Cournot's model of oligopoly

- Single good produced by n firms
- Cost to firm i of producing q_i units: $C_i(q_i)$, where C_i is nonnegative and increasing
- If firms' total output is Q then market price is $P(Q)$, where P is nonincreasing

Profit of firm i, as a function of all the firms' outputs:

$$\pi_i(q_1, \ldots, q_n) = q_i P\left(\sum_{j=1}^{n} q_j\right) - C_i(q_i).$$

Strategic game:

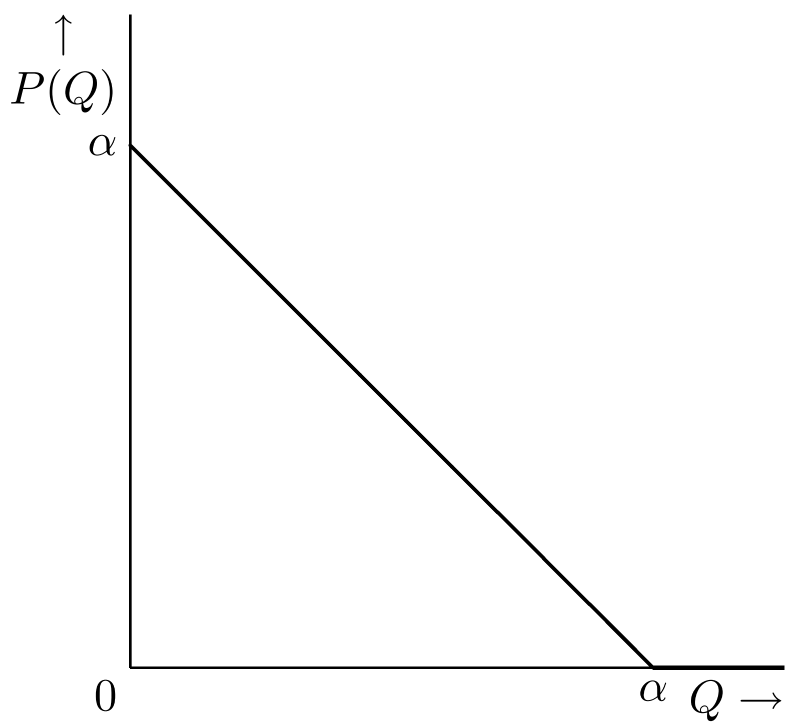
- players: firms
- each firm's set of actions: set of all possible outputs
- each firm's preferences are represented by its profit

Example

- two firms
- inverse demand:

$$P(Q) = \max\{0, \alpha - Q\} = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases}$$

- constant unit cost: $C_i(q_i) = cq_i$, where $c < \alpha$.



Nash equilibrium

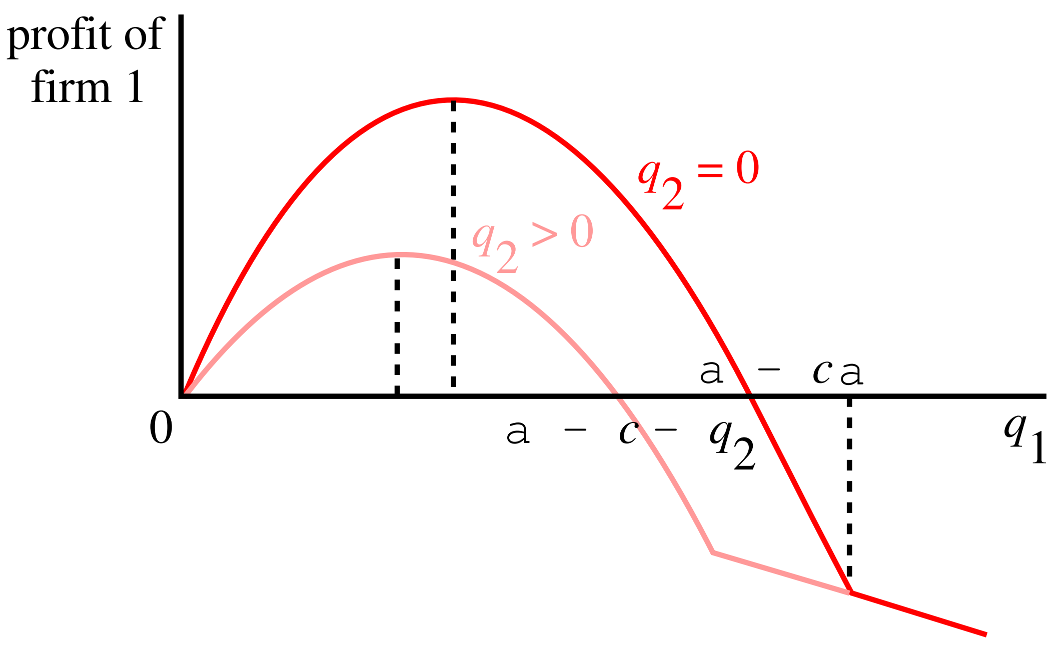
Payoff functions

Firm 1's profit is

$$\pi_1(q_1, q_2) = q_1(P(q_1 + q_2) - c)$$
$$= \begin{cases} q_1(\alpha - c - q_2 - q_1) & \text{if } q_1 \leq \alpha - q_2 \\ -cq_1 & \text{if } q_1 > \alpha - q_2 \end{cases}$$

Best response functions

Firm 1's profit as a function of q_1:



Up to $\alpha - q_2$ this function is a quadratic that is zero when $q_1 = 0$ and when $q_1 = \alpha - c - q_2$.

So when q_2 is small, optimal output of firm 1 is $(\alpha - c - q_2)/2$.

