Econ 300, Problem Set 5, Suggested Answers
Professor Cramton

10.1.8.

First we write the firm's profit function using the factor and output price information as

$$\begin{aligned}\Pi(K, L) &= 4 * 9L^{1/3}K^{1/3} - 12L - 6K \\ &= 36L^{1/3}K^{1/3} - 12L - 6K\end{aligned}$$

To find the optimal levels for each input, we write the first order conditions:

$$\begin{aligned}\Pi_K &= 12L^{1/3}K^{-2/3} - 6 = 0, \\ \Pi_L &= 12L^{-2/3}K^{1/3} - 12 = 0.\end{aligned}$$

Solving the second equation for K yields $K = L^2$. This can be substituted into the first equation:

$$2L^{1/3}(L2)^{-2/3} = 1,$$

which can be simplified to $2L^{-1} = 1$, or $L = 2$. Plugging this back into $K = L^2$ yields $K = 4$. Thus, the stationary point of the profit function is $(K, L) = (4, 2)$. To check that this stationary point is a maximum, we compute the second order derivatives of the profit function and evaluate them at the stationary point:

$$\begin{aligned}\Pi_{KK} &= -8L^{1/3}K^{-5/3} = -8 * 2^{1/3}4^{-5/3} = -1 < 0, \\ \Pi_{LL} &= -8L^{-5/3}K^{1/3} = -8 * 2^{-5/3}4^{1/3} = -4 < 0.\end{aligned}$$

We also calculate the cross partial derivative

$$\Pi_{KL} = 4L^{-2/3}K^{-2/3} = -8 * 2^{-2/3}4^{-2/3} = 1,$$

from which we see that $\Pi_{KK}\Pi_{LL} = 4 > 1 = (\Pi_{KL})^2$. We can thus conclude that the point $(K^*, L^*) = (4, 2)$ is a maximum.

10.1.10.

(a) With linear total costs, the profit function of AWL is given by

$$\begin{aligned}\Pi(Q_T, Q_M) &= (\alpha - \beta Q_T)Q_T + (\gamma - \theta Q_M)Q_M - \Psi - c(Q_T + Q_M) \\ &= -\beta Q_T^2 - \theta Q_M^2 + (\alpha - c)Q_T + (\gamma - c)Q_M - \Psi\end{aligned}$$

The first order conditions of the problem consisting of maximizing profits with respect to Q_T and Q_M are

$$\begin{aligned}\Pi_{Q_T} &= -2\beta Q_T + (\alpha - c) = 0, \\ \Pi_{Q_M} &= -2\theta Q_M + (\gamma - c) = 0.\end{aligned}$$

The first equation leads to $Q_T = (\alpha - c)/2\beta$ and the second equation leads to $Q_M = (\gamma - c)/2\theta$. These are the optimal quantities to sell in Massachusetts and in Texas (we don't need to check the second order conditions if we recognize that the profit function is a globally concave quadratic function). The profits are then given by

$$\begin{aligned}\Pi(Q_T, Q_M) &= -\frac{(\alpha-c)^2}{4\beta} - \frac{(\gamma-c)^2}{4\theta} + (\alpha - c)\frac{(\alpha-c)}{2\beta} + (\gamma - c)\frac{(\gamma-c)}{2\theta} - \Psi \\ &= \frac{(\alpha-c)^2}{4\beta} + \frac{(\gamma-c)^2}{4\theta} - \Psi\end{aligned}$$

(b) With quadratic total costs, the profit function of AWL is given by

$$\begin{aligned}\Pi(Q_T + Q_M) &= (\alpha - \beta Q_T)Q_T + (\gamma - \theta Q_M)Q_M - \Psi - c(Q_T + Q_M)^2 \\ &= -(\beta + c)Q_T^2 - (\theta + c)Q_M^2 - 2cQ_M Q_T + \alpha Q_T + \gamma Q_M - \Psi\end{aligned}$$

The first order conditions of the maximization problem are

$$\begin{aligned}\Pi_{Q_T} &= -2(\beta + c)Q_T - 2cQ_M + \alpha = 0, \\ \Pi_{Q_M} &= -2(\theta + c)Q_M - 2cQ_T + \gamma = 0.\end{aligned}$$

Solving these two equations for the two unknowns Q_T and Q_M yields the optimal quantities

$$\begin{aligned}Q_T &= \frac{(\theta + c)\alpha c^{-1} - \gamma}{2(\theta + c)(\beta + c)c^{-1} - 2c}, \\ Q_M &= \frac{(\beta + c)\gamma c^{-1} - \alpha}{2(\theta + c)(\beta + c)c^{-1} - 2c}.\end{aligned}$$

Again, we don't need to check the second order condition since the profit function is a globally concave quadratic function.

