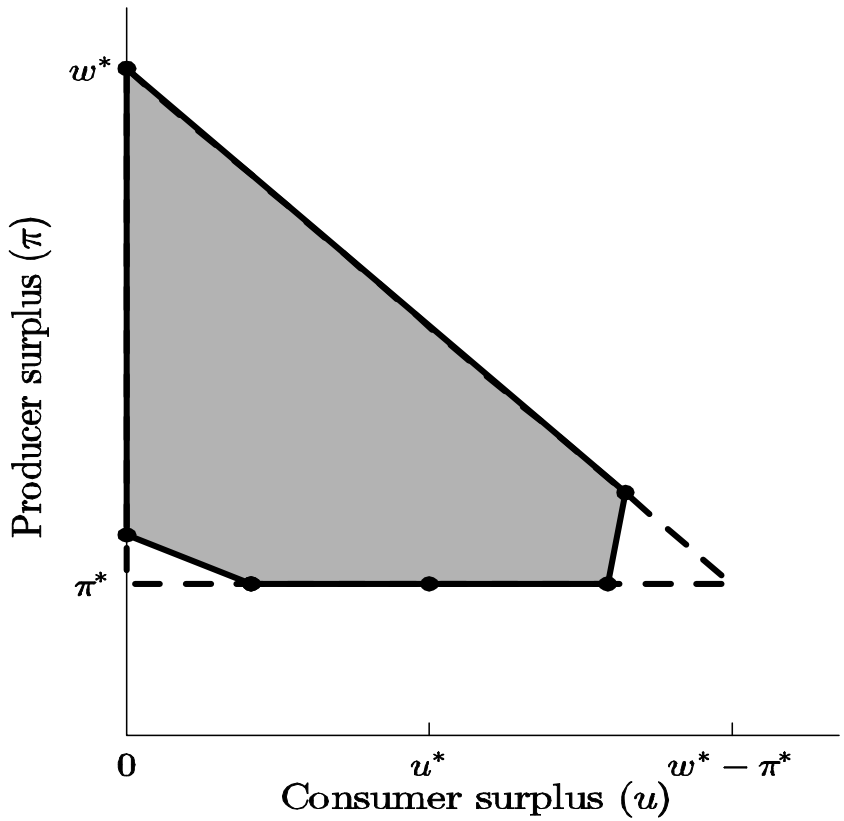
- Now it is natural to plot consumer surplus and producer surplus as a function of x, the probability of type 1







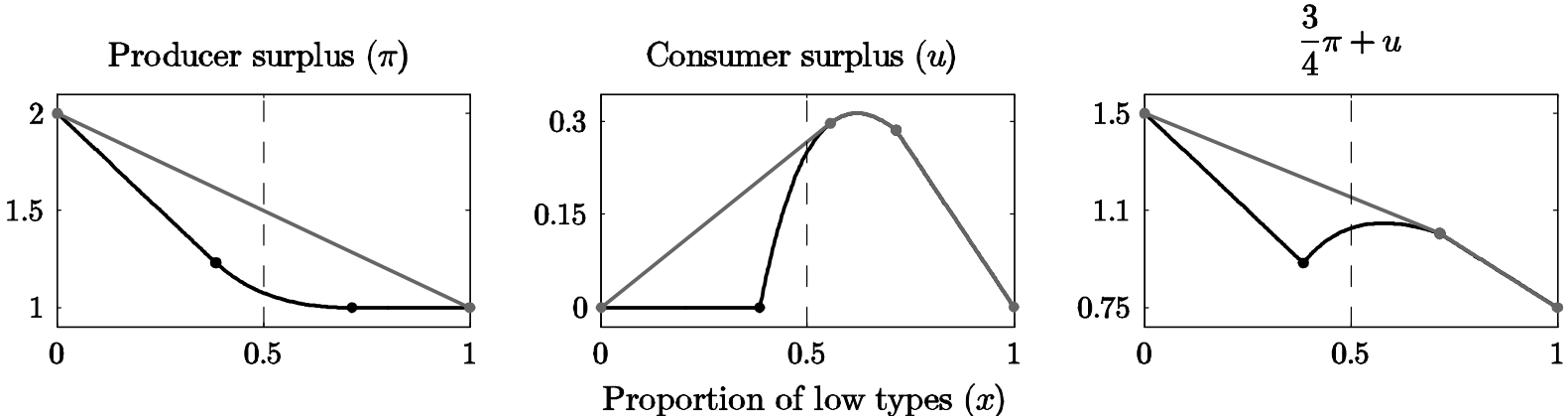
- Now solving for feasible (consumer surplus, producer surplus pairs) for $x = \frac{1}{2}$ comes from concavifying weighted sums of these expressions



- Now allow any $q \in [0,1]$
- If x is the proportion of low types, the optimal contract is now:

$$\widetilde{q}(x) = \begin{cases} 0, & \text{if} \quad x \leq \frac{1}{2+4\varepsilon} \\ \frac{1}{2} - \frac{1}{8\varepsilon}\left(2 - \frac{1}{x}\right), & \text{if} \quad \frac{1}{2+4\varepsilon} \leq x \leq \frac{1}{2-4\varepsilon} \\ 1, & \text{if} \quad x \geq \frac{1}{2-4\varepsilon} \end{cases}$$

Two Types, Continuous Output



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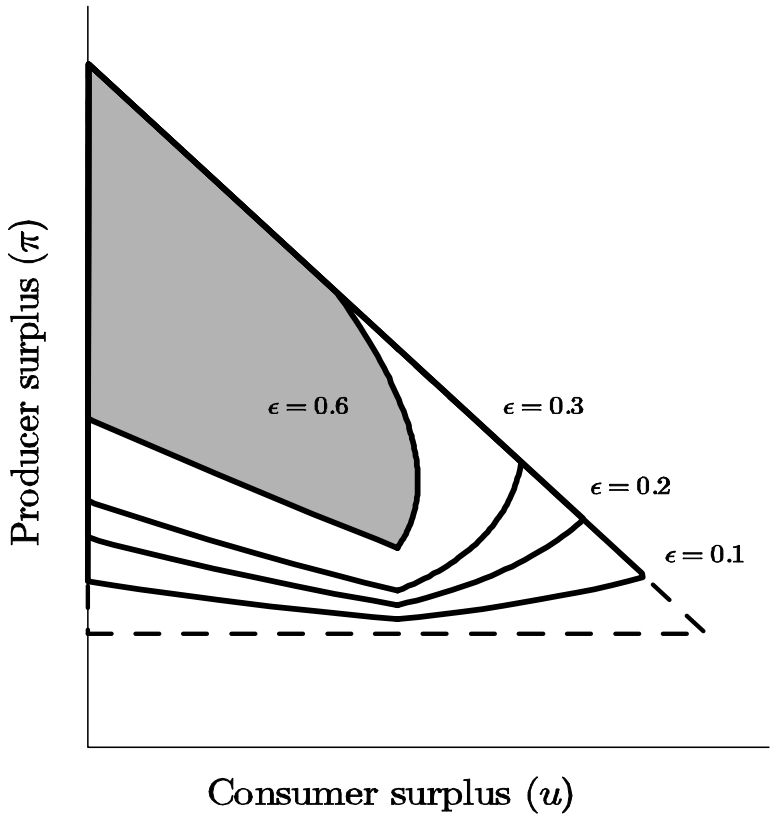


Bottom Line

1. The set of prior distributions of types where it is possible to attain bottom left and bottom right corner will shrink fast as the setting gets more complex
2. As long as there are a finite set of output levels,
    1. There is an analogous restriction to extreme points of best response regions of the simplex (geometric approach translates)
    2. The "bottom flat" survives: there is an open set of information rents consistent with principal getting uninformed profit
3. With continuum output levels
    1. The "bottom flat" goes
    2. Multiple information rents consistent with other levels of consumer profit, approaching the triangle continuously as we approach a linear case

1. Kamenica and Gentzkow (2010): Suppose that a sender could commit (before observing his type) to cheap talk signals to send to a receiver. What would he send?
2. de facto, this is what happened in Aumann and Maschler (1995) repeated games with one sided information who showed sender "concavifies" payoffs
3. We can solve for feasible surplus pairs by this method if the "sender" were a social planner maximizing a arbitrary weighted sum of consumer and producer surplus and the "receiver" were the monopolist
4. Very helpful in two type case, implicit in many type case

Many Player Version

- robust predictions research agenda....
- the set of all outcomes that could arise in Bayes Nash equilibrium in given "basic game" for all possible information structures = "Bayes correlated equilibria"
  - "The comparison of information structures in games: Bayes correlated equilibrium and individual sufficiency" (general theory)
  - "Robust predictions in games with incomplete information games" (applications in symmetric continuum player linear best response games, Ecta (2013))
- seller problem here is single player application
- this paper is by-product of many player application:
  - Bergemann, Brooks and Morris: "Extremal Information Structures in First Price Auction"

Auction Teaser

- First price auction
- Bidder i's valuations drawn according to cdf F_i
- Lower bound on interim bidder surplus of bidder with valuation v is

$$\underline{u}_i(v) = \max_b (v - b) \prod_{j \neq i} F_j(b)$$

- Lower bound on ex ante expected surplus of bidder i is

$$\underline{U}_i = \int_{v=0}^{1} \underline{u}_i(v) f_i(v)\, dv$$

- Upper bound on expected revenue is total expected surplus minus each bidder's surplus lower bound
- Claim: there is an information structure where these bounds are attained in equilibrium

Auction Teaser: Information Structure Attaining the Lower Bound

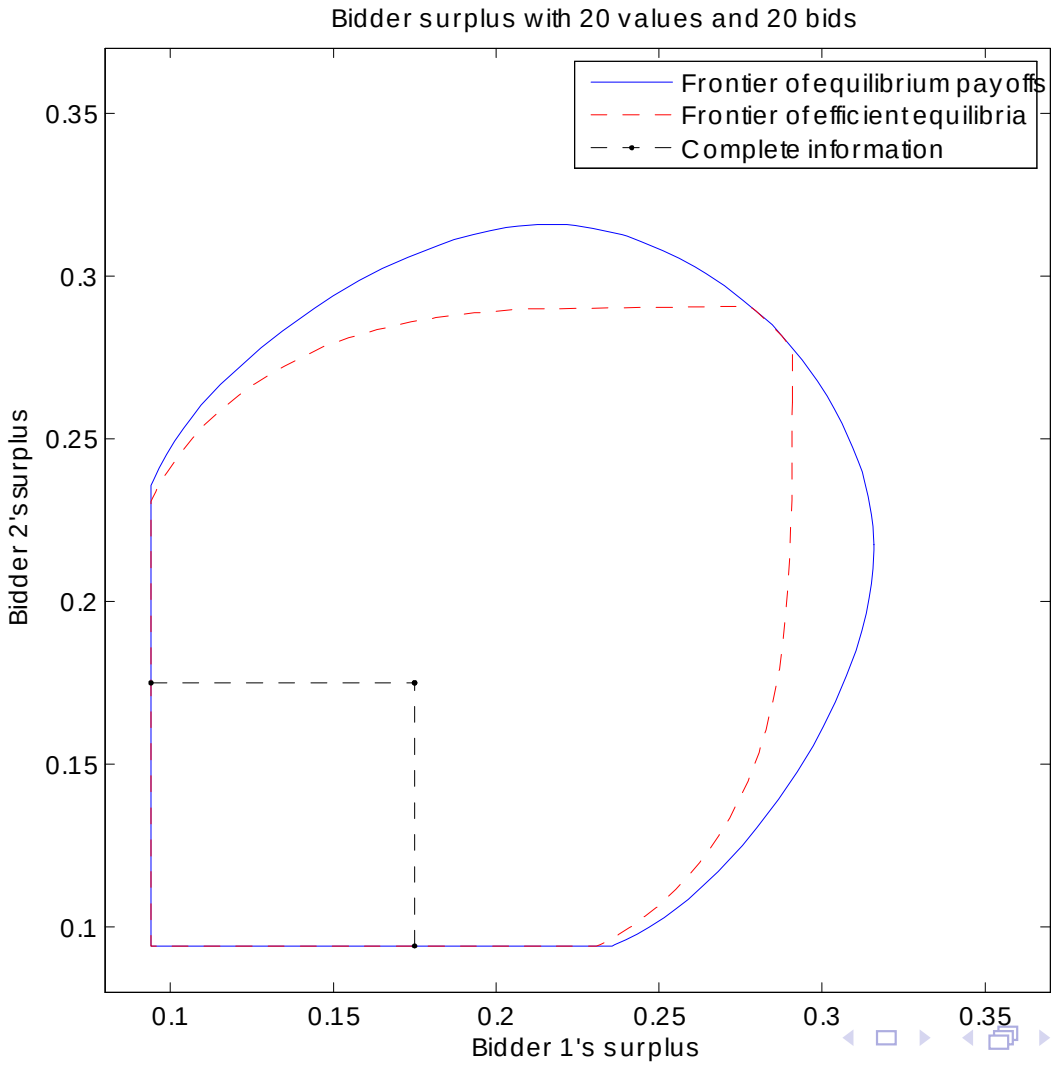
- Tell each bidder if he has the highest value or not
- Losing bidders bid their values and lose (undominated strategy)
- Winning bidder's "uniform monopoly profit" (maximum profit if he knows nothing about the losing bid) is now the lower bound $\underline{U}_i$
- Our main result states that we can provide (partial) information to the winner about highest losing bid in just such a way that he is still held down to his uniform monopoly profit and always wins

Two Bidders: Information and Revenue

- 2 bidders, valuations uniform on $[0, 1]$
- Ex ante expected surplus is $\frac{2}{3}$
- No information:
  - bid $\frac{1}{2}v$, each bidder surplus $\frac{1}{6}$, revenue $\frac{1}{3}$
- Complete information = Bertrand:
  - each bidder surplus $\frac{1}{6}$, revenue $\frac{1}{3}$
- Our intermediate information structure:
  - each bidder surplus $\frac{1}{12}$, revenue $\frac{1}{2}$

The Payoff Space of the Bidders

- distribution of bidders (surplus) and implications for revenue equivalence, ...