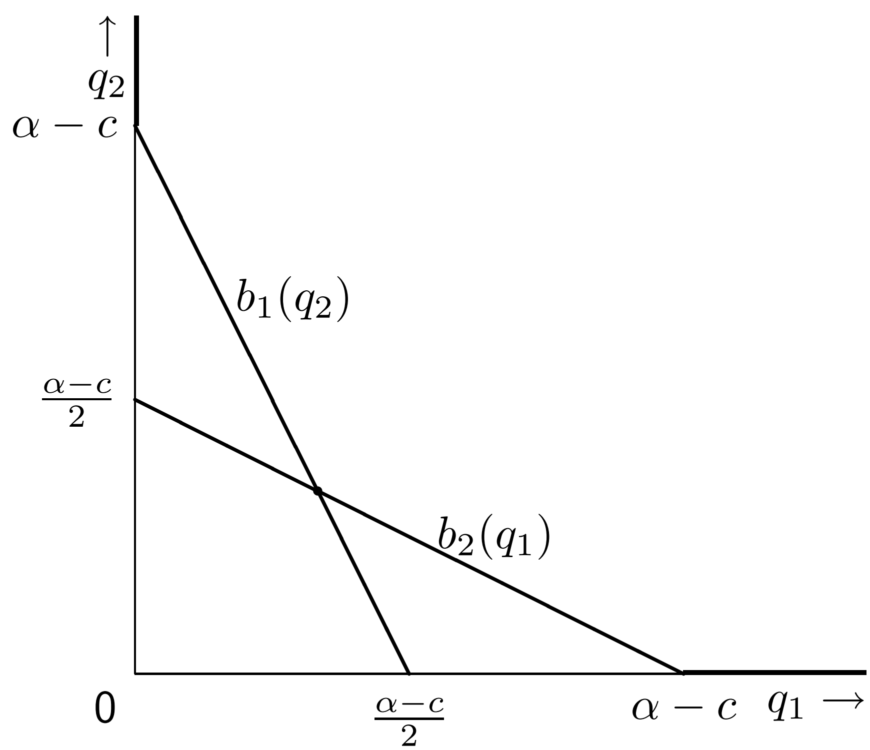
As q_2 increases this output decreases until it is zero.

It is zero when $q_2 = \alpha - c$.

Best response function is:

$$b_1(q_2) = \begin{cases} (\alpha - c - q_2)/2 & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

Same for firm 2: $b_2(q) = b_1(q)$ for all q.



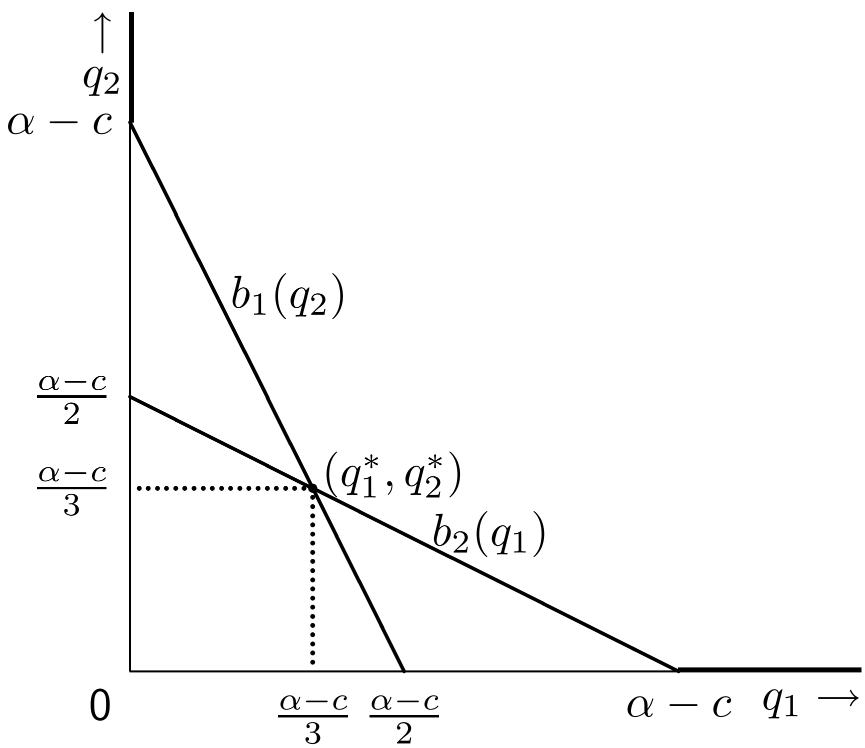
Nash equilibrium

Pair (q_1^*, q_2^*) of outputs such that

each firm's action is a best response to the other firm's action

or

$$q_1^* = b_1(q_2^*) \quad \text{and} \quad q_2^* = b_2(q_1^*):$$



or $q_1 = (\alpha - c - q_2)/2$ and $q_2 = (\alpha - c - q_1)/2$.

Solution:

$$q_1^* = q_2^* = (\alpha - c)/3.$$

Conclusion

Game has unique Nash equilibrium:

$$(q_1^*, q_2^*) = ((\alpha - c)/3, (\alpha - c)/3)$$

At equilibrium, each firm's profit is $(\alpha - c)^2/9$.

Note: Total output $2(\alpha - c)/3$ lies between monopoly output $(\alpha - c)/2$ and competitive output $\alpha - c$.

Bertrand's model of oligopoly

Strategic variable price rather than output.

- Single good produced by n firms
- Cost to firm i of producing q_i units: $C_i(q_i)$, where C_i is nonnegative and increasing
- If price is p, demand is $D(p)$
- Consumers buy from firm with lowest price
- Firms produce what is demanded

Firm 1's profit:

$$\pi_1(p_1, p_2) = \begin{cases} p_1 D(p_1) - C_1(D(p_1)) & \text{if } p_1 < p_2 \\ \frac{1}{2} p_1 D(p_1) - C_1(\frac{1}{2} D(p_1)) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

Strategic game:

- players: firms
- each firm's set of actions: set of all possible prices
- each firm's preferences are represented by its profit

