Chapter 4

Mixed Strategies and Mixed Strategy Equilibrium

Mixed Strategy

- Two kind of strategies:
  - pure
  - mixed
- Two kinds of equilibrium
  - pure strategy
  - mixed strategy
- Two games with mixed strategy equilibria:
  - Matching Pennies
  - Market Niche

Matching Pennies: The payoff matrix (All payoffs in cents)

Player 1 \ Player 2	Heads	Tails
Heads	+1, -1	-1, +1
Tails	-1, +1	+1, -1

Matching Pennies: No equilibrium in pure strategies

All Best Responses are underlined.

Player 1 \ Player 2	Heads	Tails
Heads	<u>+1</u>, -1	-1, <u>+1</u>
Tails	-1, <u>+1</u>	<u>+1</u>, -1

Computing Mixed Strategy Equilibria in 2×2 Games

- Solution criterion: each pure strategy in a mixed strategy equilibrium pays the same at equilibrium
- Each pure strategy not in a mixed strategy equilibrium pays less
- Detailed calculations for Matching Pennies and Market Niche
- An appealing condition on equilibria: payoff dominance

Matching Pennies: What about mixed strategies?

probability	1 \ 2	h (probability y)	t (probability 1- y)
x	H	+1, -1	-1, +1
1- x	T	-1, +1	+1, -1

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2	probability: y	probability: $1-y$	
	h	t	EU_1:
H	+1, -1	-1, +1	$2y - 1$
T	-1, +1	+1, -1	$1 - 2y$

$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	+1, -1	-1, +1
$1-x$	T	-1, +1	+1, -1
	EU_2:	$1 - 2x$	$2x - 1$

$EU_2(h) = x \times -1 + (1-x) \times 1 = 1 - 2x$

$EU_2(t) = x \times 1 + (1-x) \times -1 = 2x - 1$

In equilibrium, Player 1 is willing to randomize only when he is indifferent between H and T

$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$
$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$

In equilibrium: $EU_1(H) = EU_1(T)$

$\therefore \quad 2y - 1 = 1 - 2y$

$\Rightarrow \quad 4y = 2$

$\Rightarrow \quad y = ½$

$\Rightarrow \quad 1 - y = 1 - ½ = ½$

$\therefore \quad y = 1 - y = ½$

Similarly, Player 2 is willing to randomize only when she is indifferent between h and t

Player 1's Conditions:
$EU_1(H) = EU_1(T)$

Player 2's Conditions:
$EU_2(h) = x \times -1 + (1-x) \times 1 = 1 - 2x$
$EU_2(t) = x \times 1 + (1-x) \times -1 = 2x - 1$

In equilibrium: $EU_2(h) = EU_2(t)$

$\therefore \quad 1 - 2x = 2x - 1$

$\Rightarrow \quad x = ½$ and $1 - x = 1 - ½ = ½$

$\therefore \quad x = 1 - x = ½$

Matching Pennies: Equilibrium in mixed strategies

probability	1 \ 2	probability: ½	probability: ½	
		h	t	EU_1:
½	H	+1, -1	-1, +1	0
				||
½	T	-1, +1	+1, -1	0
	EU_2:	0	0	

EU_2: 0 = 0

Each is playing a best response to the other!

Mixed strategies are not intuitive: You randomize to make me indifferent.

Row randomizes to make Column indifferent.

Column randomizes to make Row indifferent.

Then each is playing a best response to the other.

Market Niche: The payoff matrix

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	100, 0
Stay Out	0, 100	0, 0

Market Niche: Two pure strategy equilibria

Mutual best responses form an equilibrium.

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	**100**, **0** ☆
Stay Out	**0**, **100** ☆	0, 0

Market Niche: What about mixed strategies?

probability	1 \ 2	e (probability y)	s (probability 1- y)
x	E	-50, -50	100, 0
1- x	S	0, 100	0, 0

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Need to calculate firm 1's expected utility from firm 2's mixed strategy

1 \ 2	e (probability y)	s (probability 1- y)	EU_1:
E	-50, -50	100, 0	100 - 150y
S	0, 100	0, 0	0