Conclusion

- It is feasible and interesting to see what happens under many information structures at once.
- This methodology generates striking new answers for classical economic questions
- In mechanism design we design the payoffs of the game, assuming the information structure is fixed
- In information design , we design the information received by the players, assuming the game is fixed.

Do We Care about Extremal Segmentations?

- extremal segmentations are "extreme"...
- might not arise exogenously....
- but suppose someone could choose segments endogenously?

Endogenous Segmentations and a Modern Perspective

- extremal segmentations are "extreme"
- might not arise exogenously
- but suppose someone could choose segments endogenously?
- Google knows everyone's values of everything (pretty much)
- Google wants to "do no evil"
- Operationalization of "do no evil": report noisy signals of values to sellers in such a way that sellers choose to price discriminate in a way that attains efficiency and gives all the efficiency gains to consumers

Bayesian Persuasion

Dirk Bergemann
(with slides courtesy of Emir Kamenica)

November 2020

Introduction

Data buyer faces a decision problem under uncertainty:

- has partial and private information;
- can acquire additional information.

Data seller offers additional information.

- How much information to provide and at what price?
- How to screen the data buyer's private information?
- How does market power distort the decision maker's actions?

Road map

- Persuasion
- Design of informational environments
- Drama-based entertainment

Persuasion

- Suppose you wish to influence someone's action

Persuasion

- Suppose you wish to influence someone's action
  - change payoffs
  - change beliefs

Persuasion

- Suppose you wish to influence someone's action
  - change payoffs: incentives
  - change beliefs: persuasion

Persuasion

- Suppose you wish to influence someone's action
  - change payoffs: incentives
  - change beliefs: persuasion
- Attempts at persuasion command substantial resources

Economic models of persuasion

- Persuasion is impossible
- Cheap talk games
  - sender can say anything he wants
- Verifiable message games
  - sender must tell the truth (but not the whole truth)

The impossibility result

- Receiver has a utility function $u(a, \omega)$
  - takes action a
  - does not know ω
- Sender has a strictly increasing utility function $v(a)$
  - knows ω

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- Verifiable messages: Sender cannot tell explicit lies
  - result: The entire truth is always revealed
- Either way: no scope for persuasion

Example

- Trial: defendant is *innocent* ($\omega = 0$) or *guilty* ($\omega = 1$)
- Shared prior: $\mu_0 = Pr(\omega = 1) = 0.3$
- Judge (Receiver)
  - *acquits* ($a = 0$) or *convicts* ($a = 1$)
  - $u(0,0) = u(1,1) = 1$
  - $u(0,1) = u(1,0) = 0$
- District Attorney (Sender)
  - $v(a) = a$

Investigations

- DA conducts an investigation
  - subpoenas
  - forensic tests
  - questions to witnesses

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  - subpoenas
  - forensic tests
  - questions to witnesses
- Outcome of the investigation observed by the judge

Investigations

- No investigation: Judge acquits
- Fully informative investigation: Judge convicts with probability 0.3

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- Can the DA do any better?

A partially informative test

- Murderer's blood, type A, found at the scene of the crime

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    - probability of guilt is ...

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$$
\begin{aligned}
Pr\left(guilt|A\right) &= \frac{Pr\left(A|G\right)Pr\left(G\right)}{Pr\left(A|G\right)Pr\left(G\right)+Pr\left(A|I\right)Pr(I)} \\
&= \frac{1\times 0.3}{1\times 0.3+0.42\times 0.7} \\
&= 0.505
\end{aligned}
$$

A partially informative test

- Possible outcomes
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- How often does the judge convict?
  - $0.3 \times 1 + 0.7 \times 0.42 \approx 0.6$
- Judge knows only 30% are guilty yet convicts 60%!

Maximum probability of conviction

- DA could have chosen other investigations
  - DNA test
  - Rh factor
  - etc.

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- Can the DA do better than the 60% conviction rate?

Maximum probability of conviction

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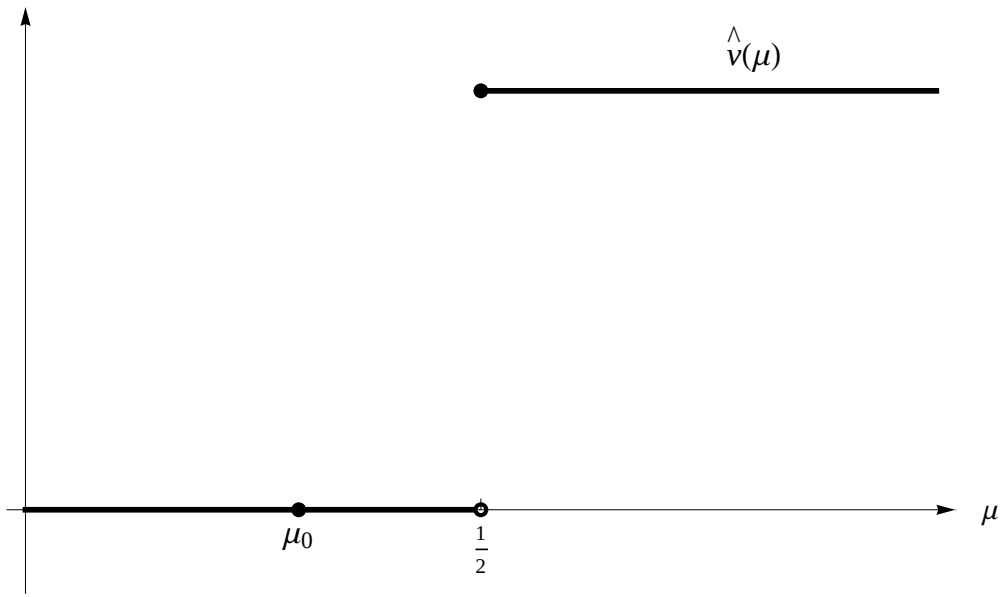
  - DNA test
  - Rh factor
  - etc.
- Can the DA do better than the 60% conviction rate?
- An investigation (signal) specifies $\pi(\cdot|\omega)$ on $\{g, i\}$
  - e.g., $\pi(g|G) = 1$, $\pi(g|I) = 0$;
  - or, $\pi(g|G) = 1 \quad \pi(g|I) = 0.42$
  - etc.

Geometry of persuasion

Value function over beliefs

- Sender's utility from a given belief μ:

$$\hat{v}(\mu) = v(a^*(\mu))$$



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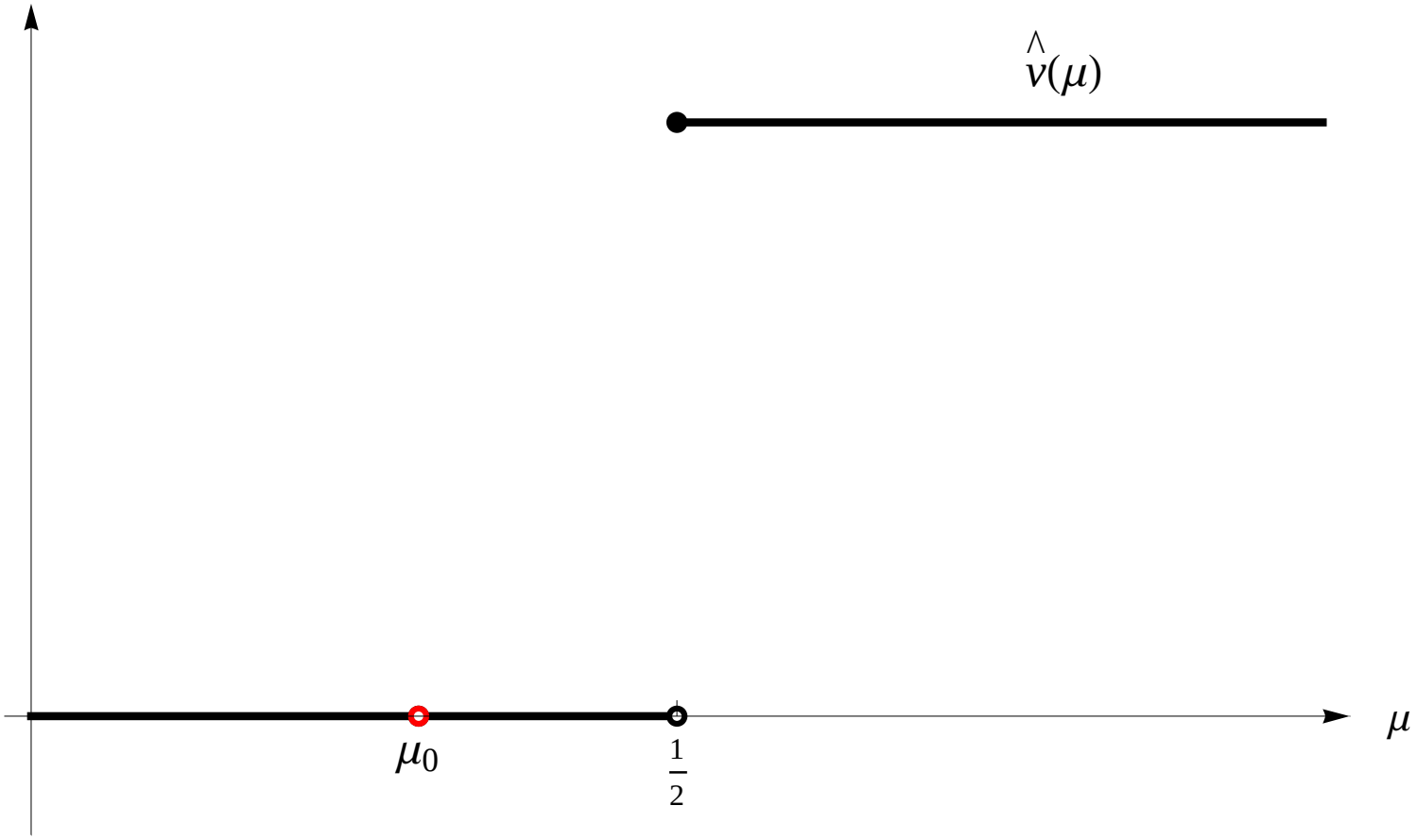
Proposition