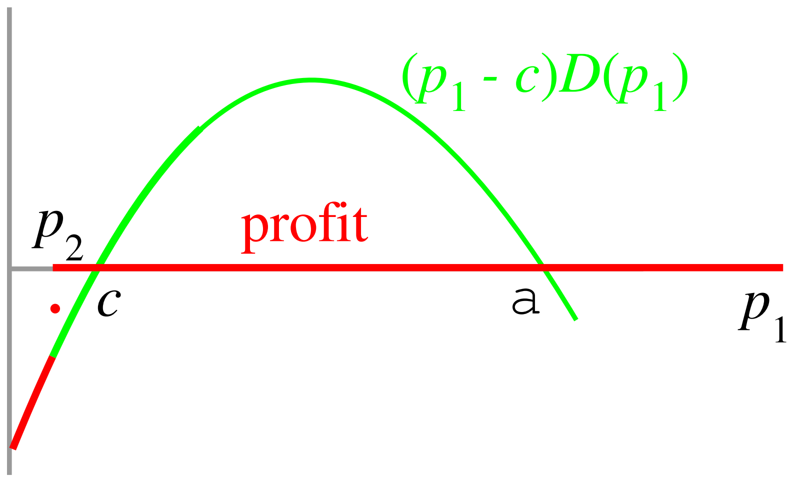
Example

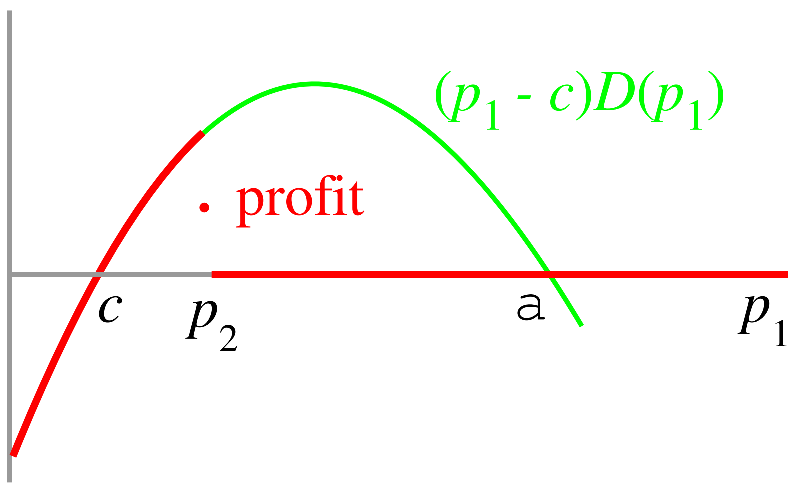
- 2 firms
- $C_i(q_i) = cq_i$ for $i = 1$, 2
- $D(p) = \alpha - p$.

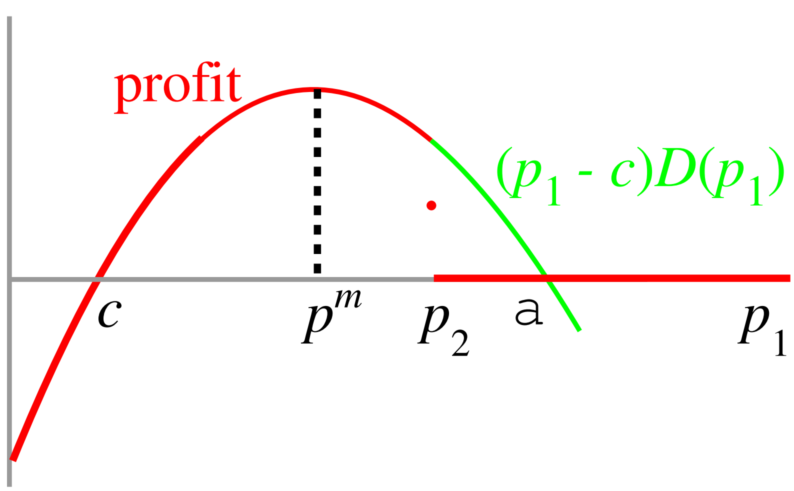
Nash equilibrium

Best response functions

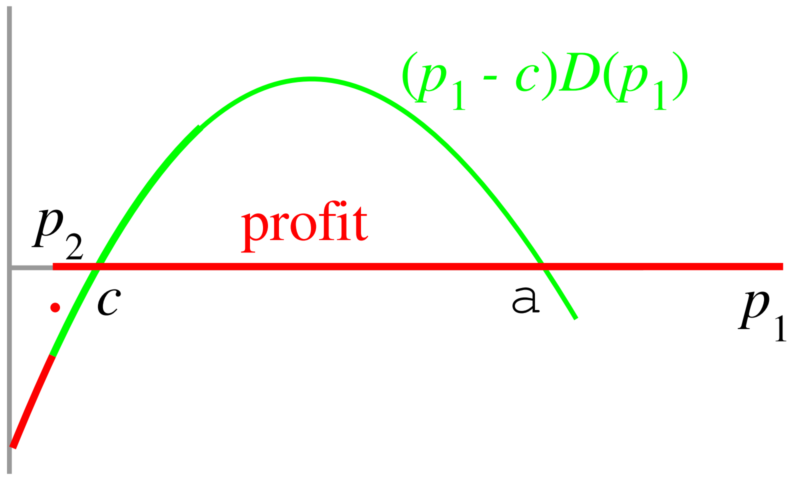
To find best response function of firm 1, look at its payoff as a function of its output, given output of firm 2.


(p1 - c)D(p1)
profit
p2
c
a
p1


(p1 - c)D(p1)
profit
c
p2
a
p1


profit
(p1 - c)D(p1)
c
p^m
p2
a
p1

$p_2 < c$



Any price greater than p_2 is a best response to p_2:

$$B_1(p_2) = \{p_1 : p_1 > p_2\}.$$

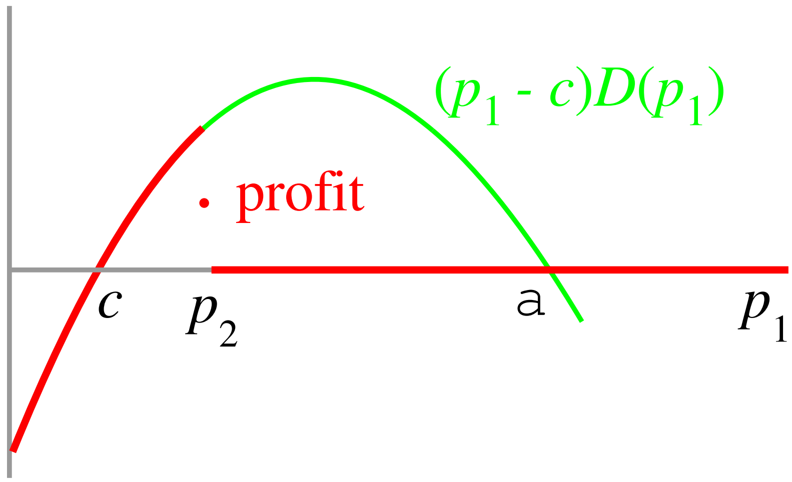
Note: a price between p_2 and c is a best response!