$EU_1(E) = y \times -50 + (1- y) \times 100 = 100 - 150y$

$EU_1(S) = y \times 0 + (1- y) \times 0 = 0$

Need to calculate firm 2's expected utility from firm 1's mixed strategy

probability	1 \ 2	e	s
x	E	-50, -50	100, 0
1- x	S	0, 100	0, 0
	EU_2:	100-150x	0

$EU_2(e) = x \times -50 + (1- x) \times 100 = 100 - 150x$

$EU_2(s) = x \times 0 + (1- x) \times 0 = 0$

In equilibrium, Firm 1 is willing to randomize only when it is indifferent between E and S

$EU_1(E) = y \times -50 + (1- y) \times 100 = 100 - 150y$

$EU_1(S) = y \times 0 + (1- y) \times 0 = 0$

In equilibrium: $EU_1(E) = EU_1(S)$

$\therefore \quad 100 - 150y = 0$

$\Rightarrow \quad 150y = 100$

$\Rightarrow \quad y = 2/3$

$\Rightarrow \quad 1 - y = 1 - 2/3 = 1/3$

$\therefore \quad y = 2/3$ and $1 - y = 1/3$

Similarly, Firm 2 is willing to randomize only when it is indifferent between h and t

Firm 1's Conditions:
$EU_1(E) = EU_1(S)$

Firm 2's Conditions:
$EU_2(e) = x \times -50 + (1-x) \times 100 = 100 - 150x$
$EU_2(s) = x \times 0 + (1-x) \times 0 = 0$

In equilibrium: $EU_2(e) = EU_2(s)$

$\therefore$ $100 - 150x = 0$

$\Rightarrow$ $150x = 100$

$\therefore$ $x = 2/3$ and $1 - x = 1/3$

Market Niche: Equilibrium in mixed strategies

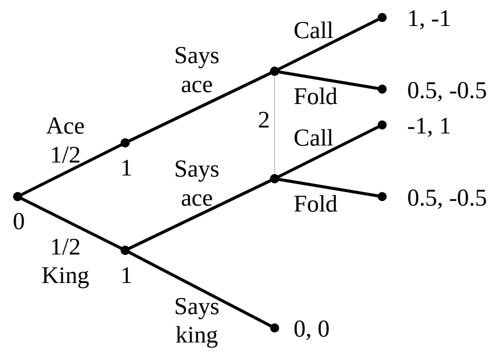
probability	1 \ 2	2/3 e	1/3 s	EU_1:
2/3	E	-50, -50	100, 0	0
				||
1/3	S	0, 100	0, 0	0
	EU_2:	0	= 0	

Each firm is playing a best response to the other!

Mixed Strategies and bluffing: Liar's Poker

- Mixed strategies as a way to be unpredictable
- Bluffing and mixed strategies
- Liar's poker, a game where bluffing pays

Liar's Poker: extensive form



Liar's Poker: normal form

1 \ 2	Call	Fold
Say A when K	0, 0	0.5, -0.5
Say K when K	0.5, -0.5	0.25, -0.25

Liar's Poker: No pure strategy equilibrium

1 \ 2	Call	Fold
Say A when K	0, **0**	**0.5**, -0.5
Say K when K	**0.5**, -0.5	0.25, **-0.25**

Liar's Poker: What about mixed strategies?

probability (1 \ 2)		y	1- y
		c	f
x	A when K	0, 0	0.5, -0.5
1- x	K when K	0.5, -0.5	0.25, -0.25

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Each player calculates his expected utility from other's mixed strategy

probability (1 \ 2)		y	1- y	
		c	f	EU_1:
x	A when K	0, 0	0.5, -0.5	0.5 - 0.5y
1- x	K when K	0.5, -0.5	0.25, -0.25	0.25 + 0.25y
	EU_2:	0.5x - 0.5	-0.25x - 0.25	

In equilibrium, player 1 is willing to randomize only when he is indifferent between A and K

$EU_1(A) = y \times 0 + (1-y) \times 0.5 = 0.5 - 0.5y$
$EU_1(K) = y \times 0.5 + (1-y) \times 0.25 = 0.25 + 0.25y$