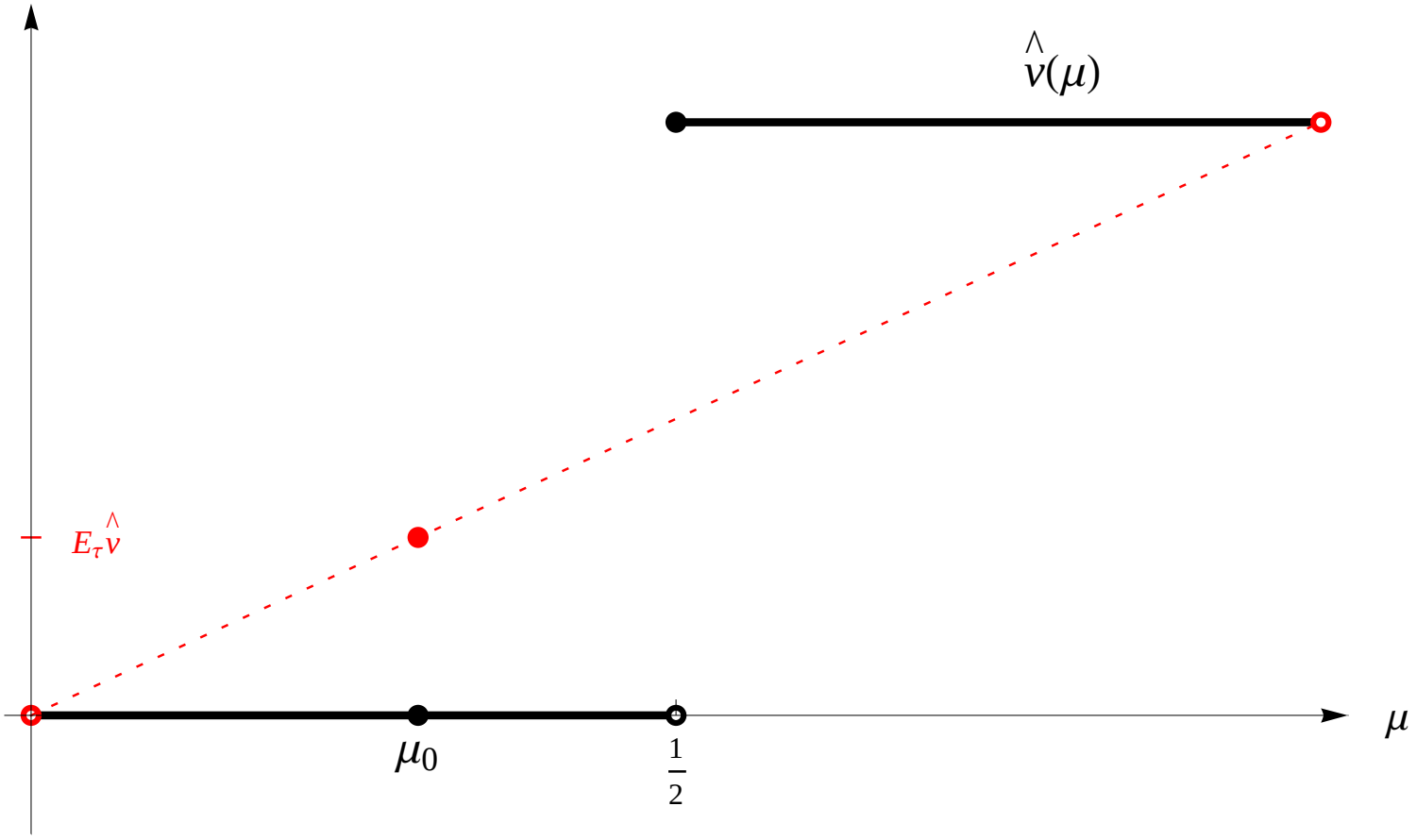
For any distribution of beliefs (whose expectation is the prior), there is an investigation that induces it.

Distributions of beliefs



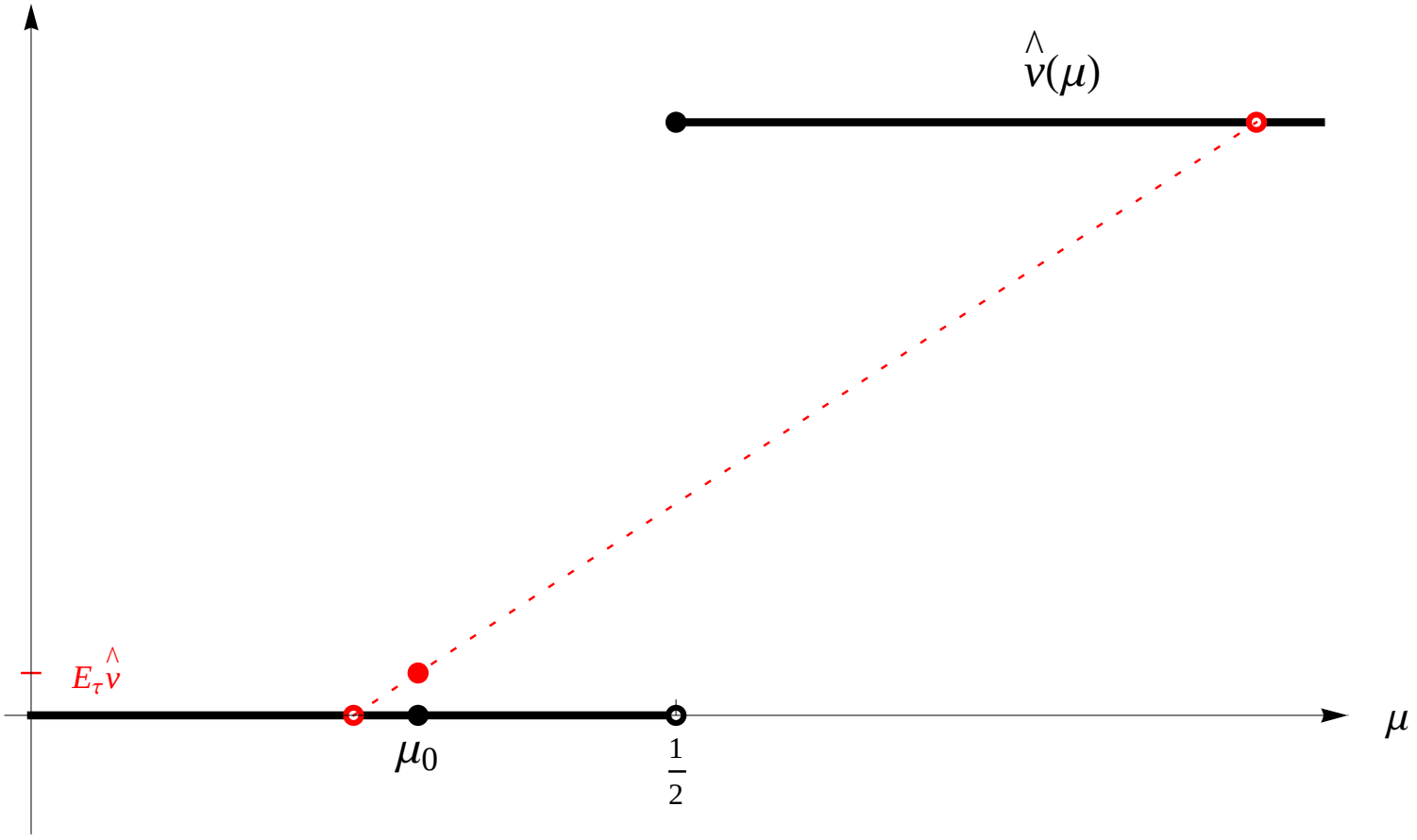
(No investigation: 0% conviction rate)

Distributions of beliefs



(Fully informative investigation: 30% conviction rate)

Distributions of beliefs



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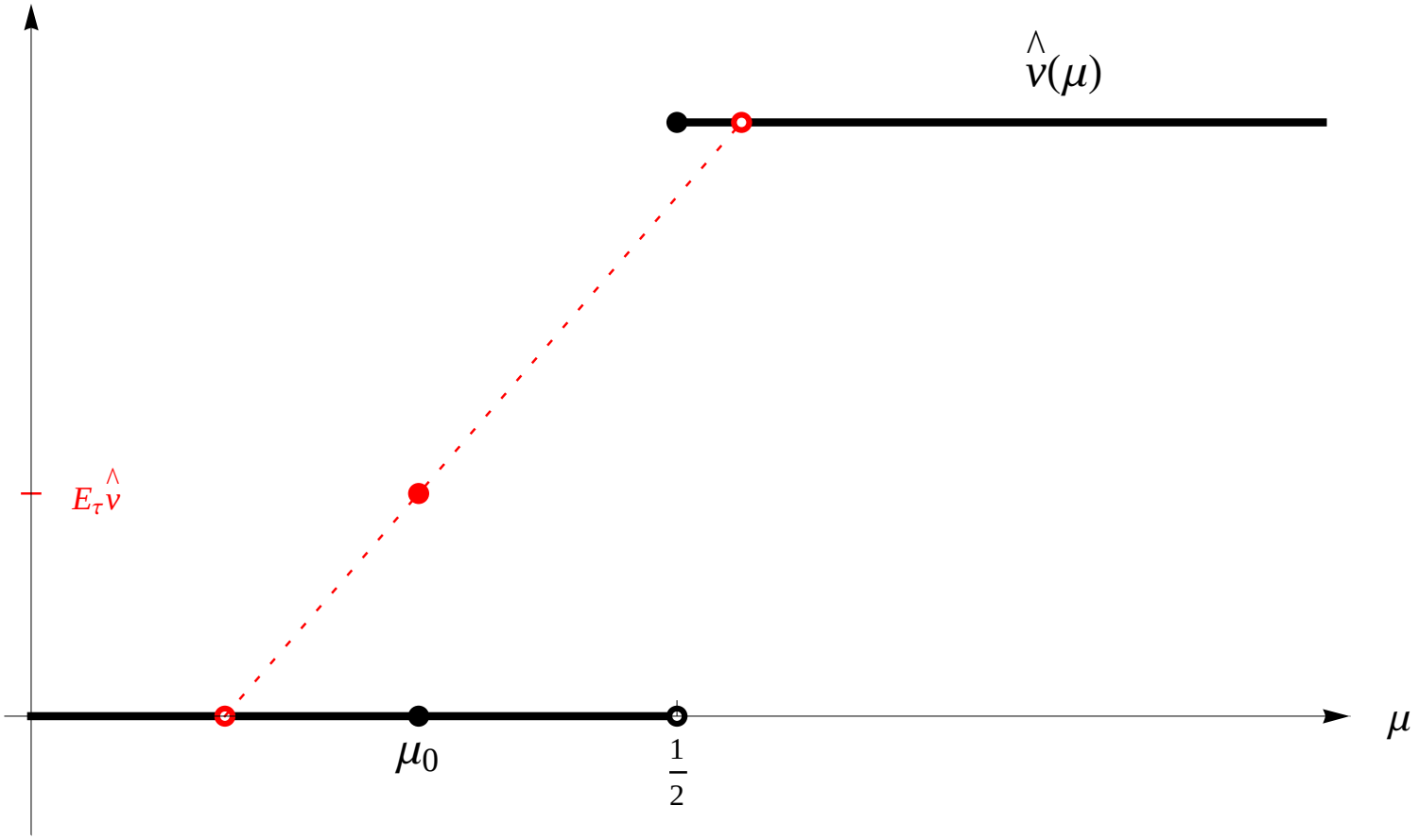

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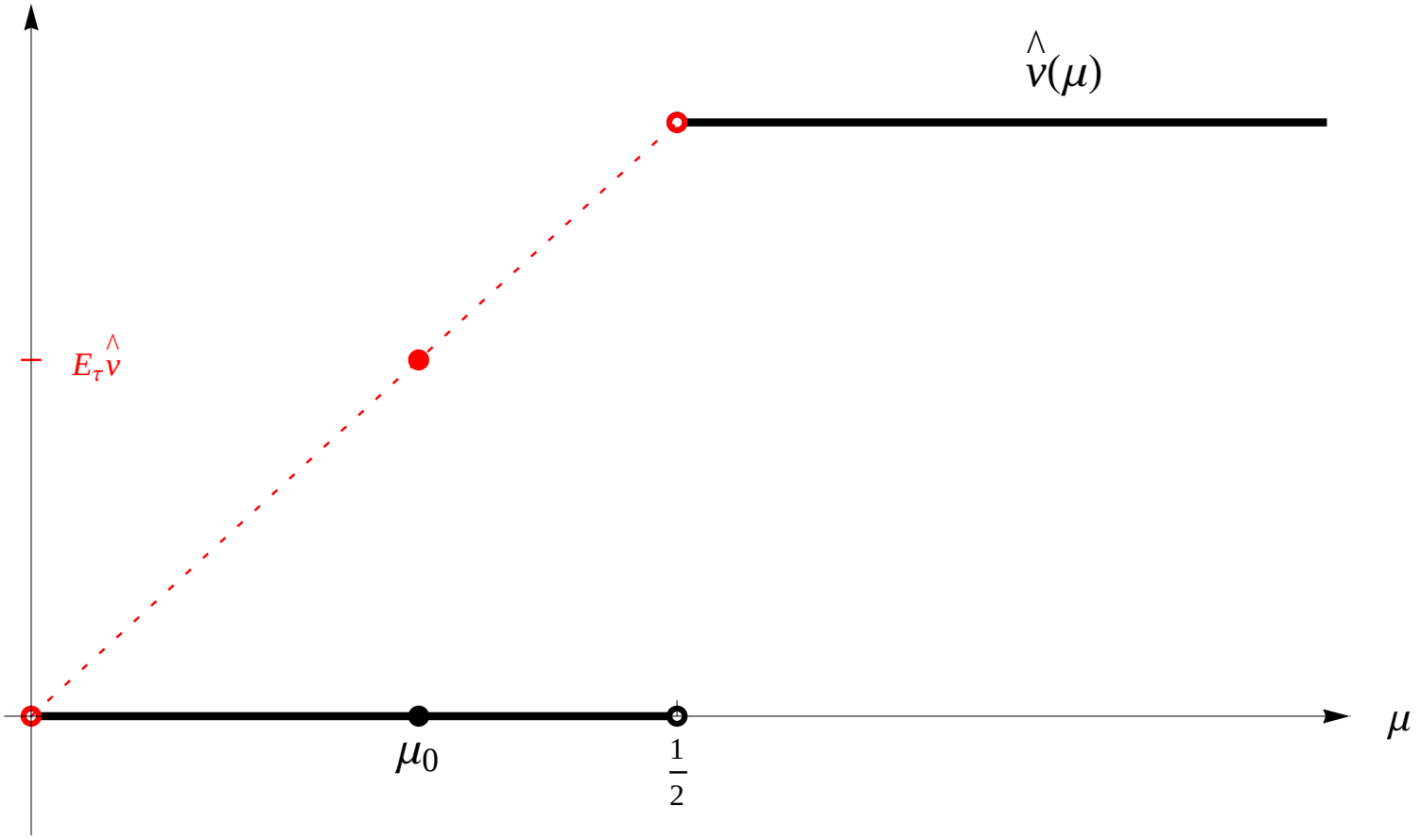