$p_2 = c$

Any price greater than or equal to p_2 is a best response to p_2:

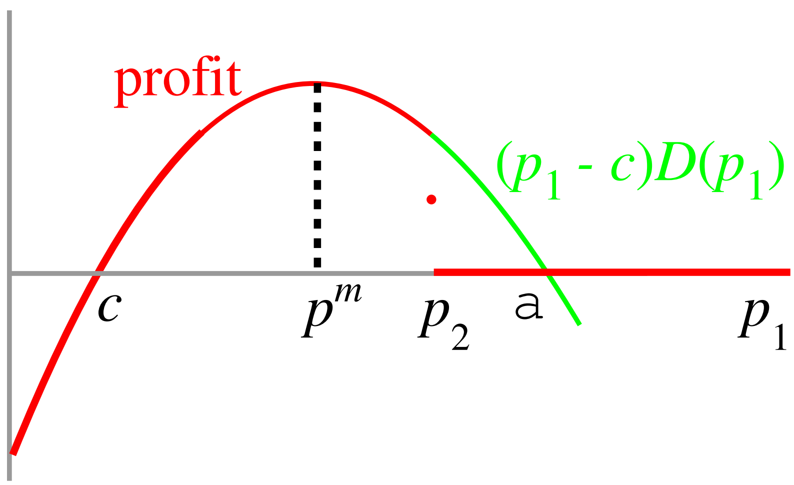
$$B_1(p_2) = \{p_1 : p_1 \geq p_2\}.$$

$c < p_2 \leq p^m$



There is no *best* response! (a bit less than p_2 is *almost* a best response).

$p^m < p_2$

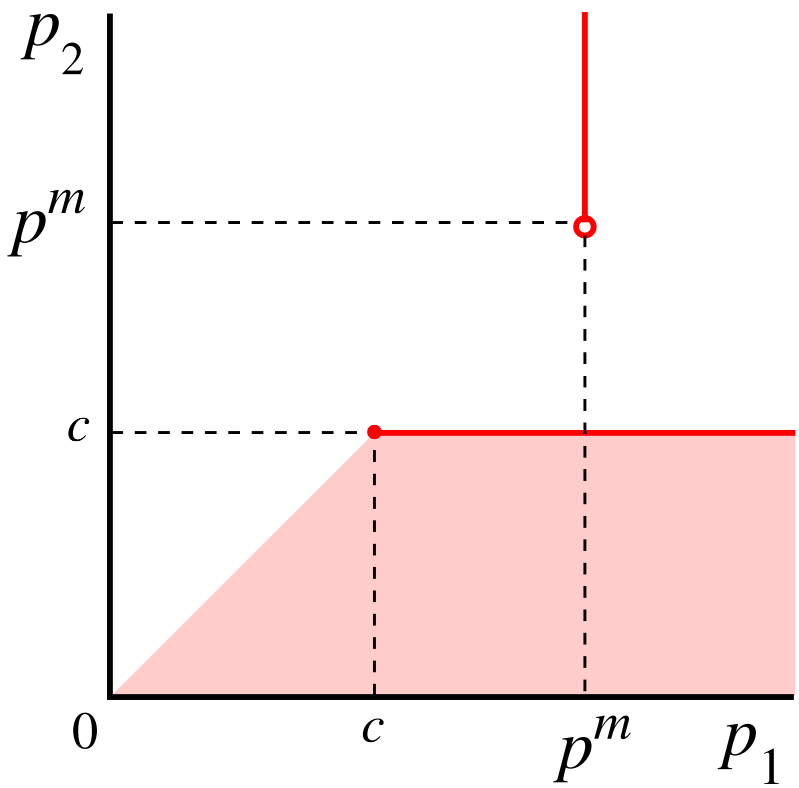


p^m is the unique best response to p_2:

$$B_1(p_2) = \{p^m\}.$$

Summary

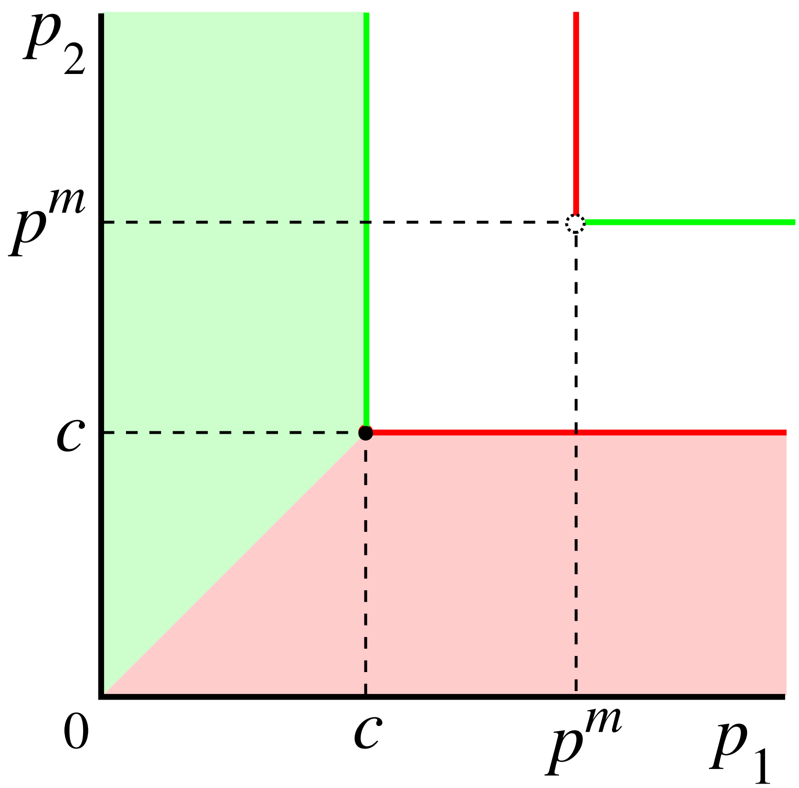
$$B_i(p_j) = \begin{cases} \{p_i : p_i > p_j\} & \text{if } p_j < c \\ \{p_i : p_i \geq p_j\} & \text{if } p_j = c \\ \varnothing & \text{if } c < p_j \leq p^m \\ \{p^m\} & \text{if } p^m < p_j. \end{cases}$$



Nash equilibrium

(p_1^*, p_2^*) such that $p_1^* \in B_1(p_2^*)$ and $p_2^* \in B_2(p_1^*)$

I.e. intersection of the graphs of the best response functions



So: unique Nash equilibrium, $(p_1^*, p_2^*) = (c, c)$.

"Direct" argument for Nash equilibrium

If each firm charges the price of c then the other firm can do no better than charge the price of c also (if it raises its price is sells no output, while if it lowers its price is makes a loss), so (c, c) is a Nash equilibrium.

No other pair (p_1, p_2) is a Nash equilibrium since

- if $p_i < c$ then the firm whose price is lowest (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price to c
- if $p_i = c$ and $p_j > c$ then firm i is better off increasing its price slightly
- if $p_i \geq p_j > c$ then firm i can increase its profit by lowering p_i to some price between c and p_j (e.g. to slightly below p_j if $D(p_j) > 0$ and to p^m if $D(p_j) = 0$).