In equilibrium: $EU_1(A) = EU_1(K)$

$\therefore$ $0.5 - 0.5y = 0.25 + 0.25y$

$\Rightarrow$ $0.75y = 0.25$

$\Rightarrow$ $y = 1/3$

$\Rightarrow$ $1 - y = 1 - 1/3 = 2/3$

$\therefore$ $y = 1/3$ and $1 - y = 2/3$

Similarly, Player 2 is willing to randomize only when she is indifferent between c and f

Player 1's Conditions:
$EU_1(A) = EU_1(K)$

Player 2's Conditions:
$EU_2(c) = x \times 0 + (1-x) \times -0.5 = 0.5x - 0.5$
$EU_2(f) = x \times -0.5 + (1-x) \times -0.25 = -0.25x - 0.25$

In equilibrium: $EU_2(c) = EU_2(f)$

$\therefore$ $0.5x - 0.5 = -0.25x - 0.25$

$\Rightarrow$ $0.75x = 0.25$

$\therefore$ $x = 1/3$ and $1 - x = 2/3$

Liar's Poker: Equilibrium in mixed strategies

probability (1 \ 2)		1/3	2/3	
		c	f	EU_1:
1/3	A	0, 0	0.5, -0.5	1/3
				||
2/3	K	0.5, -0.5	0.25, -0.25	1/3
	EU_2:	-1/3 =	-1/3	

Each player is playing a best response to the other!

Mixed Strategy Equilibria of Coordination Games and Coordination Problems

- Games with mixed strategy equilibria which cannot be detected by the arrow diagram
- The mixed strategy equilibrium of Video System Coordination is not efficient

Correlated Equilibrium

- Mixed strategy Nash equilibria tend to have low efficiency
- Correlated equilibria
  - public signal
  - Nash equilibrium in game that follows

Asymmetric Mixed Strategy Equilibria

- Making a game asymmetric often makes its mixed strategy equilibrium asymmetric
- Asymmetric Market Niche is an example

Asymmetrical Market Niche: The payoff matrix

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	150, 0
Stay Out	0, 100	0, 0

Asymmetrical Market Niche: Two pure strategy equilibria

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	**150**, **0** ☆
Stay Out	**0**, **100** ☆	0, 0

Asymmetrical Market Niche: What about mixed strategies?

probability	1 \ 2	e (y)	s (1- y)
x	E	-50, -50	150, 0
1- x	S	0, 100	0, 0

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Need to calculate each firm's expected utility from the firm's mixed strategy

probability	1 \ 2	e (y)	s (1- y)	EU_1:
x	E	-50, -50	150, 0	150 - 200y
1- x	S	0, 100	0, 0	0
	EU_2:	100 - 150x	0	

In equilibrium, Firm 1 is willing to randomize only when it is indifferent between E and S

$EU_1(E) = y \times -50 + (1-y) \times 150 = 150 - 200y$

$EU_1(S) = y \times 0 + (1-y) \times 0 = 0$

In equilibrium: $EU_1(E) = EU_1(S)$

$\therefore \quad 150 - 200y = 0$

$\Rightarrow \quad 200y = 150$

$\Rightarrow \quad y = 3/4$

$\Rightarrow \quad 1 - y = 1 - 3/4 = 1/4$

$\therefore \quad y = 3/4$ and $1 - y = 1/4$

Similarly, Firm 2 is willing to randomize only when it is indifferent between h and t

Firm 1's Conditions:
$EU_1(E) = EU_1(S)$

Firm 2's Conditions:

$EU_2(e) = x \times -50 + (1-x) \times 100 = 100 - 150x$

$EU_2(s) = x \times 0 + (1-x) \times 0 = 0$

In equilibrium: $EU_2(e) = EU_2(s)$

$\therefore \quad 100 - 150x = 0$

$\Rightarrow \quad 150x = 100$

$\therefore \quad x = 2/3$ and $1 - x = 1/3$

Asymmetrical Market Niche: Equilibrium in mixed strategies

probability (1 \ 2)		3/4 e	1/4 s	EU_1:
2/3	E	-50, -50	150, 0	0
				||
1/3	S	0, 100	0, 0	0
EU_2:		0	= 0	

Each firm is playing a best response to the other!

Asymmetrical Market Niche: Equilibrium in mixed strategies

Although the two pure strategy equilibria (E,s) and (S,e) did not change in Asymmetrical Market Niche, the mixed strategies equilibrium did change.