(Optimal investigation: 60% conviction rate)

Concave closure

- Let $V(\mu)$ to be the smallest concave function everywhere above $\hat{v}(\mu)$

Concave closure

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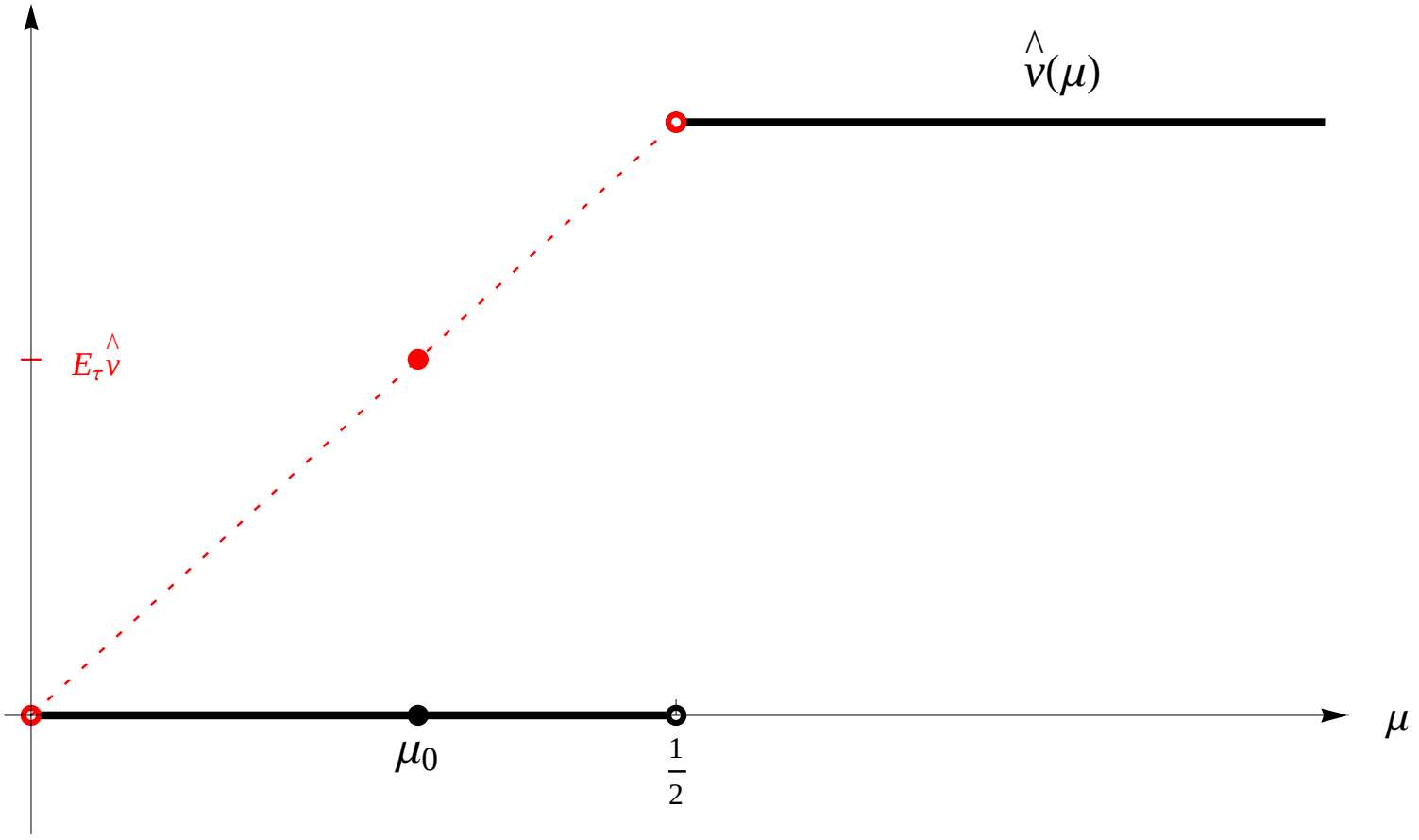
  - Maximum payoff attainable through persuasion: $V(\mu_0)$

Concave closure



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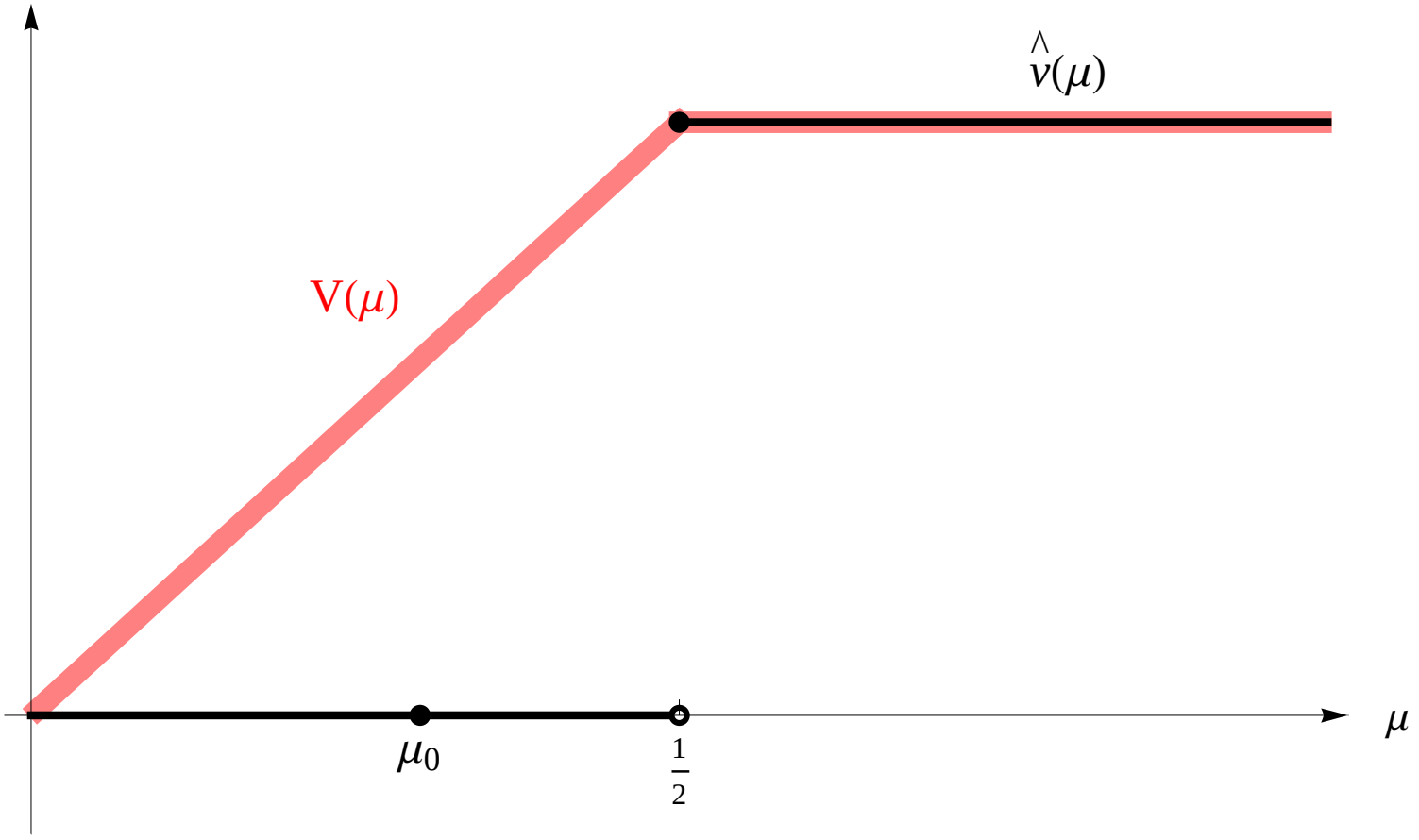

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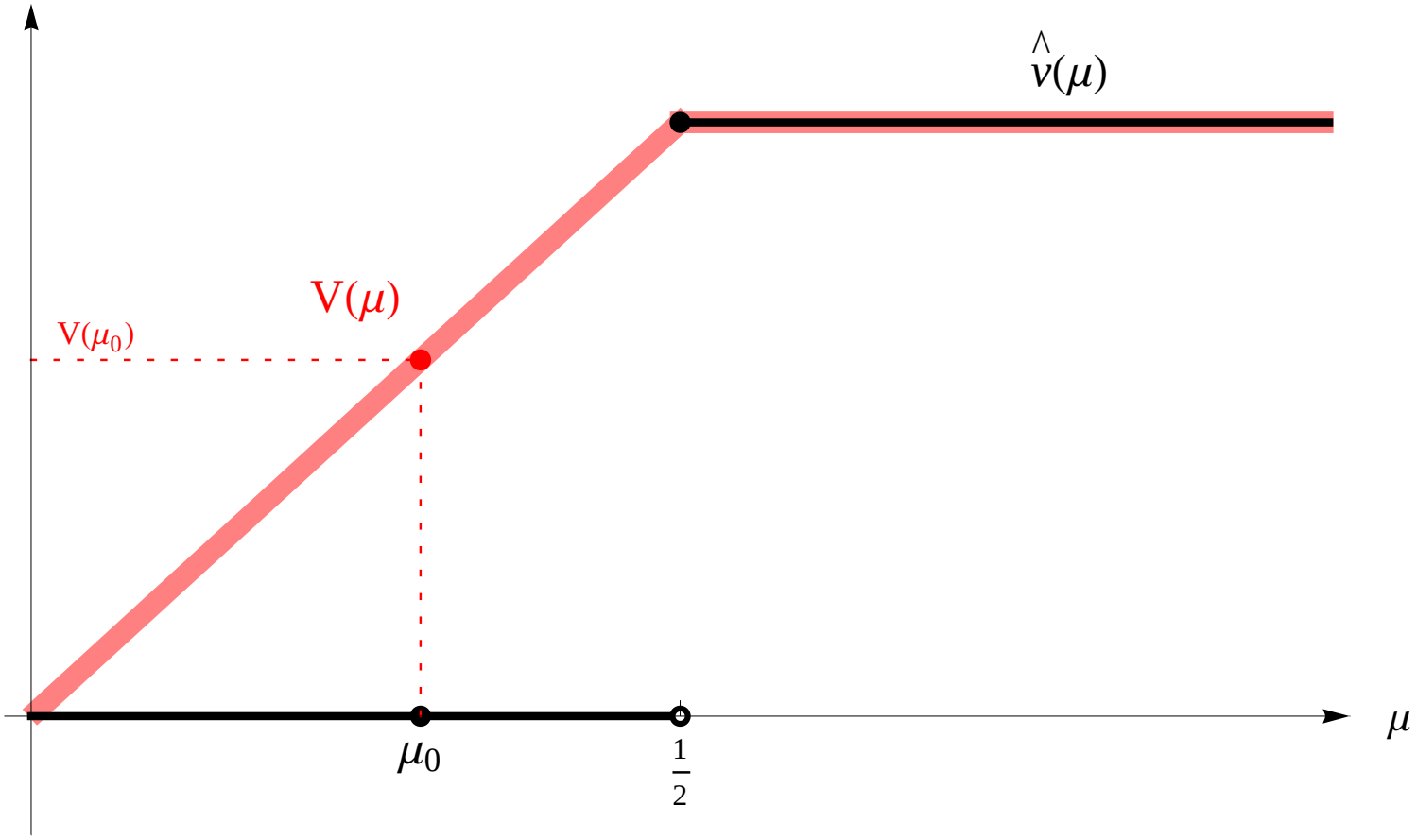