Note: to show a pair of actions is not a Nash equilibrium we need only find a *better* response for one of the players—not necessarily the *best* response.

Equilibria in Cournot's and Bertrand's models generate different economic outcomes:

- equilibrium price in Bertrand's model is c
- price associated with an equilibrium of Cournot's model is $\frac{1}{3}(\alpha + 2c)$, which exceeds c since $\alpha > c$.

Does one model capture firms' strategic reasoning better than the other?

Bertrand's model: firm changes its behavior if it can increase its profit by changing its price, on the assumption that the other firm will not change its price but the other firm's output will adjust to clear the market.

Cournot's model: firm changes its behavior if it can increase its profit by changing its output, on the assumption that the output of the other firm will not change but the price will adjust to clear the market.

If prices can easily be changed, Cournot's model may thus better capture firms' strategic reasoning.

Hotelling's model of electoral competition

- Several candidates vie for political office
- Each candidate chooses a policy position
- Each citizen, who has preferences over policy positions, votes for one of the candidates
- Candidate who obtains the most votes wins.

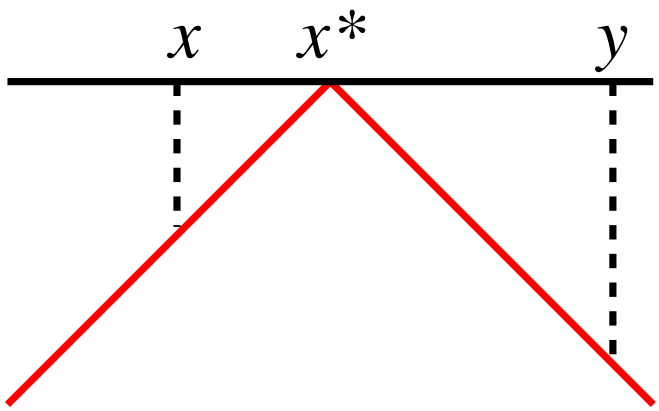
Strategic game:

- Players: candidates
- Set of actions of each candidate: set of possible positions
- Each candidate gets the votes of all citizens who prefer her position to the other candidates' positions; each candidate prefers to win than to tie than to lose.

Note: Citizens are not players in this game.

Example

- Two candidates
- Set of possible positions is a (one-dimensional) interval.
- Each voter has a single favorite position, on each side of which her distaste for other positions increases equally.

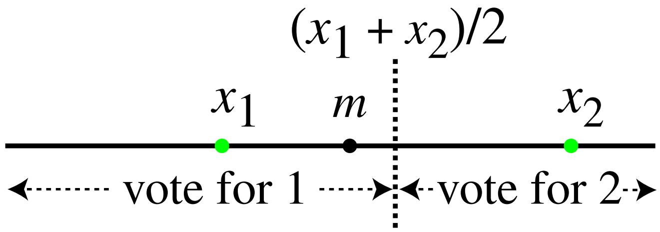


- Unique median favorite position m among the voters: the favorite positions of half of the voters are at most m, and the favorite positions of the other half of the voters are at least m.

Note: m may not be in the center of the policy space.

Positions and votes

Candidate who gets most votes wins.

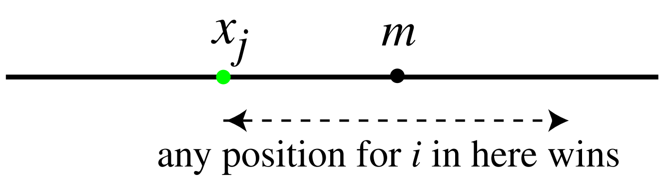


In this case, candidate 1 wins.

Best responses

Best response of candidate i to x_j:

- $x_j < m$:



  candidate i wins if $x_i > x_j$ and $\frac{1}{2}(x_i + x_j) < m$, or in other words $x_j < x_i < 2m - x_j$. Otherwise she either ties or loses. Thus every position between x_j and $2m - x_j$ is a best response of candidate i to x_j.

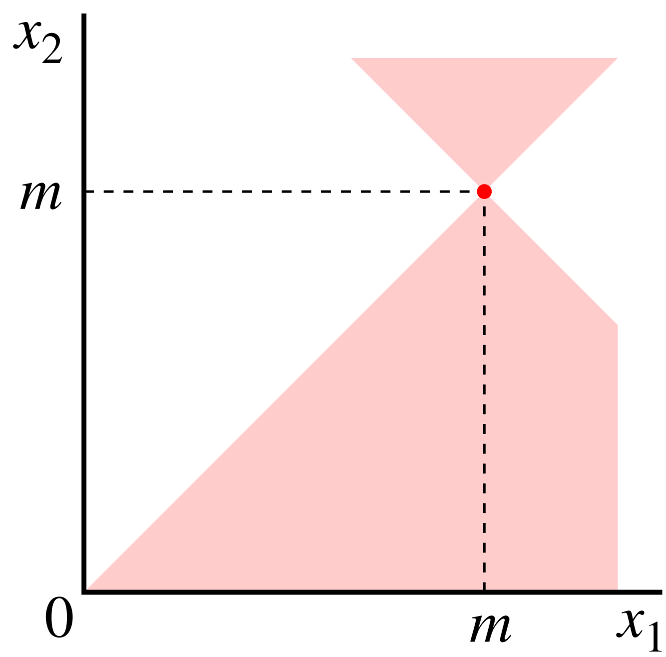
- $x_j > m$: symmetrically, every position between $2m - x_j$ and x_j is a best response of candidate i to x_j.

- $x_j = m$: if candidate i choose $x_i = m$ she ties for first place; if she chooses any other position she loses. Thus m is the unique best response of candidate i to x_j.

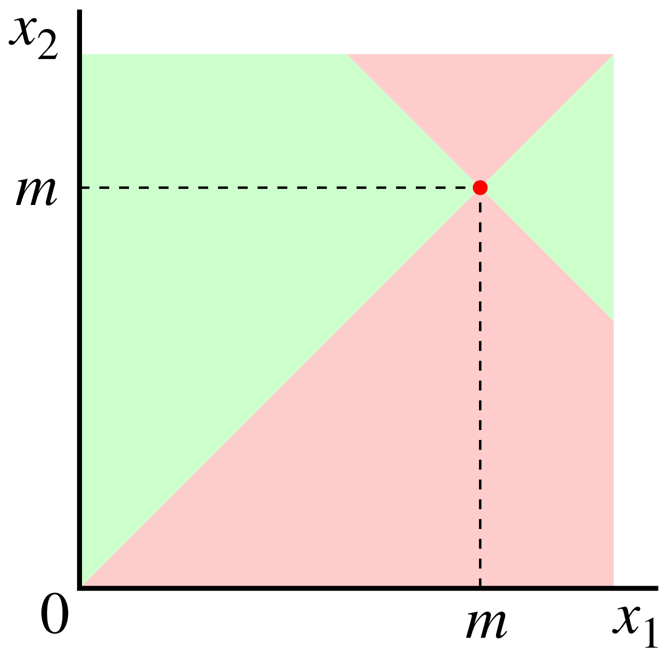
Summary

Candidate i's best response function:

$$B_i(x_j) = \begin{cases} \{x_i \colon x_j < x_i < 2m - x_j\} & \text{if } x_j < m \\ \{m\} & \text{if } x_j = m \\ \{x_i \colon 2m - x_j < x_i < x_j\} & \text{if } x_j > m. \end{cases}$$



Nash equilibrium



Unique Nash equilibrium, in which both candidates choose the position m.