Chicken

- Two drivers race toward a cliff
- Strategy choice:
  - swerve
  - straight ahead
- More general version of the game:
  - back down
  - do not back down
- Solution as in Market Niche Game

Chicken: The payoff matrix

player 1 \ player 2	drive straight ahead	swerve
drive straight ahead	-10, -10	1, -1
swerve	-1, 1	0, 0

Chicken: strategy for player 1

player 1 \ player 2	drive straight ahead	swerve
drive straight ahead	-10, -10	1, -1
swerve	-1, 1	0, 0

Chicken: strategy for player 2

player 1 \ player 2	drive straight ahead	swerve
drive straight ahead	-10, -10	1, -1
swerve	-1, 1	0, 0

Chicken: two pure strategy Nash equilibria

player 1 \ player 2	drive straight ahead	swerve
drive straight ahead	-10, -10	1, -1 ★
swerve	-1, 1 ★	0, 0

Chicken: The payoff matrix

player 1 probability \ player 2 probability		y straight	1- y swerve
x	straight	-10, -10	1, -1
1- x	swerve	-1, 1	0, 0

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Chicken: The payoff matrix

player 1 probability \ player 2 probability		y straight	1- y swerve	EU_1
x	straight	-10, -10	1, -1	1 – 11y
1- x	swerve	-1, 1	0, 0	-y
	EU_2	1 – 11x	-x	

In equilibrium, player 1 is willing to randomize only when she is indifferent between “swerve” and “straight”

$EU_1(\text{straight}) = y \times (-10) + (1-y) \times 1 = 1 - 11y$

$EU_1(\text{swerve}) = y \times (-1) + (1-y) \times 0 = -y$

In equilibrium: $EU_1(\text{swerve}) = EU_1(\text{straight})$

$\therefore \quad 1 - 11y = -y$

$\Rightarrow \quad 1 = 10y$

$\Rightarrow \quad y = 1/10$

$\Rightarrow \quad 1 - y = 1 - 1/10 = 9/3$

$\therefore \quad y = 1/10$ and $1 - y = 9/10$

Similarly, player 2 is willing to randomize only when he is indifferent between "swerve" and "straight"

Player 1's Conditions:
$EU_1(\text{swerve}) = EU_1(\text{straight})$

Player 2's Conditions:
$EU_2(\text{straight}) = x \times (-10) + (1-x) \times 1 = 1 - 11x$
$EU_2(\text{swerve}) = x \times (-1) + (1-x) \times 0 = -x$

In equilibrium: $EU_2(\text{swerve}) = EU_2(\text{straight})$

$\therefore \quad 1 - 11x = -x$

$\Rightarrow \quad x = 1/10$

$\therefore \quad x = 1/10$ and $1 - x = 9/10$

Chicken: The payoff matrix

player 1 \ player 2	probability	1/10 straight	9/10 swerve	EU_1
probability 1/10	straight	-10, -10	1, -1	-0.1
9/10	swerve	-1, 1	0, 0	-0.1
	EU_2	-0.1	-0.1	

Everyday Low Prices

- Sales are mixed strategies
- Sears' marketing campaign to do away with sales, called Everyday Low Prices
- Two types of buyers:
  - informed
  - uninformed
- A mixed strategy equilibrium tells how often to run sales

Everyday Low Pricing: The payoff matrix

Retailer 1 \ Retailer 2	Normal price np	Sale price sp
NP	7500, 7500	7500, 8500
SP	8500, 7500	5500, 5500

Everyday Low Pricing: Two pure strategy equilibria

Retailer 1 \ Retailer 2	np	sp
NP	7500, 7500	**7500, 8500** ★
SP	**8500, 7500** ★	5500, 5500

Everyday Low pricing: What about mixed strategies?

probability	1 \ 2	y np	1- y sp
x	NP	7500, 7500	7500, 8500
1- x	SP	8500, 7500	5500, 5500

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Each retailer calculates its expected utility from other's mixed strategy

probability	1 \ 2	y	1- y	
		np	sp	EU_1
x	NP	7500, 7500	7500, 8500	7500
1- x	SP	8500, 7500	5500, 5500	3000y + 5500
	EU_2:	7500	3000x + 5500	