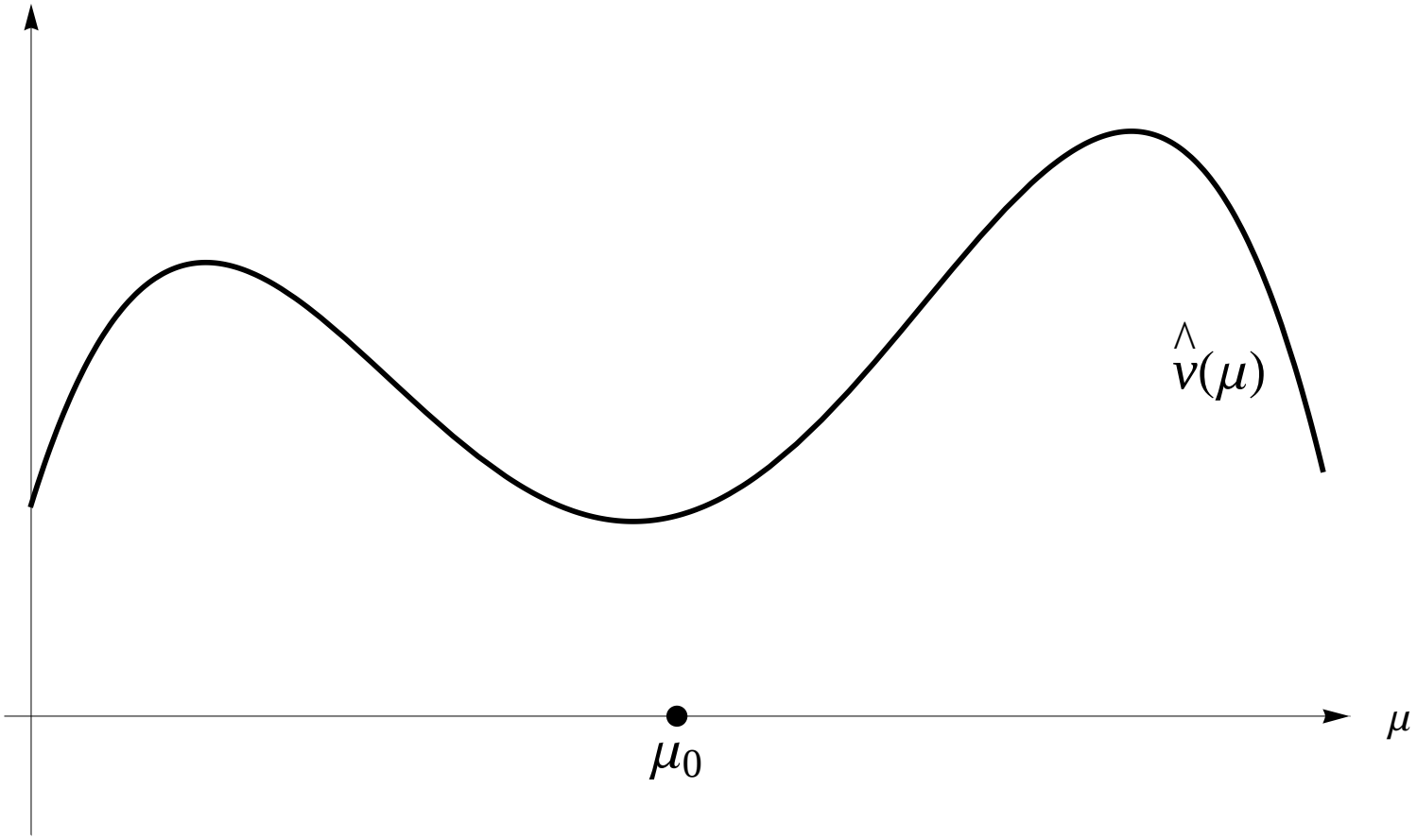
General case

- Receiver has a utility function $u(a, \omega)$
- Sender has a utility function $v(a, \omega)$
- Sender's utility from a given belief μ:

$$\hat{v}(\mu) = E_{\mu} v(a^*(\mu), \omega)$$

- Sender can attain $V(\mu_0)$ through persuasion

Another example



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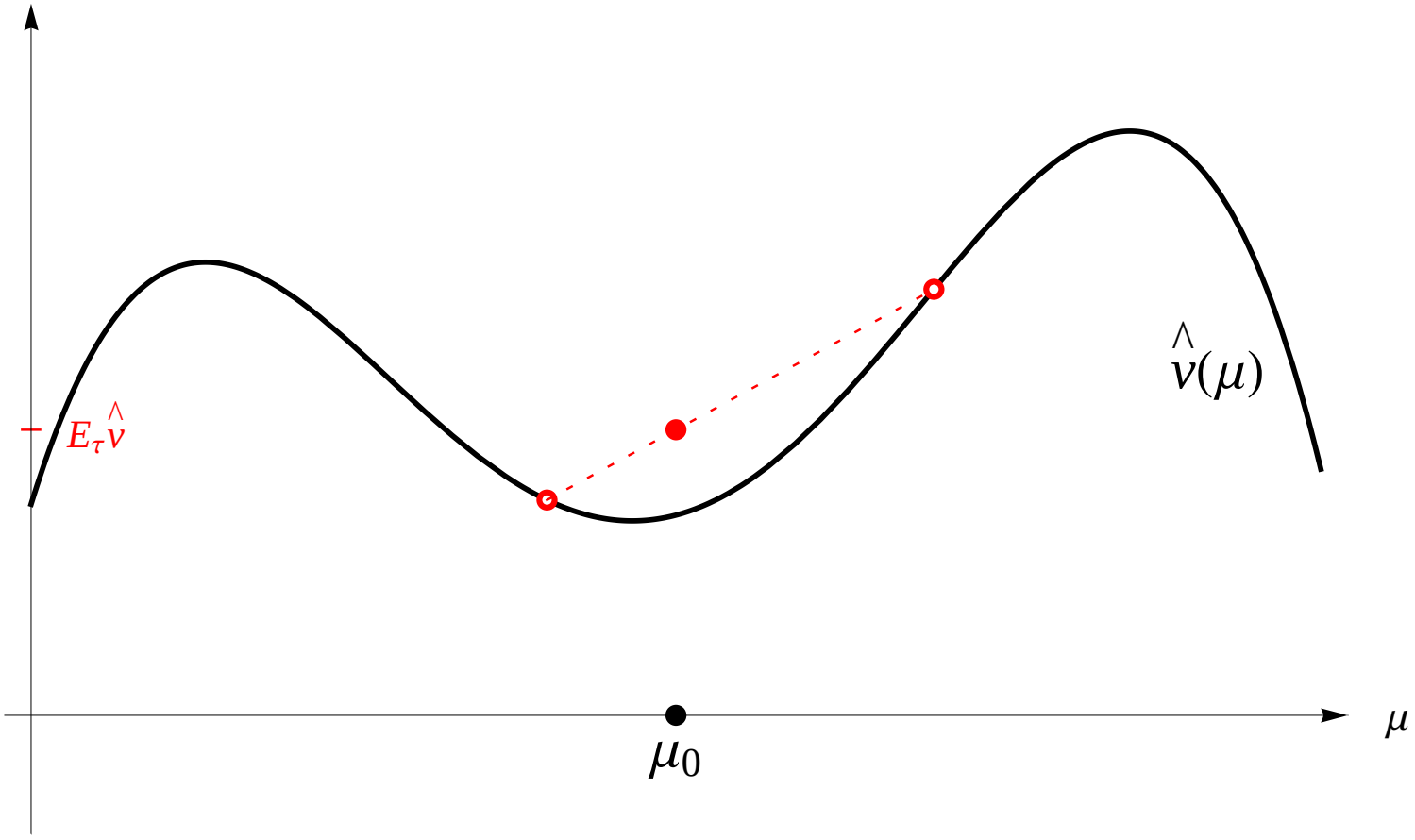

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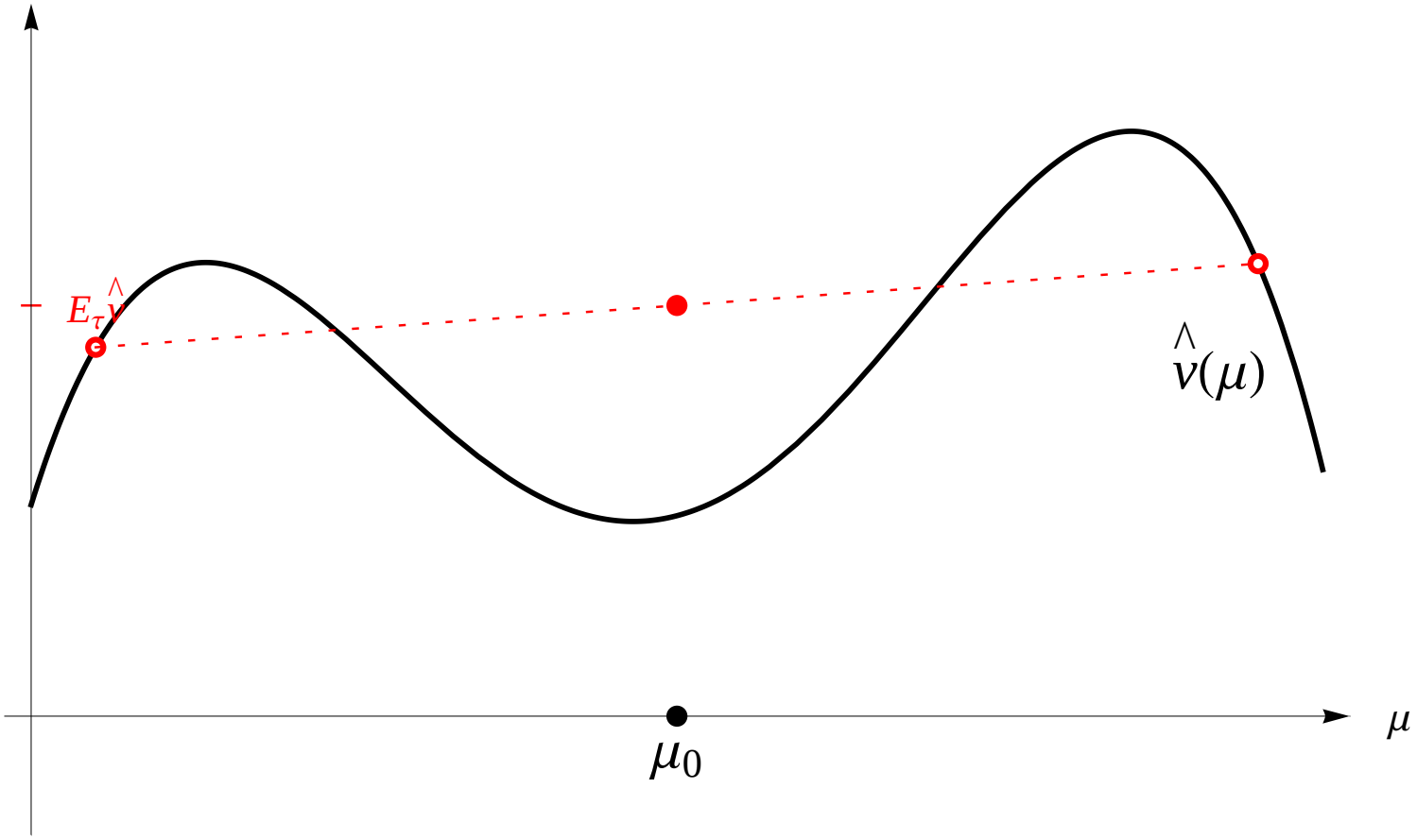