Outcome of election is tie.

Competition between the candidates to secure a majority of the votes drives them to select the same position.

Direct argument for Nash equilibrium

(m, m) is an equilibrium: if either candidate chooses a different position she loses.

No other pair of positions is a Nash equilibrium:

- if one candidate loses then she can do better by moving to m (where she either wins outright or ties for first place)
- if the candidates tie (because their positions are either the same or symmetric about m), then either candidate can do better by moving to m, where she wins outright.

The War of Attrition

- Two parties involved in a costly dispute
- E.g. two animals fighting over prey
- Each animal chooses time at which it intends to give up
- Once an animal has given up, the other obtains all the prey
- If both animals give up at the same time then they split the prey equally.
- Fighting is costly: each animal prefers as short a fight as possible.

Also a model of bargaining between humans.

Let time be a continuous variable that starts at 0 and runs indefinitely.

Assume value to party i of object in dispute is $v_i > 0$; value of half of object is $v_i/2$.

Each unit of time that passes before one of parties concedes costs each party one unit of payoff.

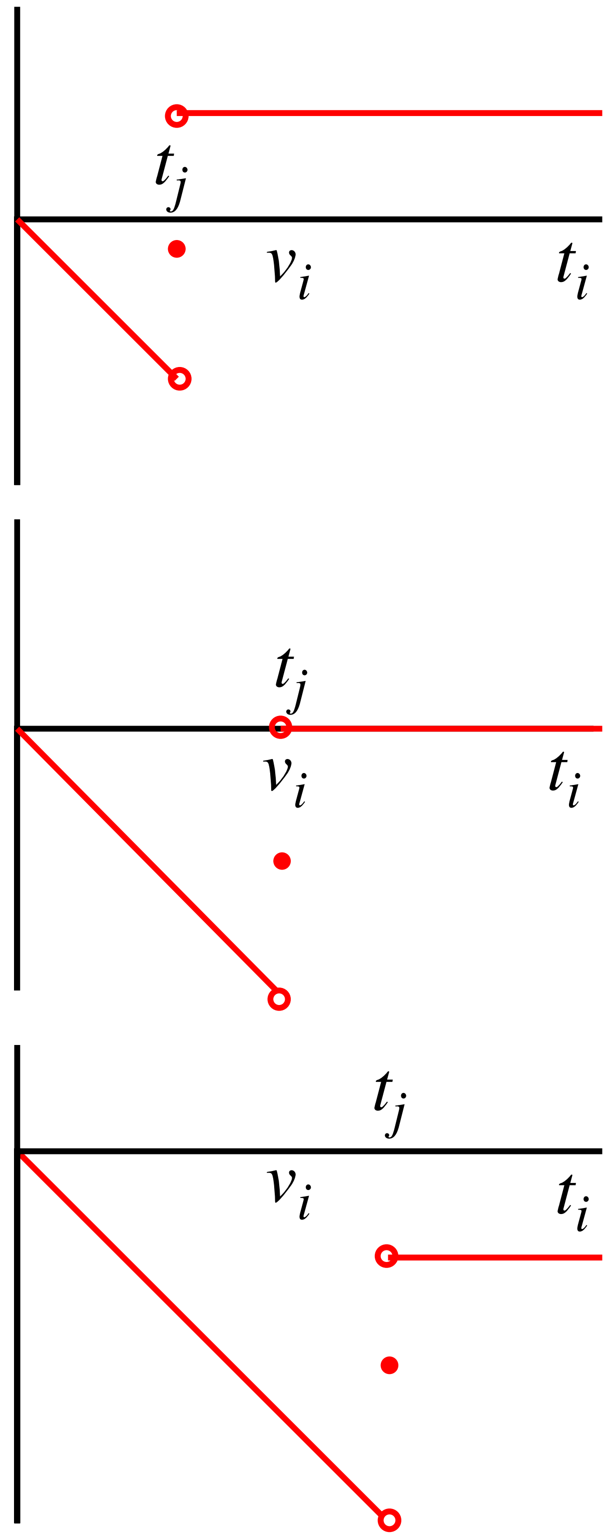
Strategic game

- players: the two parties
- each player's set of actions is $[0, \infty)$ (set of possible concession times)
- player i's preferences are represented by payoff function

$$u_i(t_1, t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j, \end{cases}$$

where j is the other player.

Payoff function:

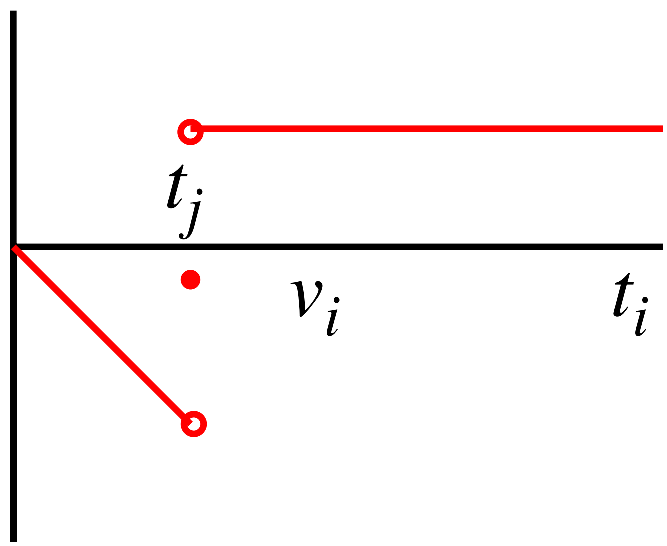


Best responses

Suppose player j concedes at time t_j:

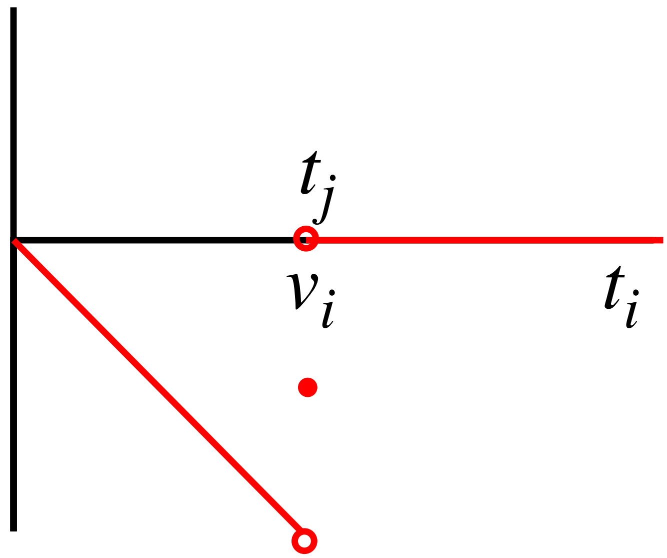
Intuitively: if t_j is small then optimal for player i to wait until after t_j; if t_j is large then player i should concede immediately.

Precisely: if $t_j < v_i$:



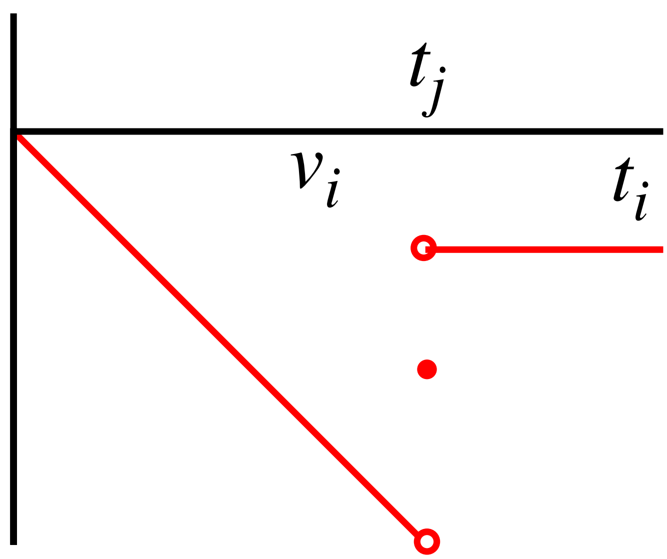
so any time after t_j is a best response to t_j

If $t_j = v_i$:



so conceding at 0 or at any time after t_j is a best response to t_j

If $t_j > v_i$:

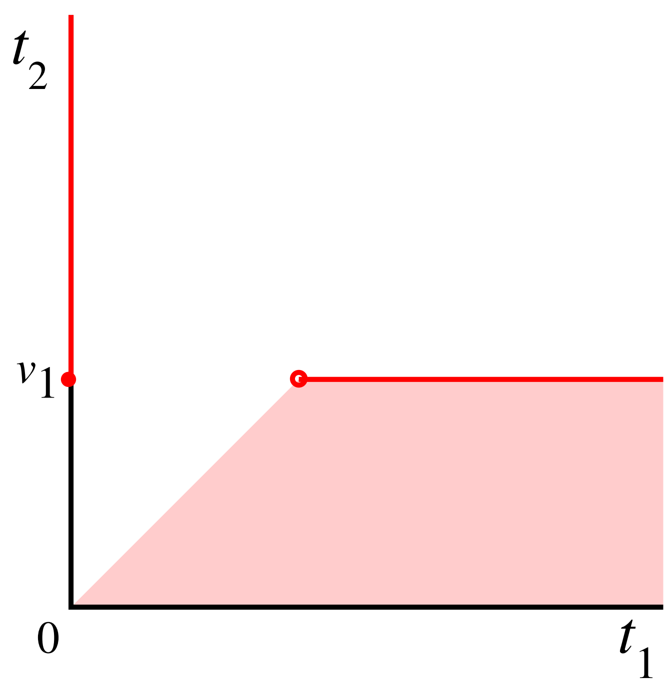


so best time for player i to concede is 0.

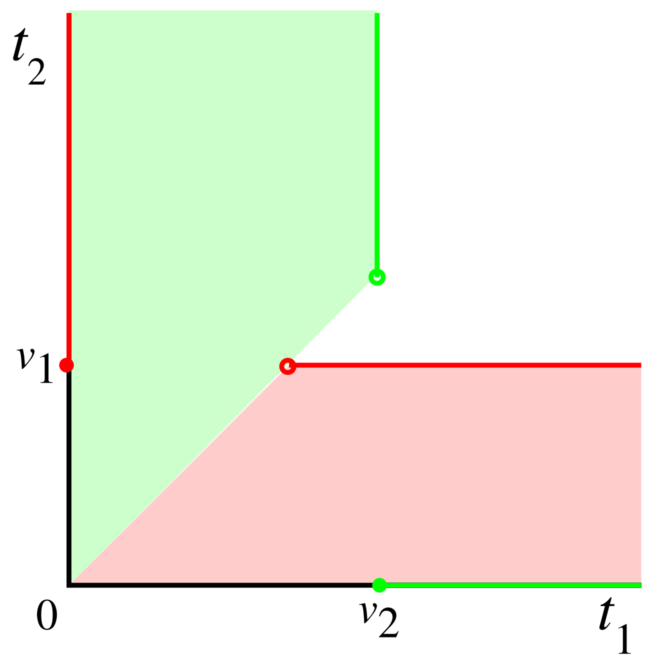
So player i's best response function:

$$B_i(t_j) = \begin{cases} \{t_i : t_i > t_j\} & \text{if } t_j < v_i \\ \{0\} \cup \{t_i : t_i > t_j\} & \text{if } t_j = v_i \\ \{0\} & \text{if } t_j > v_i. \end{cases}$$

Best response function of player 1:



Nash equilibrium



Nash equilibria: (t_1, t_2) such that either

$$t_1 = 0 \text{ and } t_2 \geq v_1$$

or

$$t_2 = 0 \text{ and } t_1 \geq v_2.$$

That is: either player 1 concedes immediately and player 2 concedes at the earliest at time v_1, or player 2 concedes immediately and player 1 concedes at the earliest at time v_2.

Note: in no equilibrium is there any fight

Note: there is an equilibrium in which either player concedes first, regardless of the sizes of the valuations.