In equilibrium, Retailer 1 is willing to randomize only when it is indifferent between NP and SP

$EU_1(NP) = y \times 7500 + (1- y) \times 7500 = 7500$
$EU_1(SP) = y \times 8500 + (1- y) \times 5500 = 3000y + 5500$

In equilibrium: $EU_1(NP) = EU_1(SP)$

$\therefore \quad 7500 = 3000y + 5500$

$\Rightarrow \quad 3000y = 2000$

$\Rightarrow \quad y = 2/3$

$\Rightarrow \quad 1 - y = 1 - 2/3 = 1/3$

$\therefore \quad y = 2/3$ and $1 - y = 1/3$

Similarly, Retailer 2 is willing to randomize only when it is indifferent between c and f

Retailer 1's Conditions:
$EU_1(SP) = EU_1(NP)$

Retailer 2's Conditions:
$EU_2(np) = x \times 7500 + (1- x) \times 7500 = 7500$
$EU_2(sp) = x \times 8500 + (1- x) \times 5500 = 3000x + 5500$

In equilibrium: $EU_2(np) = EU_2(sp)$

$\therefore \quad 7500 = 3000x + 5500$

$\Rightarrow \quad 3000x = 200$

$\therefore \quad x = 2/3$ and $1 - x = 1/3$

Everyday Low Pricing: Equilibrium in mixed strategies

probability	1 \ 2	2/3	1/3	
		np	sp	EU_1:
2/3	NP	7500, 7500	7500, 8500	7500
				||
1/3	SP	8500, 7500	5500, 5500	7500
	EU_2:	7500 =	7500	

Each player is playing a best response to the other!

Mixed strategies are not intuitive: You randomize to make me indifferent.

R
E
M
I
N
D
E
R

Row randomizes to make Column indifferent.

Column randomizes to make Row indifferent.

Then each is playing a best response to the other.

Appendix: Bluffing in 1-card Stud Poker

- A version of poker with 3 kinds of cards (ace, king, and queen), 1-card hands, and players who see their cards
- For some ratios of the ante to the bet, 1-card stud poker has a unique equilibrium which is in mixed strategies
- Equilibrium play in poker usually calls for some bluffing
- The solution of poker has all players breaking even

One-card Stud Poker
Payoff matrix, player 1

Player 1 \ Player 2	I: Bet AKQ	II: Bet AK	III: Bet AQ	IV: Bet A
I: Bet AKQ	0, 0	(a-2b)/9, (2b-a)/9	3a/9, -3a/9	(4a-2b)/9, (2b-4a)/9
II: Bet AK	(2b-a)/9, (a-2b)/9	0, 0	(a+b)/9, -(a+b)/9	(2a-b)/9, (b-2a)/9
III: Bet AQ	-3a/9, 3a/9	-(a+b)/9, (a+b)/9	0, 0	(2a-b)/9, (b-2a)/9
IV: Bet A	(2b-4a)/9, (4a-2b)/9	(b-2a)/9, (2a-b)/9	(b-2a)/9, (2a-b)/9	0, 0

One-card Stud Poker.
Payoff matrix, player 1, a=$1, b=$1

Player 1 \ Player 2	I: Bet AKQ	II: Bet AK	III: Bet AQ	IV: Bet A
I: Bet AKQ	0, 0	-1/9, 1/9	3/9, -3/9	2/9, -2/9
II: Bet AK	1/9, -1/9	0, 0 ★	2/9, -2/9	1/9, -1/9
III: Bet AQ	-3/9, 3/9	-2/9, 2/9	0, 0	1/9, -1/9
IV: Bet A	-2/9, 2/9	-1/9, 1/9	-1/9, 1/9	0, 0

One-card Stud Poker.
Payoff matrix, player 1, a=$1 b=$2

Player 1 \ Player 2	I: Bet AKQ	II: Bet AK	III: Bet AQ	IV: Bet A
I: Bet AKQ	0, 0	-3/9, 3/9	3/9, -3/9	0, 0
II: Bet AK	3/9, -3/9	0, 0 ★	3/9, -3/9	0, 0 ★
III: Bet AQ	-3/9, 3/9	-3/9, 3/9	0, 0	0, 0
IV: Bet A	0, 0	0, 0 ★	0, 0	0, 0 ★