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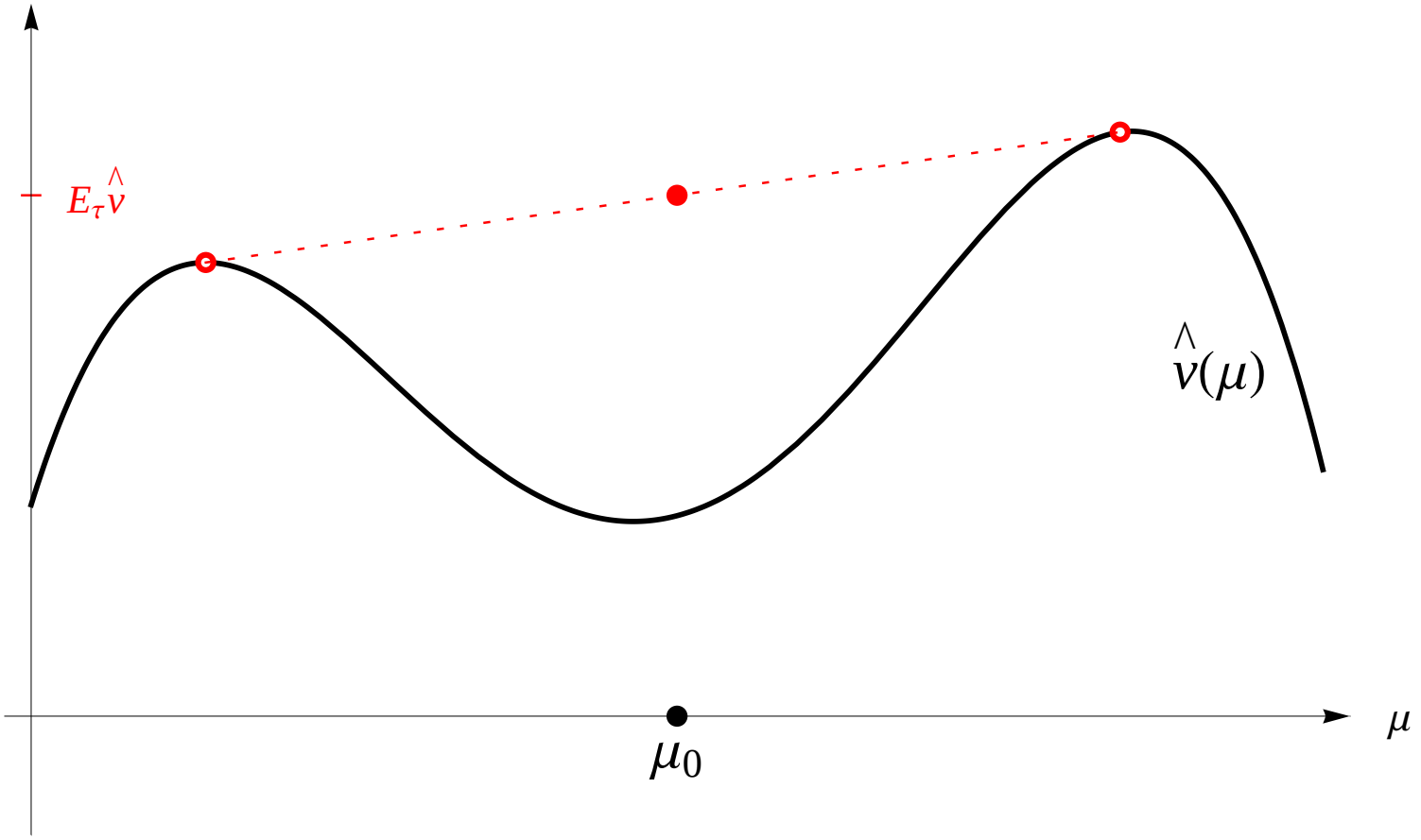

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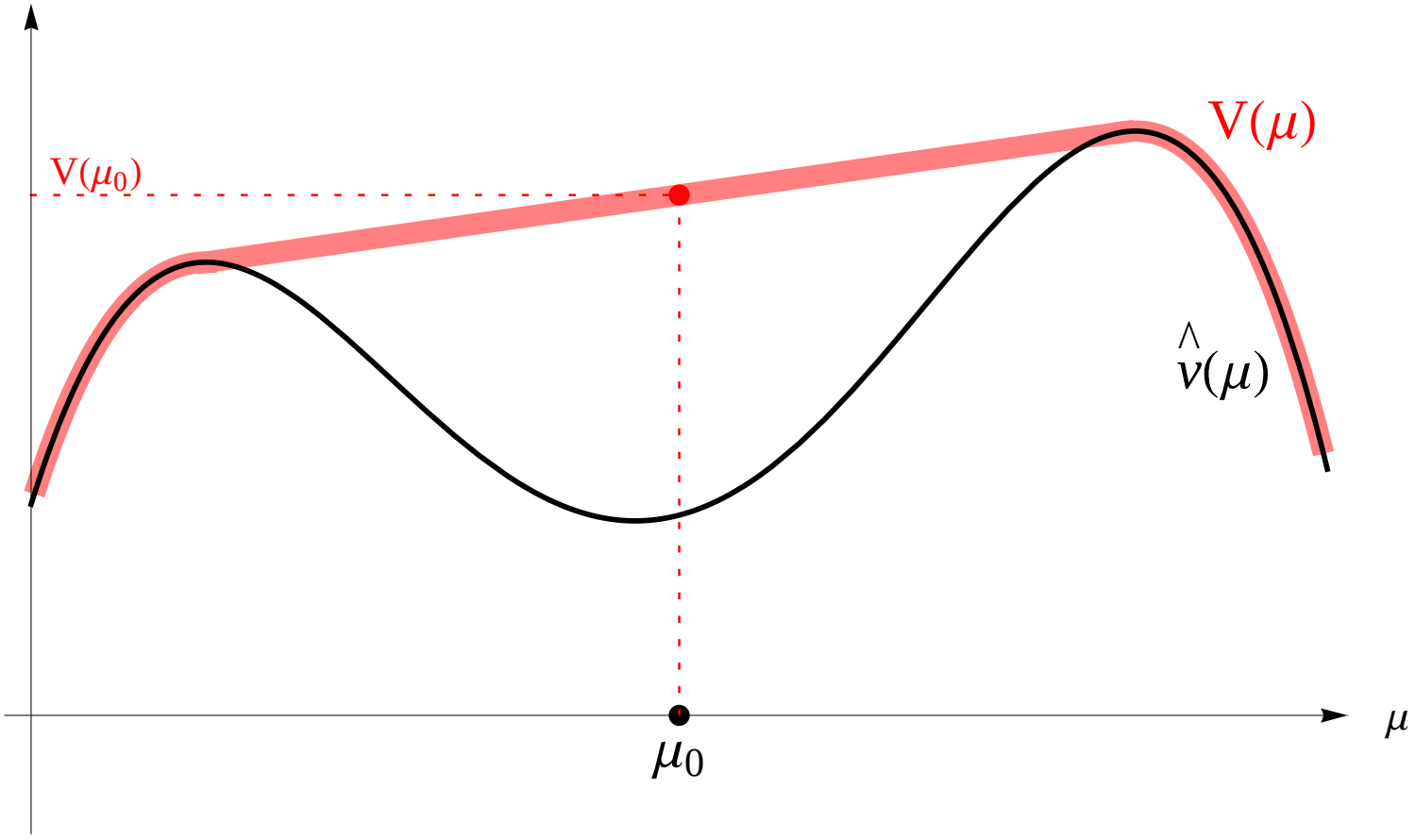

Taking stock

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- Applications of Bayesian persuasion:
  - marketing: free samples / test drives / trial versions of software
  - organizational economics: employee performance reviews
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  - accounting: financial reporting
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- New toolbox:
  - geometric solution to the problem of information design
  - application to entertainment industries

Dynamic Mechanism Design: The Dynamic Pivot Mechanism

Dirk Bergemann

November 2020

Intertemporal Efficiency with Private Information

- random arrival of buyers, sellers and/or objects
  - selling seats for an airplane with random arrival of buyers
  - bidding on ebay
  - bidding for construction projects with uncertain arrival of new projects
- bidding for links in sponsored search (Google, Yahoo, etc.)
  - initial uncertainty and learning about click-through probability
  - initial uncertainty and learning about conversion probability
- leasing resource over time
  - auction of renewable license, right, capacity over time
  - web serving, computational resource (bandwidth, CPU)

Static Efficiency with Private Information

- Vickrey (1961), Clarke (1971), Groves (1973) (VCG) establish incentive compatibility in dominant strategies in private value environments
  - agent i internalizes the social objective
- Clarke (1971) and Laffont (1977) analyze specific VCG mechanism: pivotal mechanism
- externality cost of i

utility of $I \setminus i$ when i is absent - utility of $I \setminus i$ when i is present

- i internalizes social objective as i pays her externality cost
- marginal contribution of i = utility of i - externality cost of i

Dynamic Pivotal Mechanism

- payoff in Pivot mechanism = marginal contribution
- develop marginal contribution mechanism in intertemporal environments with new arrival of information regarding:
  - preferences
  - agents
  - allocations
- design sequence of payments so that each agent receives **flow** marginal contribution in every period

...but wait...

- solve intertemporal problem as a completely contingent plan
- embed intertemporal problem in a static problem (as in an Arrow Debreu economy) ...
- ... and then appeal to the classic VCG results.
- but the contingent view fails to account for strategic possibilities of the agents in the sequential model

Sequential Incentive and Participation Constraints

- information arrives over time
- report of agent *i* in period *t* responds to private information of agent *i*, but may also respond to past reports of other agents (possibly inferred from allocative decisions)
- truthtelling (generally) fails to be a weakly dominant strategy
- with forward looking agents, participation constraint is required to be satisfied at every point in time
  (and not only in the initial period)

Results

- dynamic pivotal mechanism is socially efficient
- periodic ex post: with respect to information in period *t*
- satisfies (periodic) ex post incentive constraints
- satisfies (periodic) ex post participation constraints
- adding efficient exit condition (weak “online” condition) uniquely identifies dynamic pivotal mechanism:

  if *i* doesn’t impact future allocative decisions,
  then *i* doesn’t have future monetary transfers

Literature

- Dolan (RAND 1978):
  priority queuing
- Parkes et al. (2003):
  delayed VCG without participation or budget balance constraints
- Bergemann & Valimaki (JET 2006):
  complete information, repeated allocation of single object over time, first price bidding
- Athey & Segal (2007):
  balanced budget rather than participation constraints

Scheduling

- scheduling tasks
- discrete time, infinite horizon: $t = 0, 1, ...$
- common discount factor δ
- finite number of agents: $i \in \{0, 1, ..., I\}$
- each agent i has a single task
- value of task for i is:

$$v_i > 0$$

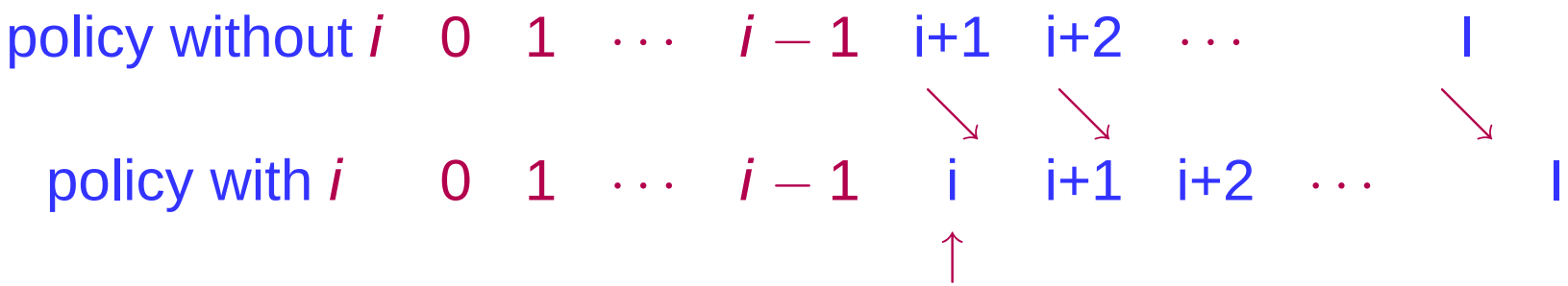
- quasilinear utility: $v_i - p_i$

Assignment

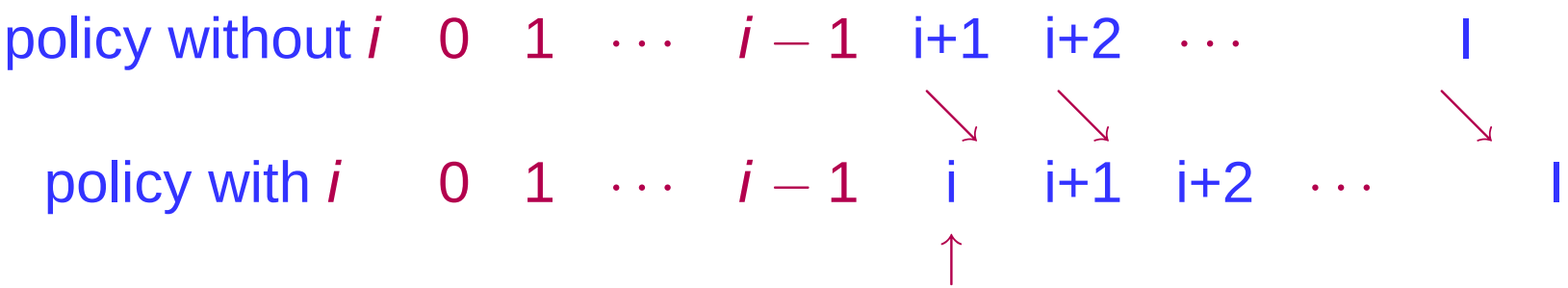
- values are given wlog in descending order:

$$v_0 > v_1 > \cdots > v_I > 0$$

- marginal contribution of task i :
  difference in welfare with i and without i
- efficient task assignment policy:



Marginal Contribution



- insert valuable task i:
- raise the value of all future tasks: $t > i$
- marginal contribution M_i:

$$M_i = \sum_{t=0}^{I} \delta^t v_t - \left(\sum_{t=0}^{i-1} \delta^t v_t + \sum_{t=i+1}^{I} \delta^{t-1} v_t \right)$$

or

$$M_i = \sum_{t=i}^{I} \delta^t \left(v_t - v_{t+1} \right) \geq 0$$

Externality

- from marginal contribution to externality pricing:

$$M_i = v_i - p_i$$

- externality cost of task i is:

$$p_i = v_{i+1} - \sum_{t=i+1}^{I} \delta^{t-i} \overbrace{(v_t - v_{t+1})}^{>0}$$