Note: equilibria are *asymmetric*, even when $v_1 = v_2$, in which case the game is symmetric.

E.g. could be a stable social norm that the current owner of the object concedes immediately; or that the challenger does so.

Single population case: only symmetric equilibria are relevant, and there are none!

Auctions

Common type of auction:

- people sequentially bid for an object
- each bid must be greater than previous one
- when no one wishes to submit a bid higher than current one, person making current bid obtains object at price she bid.

Assume everyone is certain about her valuation of the object before bidding begins, so that she can learn nothing during the bidding.

Model

- each person decides, before auction begins, maximum amount she is willing to bid
- person who bids most wins
- person who wins pays the *second highest* bid.

Idea: in a dynamic auction, a person wins if she continues bidding after everyone has stopped—in which case she pays a price slightly higher than the price bid by the last person to drop out.

Strategic game:

- players: bidders
- set of actions of each player: set of possible bids (nonnegative numbers)
- preferences of player i: represented by a payoff function that gives player i $v_i - p$ if she wins (where v_i is her valuation and p is the second-highest bid) and 0 otherwise.

This is a *sealed-bid second-price auction*.

How to break ties in bids?

Simple (but arbitrary) rule: number players $1, \ldots, n$ and make the winner the player with the lowest number among those that submit the highest bid.

Assume that $v_1 > v_2 > \cdots > v_n$.

Nash equilibria of second-price sealed-bid auction

One Nash equilibrium

$(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$

Outcome: player 1 obtains the object at price v_2; her payoff is $v_1 - v_2$ and every other player's payoff is zero.

Reason:

- Player 1:
  - if she changes her bid to some $x \geq b_2$ the outcome does not change (remember she pays the *second* highest bid)
  - if she lowers her bid below b_2 she loses and gets a payoff of 0 (instead of $v_1 - b_2 > 0$).
- Players 2, ..., n:
  - if she lowers her bid she still loses
  - if she raises her bid to $x \leq b_1$ she still loses
  - if she raises her bid above b_1 she wins, but gets a payoff $v_i - v_1 < 0$.

Another Nash equilibrium

$(v_1, 0, \ldots, 0)$ is also a Nash equilibrium:

Outcome: player 1 obtains the object at price 0; her payoff is v_1 and every other player's payoff is zero.

Reason:

- Player 1:
  - any change in her bid has no effect on the outcome
- Players 2, ..., n:
  - if she raises her bid to $x \leq v_1$ she still loses
  - if she raises her bid above v_1 she wins, but gets a negative payoff $v_i - v_1$.

Another Nash equilibrium

$(v_2, v_1, 0, \ldots, 0)$ is also a Nash equilibrium:

Outcome: player 2 gets object at price v_2; all payoffs 0.

Reason:

- Player 1:
  - if she raises her bid to $x < v_1$ she still loses
  - if she raises her bid to $x \geq v_1$ she wins, and gets a payoff of 0
- Player 2
  - if she raises her bid or lowers it to $x > v_2$, outcome remains same
  - if she lowers her bid to $x \leq v_2$ she loses and gets 0
- Players 3, ..., n:
  - if she raises her bid to $x \leq v_1$ she still loses
  - if she raises her bid above v_1 she wins, but gets a negative payoff $v_i - v_1$.

Player 2's may seem "risky"—but isn't if the other players adhere to their equilibrium actions.

Nash equilibrium requires only that each player's action be optimal, *given* the other players' actions.

In a dynamic setting, player 2's bid isn't credible (why would she keep bidding above v_2?) [Will study this issue later.]

Distinguishing between equilibria

For each player i the action v_i *weakly dominates* all her other actions

That is: player i can do no better than bid v_i *no matter what the other players bid*.

Argument:

- If the highest of the other players' bids is at least v_i, then
  - if player i bids v_i her payoff is 0
  - if player i bids $x \neq v_i$ her payoff is either zero or negative.
- If the highest of the other players' bids is $\overline{b} < v_i$, then
  - if player i bids v_i her payoff is $v_i - \overline{b}$ (she obtains the object at the price $\overline{b}$)
  - if player i submits some other bid then she either obtains the good and gets the same payoff, or does not obtain the good and gets the payoff of zero.

Summary

Second-price auction has many Nash equilibria, but the equilibrium $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ is the only one in which every players' action weakly dominates all her other actions.

First-price auction

Another auction form:

- auctioneer begins by announcing a high price
- price is gradually lowerered until someone indicates a willingness to buy the object at that price.

Model

Strategic game:

- players: bidders
- actions of each player: set of possible bids (nonnegative numbers)
- preferences of player i: represented by a payoff function that gives player i $v_i - p$ if she wins (where v_i is her valuation and p is her bid) and 0 otherwise.

This is a *first-price sealed-bid auction*.

One Nash equilibrium

$(v_2, v_2, v_3, \ldots, v_n)$

Outcome: player 1 obtains the object at the price v_2.

Why is this a Nash equilibrium?

Property of all equilibria

In all equilibria the object is obtained by the player who values it most highly (player 1)

Argument:

- If player $i \neq 1$ obtains the object then we must have $b_i > b_1$.
- But there is no equilibrium in which $b_i > b_1$:
  - if $b_i > v_2$ then i's payoff is negative, so she can do better by reducing her bid to 0
  - if $b_i \leq v_2$ then player 1 can increase her payoff from 0 to $v_1 - b_i$ by bidding b_i.

Another equilibrium

$(v_1, v_1, v_3, \ldots, v_n)$

Outcome: player 1 obtains the object at the price v_1.

As before, player 2's action may seem "risky": if there is any chance that player 1 submits a bid less than v_1 then there is a chance that player 2's payoff is negative.

Domination

As in a second-price auction, any player i's action of bidding $b_i > v_i$ is weakly dominated by the action of bidding v_i:

- if the other players' bids are such that player i loses when she bids b_i, then it makes no difference to her whether she bids b_i or v_i
- if the other players' bids are such that player i wins when she bids b_i, then she gets a negative payoff bidding b_i and a payoff of 0 when she bids v_i.

However, in a first-price auction, unlike a second-price auction, a bid by a player *less than* her valuation is *not* weakly dominated.

Reason: if player i bids $v_i' < v_i$ and the highest bid of the other players is $< v_i'$, then player i is better off than she is if she bids v_i.

Revenue equivalence

The price at which the object is sold, and hence the auctioneer's revenue, is the same in the equilibrium $(v_1, \ldots, v_n)$ of the second-price auction as it is in the equilibrium $(v_2, v_2, v_3, \ldots, v_n)$ of the first-price auction.