- task i directly replaces task $i+1$, but also:
- task i raises the value of all future tasks

Incomplete Information

- v_i is private information to agent i at $t = 0$
- incentive compatibility and efficient sorting: when would i like to win against j versus $j + 1$:

$$\begin{aligned} & \left(v_i - v_j\right) - \sum_{t=j}^{I} \delta^{t-(j-1)} \left(v_t - v_{t+1}\right) \\ \geq \quad & \delta \left(v_i - v_{j+1}\right) - \sum_{t=j+1}^{I} \delta^{t-j} \left(v_t - v_{t+1}\right) \end{aligned}$$

- reduces to cost of delay:

$$\left(1 - \delta\right) v_i \geq \left(1 - \delta\right) v_j .$$

- report thruthful if others report truthful: ex post equilibrium

General Model

- agent $i = 1, ..., I$
- time $t = 0, 1.....$
- common discount factor $\delta \in (0, 1)$
- quasilinear flow utility of agent i in period t :

$$v_i\left(a_t, \theta_{i,t}\right) - p_{i,t}$$

- allocation $a_t \in A$
- markovian state $\theta_t = \left(\theta_{1,t}, ..., \theta_{I,t}\right) \in \Theta$
- monetary transfer $p_{i,t}$
- private (Markovian) signal $\theta_{i,t+1}$ of i is generated by conditional distribution function:

$$\theta_{i,t+1} \sim P_i\left(\cdot \left| a_t, \theta_{i,t} \right.\right)$$

Dynamic Direct Mechanism

- agent i reports $r_{i,t} \in \Theta_i$ in every period t
- inductively, a history of publicly observable reports:

$$h_t = \left(h_{t-1},\ r_{1,t}, ..., r_{I,t}\right)$$

- socially efficient allocation rule (after all histories h_t)

$$a_t^* : H_t \rightarrow [0,1]^I$$

- transfer (or pricing) rule is given by:

$$p_t : H_t \rightarrow \mathbb{R}^I$$

Strategies

- private history of agent i:

$$h_{i,t} = \left(\theta_{i,0}, r_0, , ...\theta_{i,t-1}, r_{t-1}, \theta_{i,t}\right)$$

- reporting strategy for agent i:

$$r_{i,t} : H_{i,t} \to \Theta_i$$

- expected discounted payoff for bidder i :

$$\mathbb{E} \sum_{t=0}^{\infty} \delta^t \left[v_i \left(a^* \left(h_t \right), \theta_{i,t} \right) - p_{i,t} \left(h_t \right) \right].$$

- strategy of i solves sequential problem $V_i(h_{i,t})$:

$$\max_{r_{i,t} \in \Theta_i} \mathbb{E} \left\{ v_{i,t} \left(a^* \left(h_t \right), \theta_{i,t} \right) - p_{i,t} \left(h_t \right) + \delta V_i \left(h_{i,t+1} \right) \right\}$$

- taking expectation $\mathbb{E}$ wrt $r_{-i,t}$

Equilibrium

- denote by $h_{-i,t} \triangleq h_t \setminus r_{i,t}$
- *Bayesian incentive compatible* if $r_{i,t} = \theta_{i,t}$ solves

$$\max_{r_{i,t} \in \Theta_i} \mathbb{E}\left\{v_i\left(a^*\left(r_{i,t}, h_{-i,t}\right), \theta_{i,t}\right) - p_{i,t}\left(r_{i,t}, h_{-i,t}\right) + \delta V_i\left(h_{i,t+1}\right)\right\}$$

- periodic ex post: with respect to all information in period t
- (periodic) *ex post incentive compatible* if $r_{i,t} = \theta_{i,t}$ solves

$$\max_{r_{i,t} \in \Theta_i} \left\{v_i\left(a^*\left(r_{i,t}, h_{-i,t}\right), \theta_{i,t}\right) - p_{i,t}\left(\theta_{i,t}, h_{-i,t}\right) + \delta V_i\left(h_{i,t+1}\right)\right\}$$

for all histories $h_t \in H_t$

Marginal Contribution

- value of social program after removing agent *i*

$$W_{-i}(\theta_u) = \max_{\{a_{-i,t}\}_{t=u}^{\infty}} \mathbb{E} \sum_{t=u}^{\infty} \sum_{j \neq i} \delta^{t-u} a_{j,t}(\theta_t) v_j(\theta_{j,t})$$

- *marginal contribution* $M_i(\theta_t)$ of agent *i* at state θ_t is:

$$M_i(\theta_t) = W(\theta_t) - W_{-i}(\theta_t)$$

- value M_i conditional on state θ_{t+1} and allocation a_t:

$$M_i(\theta_{t+1} | a_t)$$

- *flow marginal contribution* $m_i(\theta_t)$:

$$M_i(\theta_t) = m_i(\theta_t) + \delta M_i(\theta_{t+1} | a_t^*)$$

Flow Marginal Contribution

- flow marginal contribution:

$$m_i(\theta_t) = M_i(\theta_t) - \delta M_i(\theta_t, a_t^*)$$

- expanding flow term with respect to time

$$m_i(\theta_t) = \overbrace{(W(\theta_t) - W_{-i}(\theta_t))}^{\text{M}_\text{i} \text{ starting at t}} - \overbrace{\delta(W(\theta_{t+1}|a_t^*) - W_{-i}(\theta_{t+1}|a_t^*))}^{\text{M}_\text{i} \text{ starting at t+1 and } a_t^*}$$

- expanding flow term with respect to identity (rearranging)

$$m_i(\theta_t) = \overbrace{(W(\theta_t) - \delta W(\theta_{t+1}|a_t^*))}^{\text{current value with i}} - \overbrace{(W_{-i}(\theta_t) - \delta W_{-i}(\theta_{t+1}|a_t^*))}^{\text{current value without i but } a_t^*}$$

Dynamic Pivotal Mechanism

Theorem (Dynamic Pivotal Mechanism)

The socially efficient allocation rule $\{a^*\}$ *satisfies ex post incentive and ex post participation constraint with payment* p^**:*

$$p^*_{i,t}\left(a^*\left(\theta_t\right),\theta_{-i,t}\right) = v_i\left(a^*\left(\theta_t\right),\theta_{i,t}\right) - m_i\left(\theta_t\right).$$

- efficiency, marginal contribution pricing
- static environment: incentive compatibility in weakly dominant strategies
- dynamic environment: dominant incentive compatibility fails to hold as it removes the ability to respond to past announcements
- yet ex post incentive compatibility can be satisfied in dynamic mechanism

Uniqueness

- many transfer rules support ex post incentive and ex post participation constraints in dynamic setting
- temporal separation between allocative influence and monetary payments is undesirable for many reason:

1. agent *i* may leave the mechanisms and break her commitment *after* she ceases to have a pivotal role but *before* her payments come due
2. if agent *i* ceases to influence current or future allocative decisions, why communicate for monetary transfers only

Efficient Exit

- presence of i is not influential for the future, given θ_τ, if

$$\Pr\left(a_t^*\left(\theta_t\right) \neq a_{-i,t}^*\left(\theta_t\right),\ \forall t \geq \tau \,|\theta_\tau\right) = 0.$$

Definition (Efficient Exit)

A mechanism satisfies the efficient exit condition if

$$\Pr\left(a_t^*\left(\theta_t\right) \neq a_{-i,t}^*\left(\theta_t\right),\ \forall t \geq \tau \,|\theta_\tau\right) = 0,$$

then

$$p_{i,\tau}\left(\theta_\tau\right) = 0.$$

- weak online requirement
- decisions regarding agent i have to be made as long as agent i is pivotal with positive probability

Diverse Preferences

- diverse flow valuations:

Definition (Diverse Valuations)

The flow valuations are diverse if

1. $\forall i, \exists \theta_i = \theta_i^0$ s. th. for $\forall a_t$

$$v_{i,t}\left(a_t, \theta_i^0\right) = 0 \text{ and } P_i\left(\theta_i^0; a_t, \theta_i^0\right) = 1;$$

2. $\forall i, a, x \in \mathbb{R}_+, \exists \theta_i$ s.th.

$$v_{i,t}\left(a_t, \theta_i\right) = \begin{cases} x & \text{if } a_t = a, \\ 0 & \text{if } a_t \neq a, \end{cases} \quad \text{and } P_i\left(\theta_i^0; a_t, \theta_i\right) = 1.$$

Uniqueness of Dynamic Pivotal Mechanism

Theorem (Uniqueness)

With diverse valuations, if a mechanism satisfies ex post incentive and participation constraints and the efficient exit condition, then it is the dynamic pivotal mechanism.

- incentive compatibility and earlier result of marginal contribution payoff can only be reconciled with transfers of the dynamic pivotal mechanism
- proof by contradiction: violation of EPIC or EPIR

Conclusion

- direct dynamic mechanism in private value environments with transferable utility
- design of monetary transfers relies on notions of marginal contribution and flow marginal contribution
- transfer the insights of pivotal mechanism from static to dynamic settings
- interesting question: interdependent value environments

Dynamic Mechanism Design:

Sequential Screening

Dirk Bergemann

November 2020

Dynamics and Revenue Maximization

- $t = 0, 1, \ldots, T, \ldots$ i.e., finite or infinite time horizon
- $0 < \delta \leq 1$ discount factor
- $\theta_{i,t}$ "value" player i at time t, $i = 1, \ldots, I$
  Note that the type of i is evolving over time
- evolution of type

$$p(\theta_{i,t+1}|\theta_{i,t}) \neq p(\theta_{i,t+1})$$

  i.e., we have some persistence.
  Special case: fully persistent type $\theta_{i,t+1} = \theta_{i,t}$
- dynamics come from
  - evolution of valuations
  - evolution of beliefs

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- Coase formulated a conjecture in 1937 his conjecture about durable goods; Gul-Sonnenschein-Wilson (JET 1986)

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- In the static problem

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  which is the monopolist problem.
- Now if the monopolist sets p_0 and the buyer does not buy, in the next period the seller will want to lower the price

$$\max_p \{p(1 - F_1(p))\}$$

  where F_1 is the truncated prior.

No Commitment Benchmark

- Note that the monopolist faces competition, i.e., himself tomorrow. Without commitment, the selling policy tomorrow competes with the selling policy today.

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- In this context, Coase conjecture is that as $\delta \longrightarrow 1 \Rightarrow \pi = \underline{\theta}$.
- GSW prove as long as $\underline{\theta} > 0$.
- Robust to many variations

Commitment: Dynamic Information / Static Decisions

- Courty and Lee (2000); "Sequential Screening" (Borgers text book)
- Two periods $t = 0, 1$.
- Two types of consumers will make a single purchase (i.e., plane ticket) at a fixed date. Information about the value of the trip increases over time
- Let the marginal cost of a seat in a plane be $c = 1$.

Example

- There are two types of travelers: leisure and business travelers. At time $t = 0$,
  - leisure travelers have a valuation distributed $U[1, 2]$,
  - business travelers have a valuation distributed $U[0, 1] \cup U[2, 3]$
- The airlines knows that $\frac{1}{3}$ are leisure travelers and $\frac{2}{3}$ business travelers.

Example

- At time $t = 1$ all information is realized privately to buyers
- What would be the optimal selling policy of the airline at $t = 1$?

- Valuation in the market will be distributed $U[0, 3]$, hence the problem of the firm is

$$\begin{aligned} & \max_p (p - c)(1 - F(p)) \\ = \quad & \max_p (p - 1)(1 - \frac{1}{3}p) \end{aligned}$$

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- The optimal price will be $p^* = 2$, with profits $\pi^* = 1 \times \frac{1}{3} = \frac{1}{3}$.

Example

- Note that at $t = 1$ we have a lot of types, in fact the continuum $[0, 3]$. But at $t = 0$ we only have 2 types (leisure and business).
- Claim:
  - at $t = 0$ the firm will offer for leisure travelers an unconditional ticket (i.e., with no refund) at price $p_l = 1.5 = E[\theta_{i,l}]$.

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- Claim:
  - at $t = 0$ the firm will offer for leisure travelers an unconditional ticket (i.e., with no refund) at price $p_l = 1.5 = E[\theta_{i,l}]$.
  - And for business travelers it will offer a ticket at a higher price p_b but with the option to a refund if the realized value at $t = 1$ is low.

Example

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- Hence the following must hold for business travelers

$$\begin{aligned} & \frac{1}{2}\left(\mathrm{E}[\theta_{i,b}|\text{high value meeting}] - p_b\right) + \frac{1}{2}\left(r_b - p_b\right) \\ = \quad & \frac{1}{2}2.5 - p_b + \frac{1}{2}r_b \\ \geq \quad & 0 \end{aligned}$$

Example

- Under efficient exit, $r_b = 1$. Thus,

$$\begin{aligned} p_b &= 1.75 \\ r_b &= 1 \end{aligned}$$

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- And the profits will be

$$\begin{aligned} \pi_0 &= \frac{1}{3}(\underbrace{1.5}_{p_l} - \underbrace{1}_{\text{cost}}) + \frac{2}{3}\left[\frac{1}{2}(\underbrace{1.75}_{p_b} - \underbrace{1}_{\text{cost}}) + \frac{1}{2}(\underbrace{1.75}_{p_b} - \underbrace{1}_{r_b})\right] \\ &= \frac{2}{3} \\ &> \frac{1}{3} \end{aligned}$$

Example

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1. Over time, incentive constraints multiply, thus more private info $\Rightarrow$ more info rent $\Rightarrow$ firm wants to sell the object earlier to avoid the info rent
2. Selling earlier gives less efficient decision

- This is a fundamental trade-off in dynamic mechanism design.
- Also note that we need commitment, otherwise if the firm offers

$$\begin{cases} p_l^* = 1.5 \\ p_b^* = 1.75, r_b^* = 1 \end{cases}$$

at both times, everyone will buy leisure ticket at 1.5, anticipating that the firm will refund you $1 - \varepsilon$ if you dont want to use your ticket.

Dynamic Screening

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- Utility $u(q, \theta)$ [e.g., $q\theta$, with $q \in [0, 1]$
- Direct Mechanism:

$$\begin{aligned} q &: [\underline{\tau}, \overline{\tau}] \times [\underline{\theta}, \overline{\theta}] \to [0, 1] \\ t &: [\underline{\tau}, \overline{\tau}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R} \end{aligned}$$

Incentive Compatibility

- no incentive to deviate ex post

$$u\left(q\left(\tau,\theta\right),\theta\right)-t\left(\tau,\theta\right)\geq u\left(q\left(\tau,\theta'\right),\theta\right)-t\left(\tau,\theta'\right)$$

for all $\tau,\theta,\theta^{'}$

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- no incentive to deviate ex ante (or double deviations)

$$\int_{\theta\in\left[\underline{\theta},\overline{\theta}\right]}\left[u\left(q\left(\tau,\theta\right),\theta\right)-t\left(\tau,\theta\right)\right]f\left(\theta|\tau\right)d\theta$$

$$\geq\int_{\theta\in\left[\underline{\theta},\overline{\theta}\right]}\left[u\left(q\left(\tau',\theta^{r}\left(\theta\right)\right),\theta\right)-t\left(\tau',\theta^{r}\left(\theta\right)\right)\right]f\left(\theta|\tau\right)d\theta$$

for all τ,τ' and $\theta^{r}:\left[\underline{\theta},\overline{\theta}\right]\rightarrow\left[\underline{\theta},\overline{\theta}\right]$

Individual Rationality

- ex ante incentive to participate:

$$\int_{\theta \in \left[\underline{\theta}, \overline{\theta}\right]} \left[u\left(q\left(\tau, \theta\right), \theta\right) - t\left(\tau, \theta\right)\right] f\left(\theta | \tau\right) d\theta \geq 0$$

  for all τ

Simplifying Incentive Compatibility

- Because - but only because - θ contains all payoff-relevant information, can relax ex ante incentive constraint to not allow double deviations:

$$\int_{\theta \in \left[\underline{\theta}, \overline{\theta}\right]} \left[u\left(q\left(\tau, \theta\right), \theta\right) - t\left(\tau, \theta\right)\right] f\left(\theta | \tau\right) d\theta$$

$$\geq \int_{\theta \in \left[\underline{\theta}, \overline{\theta}\right]} \left[u\left(q\left(\tau', \theta\right), \theta\right) - t\left(\tau', \theta\right)\right] f\left(\theta | \tau\right) d\theta$$

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$$\geq \int_{\theta\in\left[\underline{\theta},\overline{\theta}\right]} \left[u\left(q\left(\tau',\theta\right),\theta\right)-t\left(\tau',\theta\right)\right] f\left(\theta|\tau\right) d\theta$$

- ex post incentive constraints can be integrated up to replace double deviations

Incentive Compatibility as Property of Allocations

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    is increasing in τ.
  - only bottom type's IR constraint binds:

$$\int_{\theta \in \left[\underline{\theta}, \overline{\theta}\right]} \left[u\left(q\left(\underline{\tau}, \theta\right), \theta\right) - t\left(\underline{\tau}, \theta\right)\right] f\left(\theta | \underline{\tau}\right) d\theta = 0$$

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- regularity: this expression is increasing at all τ and θ.

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- restrict attention to case with $u(q, \theta) = \theta q$ and $q \in [0, 1]$
- marginal return to selling at (τ, θ) is

$$\psi(\tau, \theta) = \theta - \frac{1 - G(\tau)}{g(\tau)} \frac{\partial F(\theta|\tau) / \partial \tau}{f(\theta|\tau)}$$

- regularity: this expression is increasing at all τ and θ.
- informativeness measure

$$\frac{\partial F(\theta|\tau) / \partial \tau}{f(\theta|\tau)}$$

captures informativeness of initial signal about later value

Optimal Mechanism

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  - pay $t_0(\tau)$ upfront and $p(\tau)$ if $\psi(t, \theta) \geq 0$

Dynamic Allocations and Dynamic Information

- Two types: Battaglini (2005)
- Consumer has value θ_H or θ_L in any period.
- Cost $\frac{1}{2}q^2$ to produce q units
- Value $\theta_i q$
- Efficient sale: $q = \theta_k$
- Value evolves through time according to Markov process: α_{kl} is probability θ_k conditional on θ_l in the previous period
- Initial probabilities μ_H and μ_L
- long term contracts

Perfect Correlation

$$\alpha_{HH} = \alpha_{LL} = 1$$

$$q^*(h_t, \theta) = \begin{cases} \theta_H, \text{ if } \theta = \theta_H \\ \theta_L - \frac{\mu_H}{\mu_L}(\theta_H - \theta_L), \text{ if } \theta = \theta_L \end{cases}$$

$$p^*(h_t, \theta) = \begin{cases} \theta_H q_H + \theta_H (q_H - q_L), \text{ if } \theta = \theta_H \\ \theta_L q_L, \text{ if } \theta = \theta_L \end{cases}$$

argument

maximize

$$\mu_H \left(p_H - \frac{1}{2} q_H^2 \right) + \mu_L \left(p_L - \frac{1}{2} q_L^2 \right)$$

subject to

$$\begin{aligned} (IC_H)\ \theta_H q_H - p_H &\geq \theta_H q_L - p_L \\ (IR_L)\ \theta_L q_L - p_L &\geq 0 \end{aligned}$$

argument

$$\begin{aligned} IR_L \text{ binding} &\Rightarrow p_L = \theta_L q_L \\ IC_H \text{ binding} &\Rightarrow p_H = \theta_L q_L + \theta_H (q_H - q_L) \end{aligned}$$

unconstrained maximize

$$\mu_H \left(\theta_L q_L + \theta_H (q_H - q_L) - \frac{1}{2} q_H^2 \right) + \mu_L \left(\theta_L q_L - \frac{1}{2} q_L^2 \right)$$

FOC

$$\begin{aligned} \mu_H (\theta_H - q_H) &= 0 \\ -\mu_H (\theta_H - \theta_L) + \mu_L (\theta_L - q_L) &= 0 \end{aligned}$$

Recursive Maximization Problem

Maximize expected profits given recursively by

$$\left[p\left(h_t\right) - \frac{q^2\left(h_t\right)}{2} + \delta E\left[\Pi\left(h_t\right)\right]\right]$$

Constraints

- Incentive Constraints

$$\begin{aligned} & q\left(\theta_i|h_t\right)\theta_i - p\left(\theta_i|h_t\right) + \delta\left[U\left(h_t,\theta_i\right)\right] \\ \geq \quad & q\left(\theta_j|h_t\right)\theta_i - p\left(\theta_j|h_t\right) + \delta\left[U\left(h_t,\theta_j\right)\right] \end{aligned}$$

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- Participation Constraints: first period only is enough
- Binding constraints: at histories where an agent has not had a high value, current high value type does not want to mimic low value type

Optimal Allocation

$$q^*(h_t, \theta) = \begin{cases} \theta_H, \text{ if } \theta = \theta_H \\ \theta_L, \text{ if } \theta = \theta_L \text{ and } h_t \neq (\theta_L, ..., \theta_L) \\ \theta_L - (\theta_H - \theta_L) \frac{\mu_H}{\mu_L} \left(\frac{\alpha_{HH} - \alpha_{HL}}{\alpha_{LL}} \right)^{t-1} \text{ if } h_t = (\theta_L, ..., \theta_L) \end{cases}$$

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history dependence

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- as time goes on, buyer has less information about own type, less rent

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- History dependence of types causes problems
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- $\tau \in [\underline{\tau}, \overline{\tau}]$; prior G
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- Eso and Szentes (2017): do ex ante contracting conditional on independent innovations; no info from yet to be realized innovations

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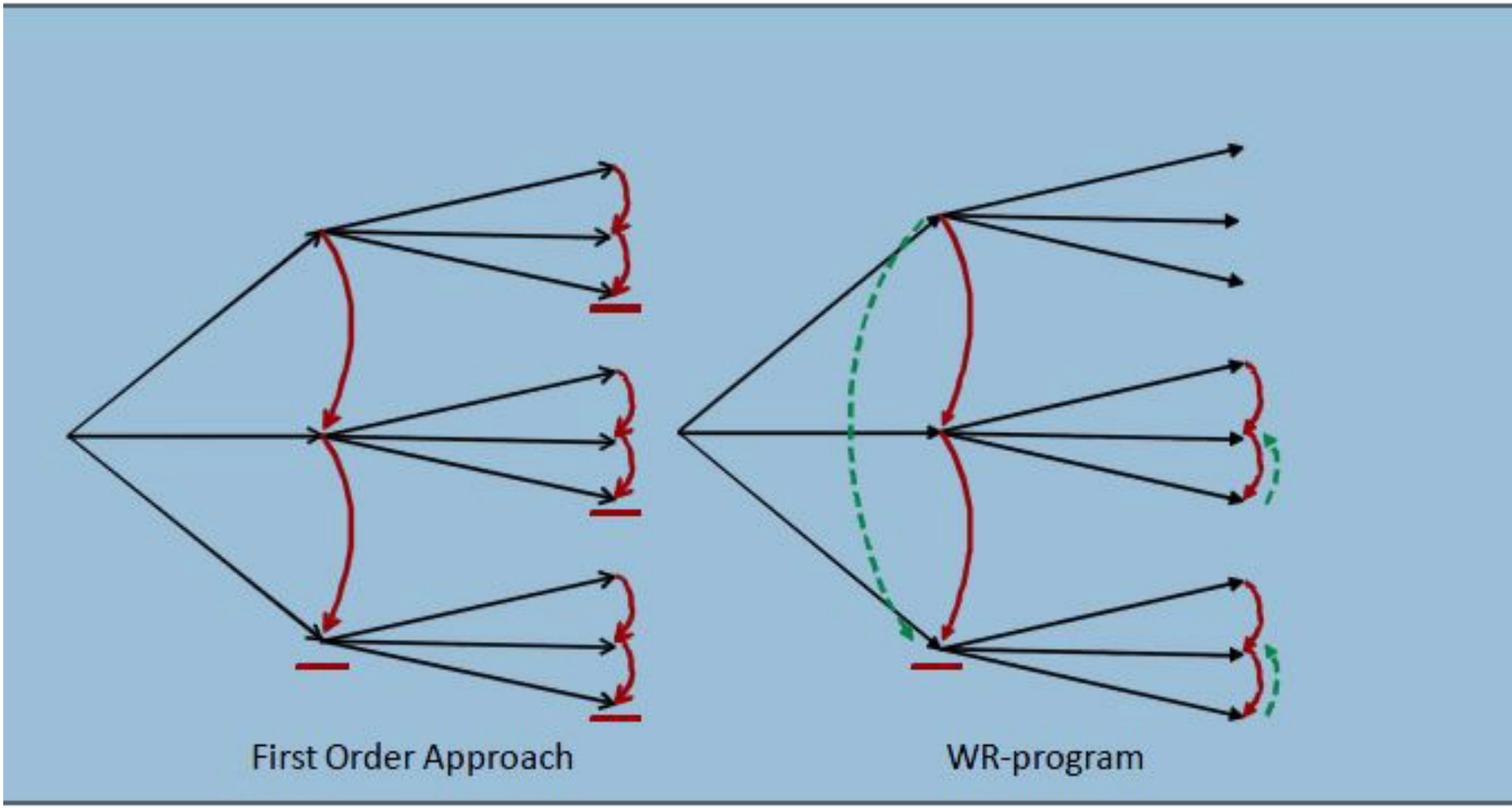
- in general (e.g., PST) very complicated condition using continuum types

Failure of First Order Approach

- Battaglini and Lamba (2015)
- Full Solution for two period three type case (corresponding to Battaglini 2005)
- Suppose that the probability of same state is $\alpha < 1$ and transit to each other state with probability $\frac{1-\alpha}{2}$

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