(c) With linear costs and a single overall market, the profit function of AWL is given by

$$\begin{aligned}\Pi(Q) &= \left[\left(\frac{\alpha\theta + \gamma\beta}{\beta + \theta}\right) - \left(\frac{\beta\theta}{\beta + \theta}\right)Q\right]Q - \Psi - cQ \\ &= -\left(\frac{\beta\theta}{\beta + \theta}\right)Q^2 + \left[\left(\frac{\alpha\theta + \gamma\beta}{\beta + \theta}\right) - c\right]Q - \Psi\end{aligned}$$

The first order condition of the maximization problem is

$$\Pi'(Q) = -2\left(\frac{\beta\theta}{\beta + \theta}\right)Q + \left[\left(\frac{\alpha\theta + \gamma\beta}{\beta + \theta}\right) - c\right] = 0$$

Solving this equation for Q leads to the optimal quantity

$$Q = \frac{\left(\frac{\alpha\theta + \gamma\beta}{\beta + \theta}\right) - c}{2\left(\frac{\beta\theta}{\beta + \theta}\right)} = \frac{\alpha\theta + \gamma\beta - c(\beta + \theta)}{2\beta\theta}.$$

Here too, because the profit function is a globally concave function, we can claim to have found the maximum without having to check the second order conditions.

11.1.2.

The problem is to minimize the objective function

$$C = 3x^2 - 4xz + 9z^2 - 8z + 36,$$

subject to the constraint x + z = 12. To use the substitution method, write the constraint as x = 12 - z, and plug it into the objective function to get:

$$\begin{aligned} C &= 3(12-z)^2 - 4(12-z)z + 9z^2 - 8z + 36 \\ &= 3 * 12^2 - 72z + 3z^2 - 48z + 4z^2 + 9z^2 - 8z + 36 \\ &= 16z^2 - 128z + 468 \end{aligned}$$

The first order condition of this univariate minimization problem is

$$C' = 32z - 128 = 0,$$

which can be solved for the optimal z to be $z = 4$. Substituting this value back into the constraint, we get $x = 8$. The combination of crops that minimizes the cost of fulfilling the contract is thus $(x, z) = (8, 4)$.

11.2.6.

(a) The Lagrangian function is

$$\mathcal{L} = AK^\alpha L^{1-\alpha} - \lambda(wL + rK - \bar{C}).$$

The first order conditions of the problem are

$$\begin{aligned} \mathcal{L}_L &= A(1-\alpha)K^\alpha L^{-\alpha} - \lambda w = 0, \\ \mathcal{L}_K &= A\alpha K^{\alpha-1} L^{1-\alpha} - \lambda r = 0, \\ \mathcal{L}_\lambda &= wL + rK - \bar{C} = 0. \end{aligned}$$

Now, solving each of the first two first order condition for λ, and setting them equal to each other yields

$$\frac{A(1-\alpha)K^\alpha L^{-\alpha}}{w} = \frac{A\alpha K^{\alpha-1} L^{1-\alpha}}{r}. \tag{1}$$

Solving this new equation for L yields

$$L = \frac{(1-\alpha)rK}{\alpha w}.$$

This can be substituted into the third first order condition (which is in fact the original constraint) to get

$$K = \frac{\alpha \bar{C}}{r}.$$

Combining this optimal level of capital with the budget constraint yields the optimal level of labor

$$L = \frac{(1-\alpha)\bar{C}}{w}.$$

(b) The Lagrangian function is

$$\mathcal{L} = wL + rK - \lambda(\bar{Q} - AK^{\alpha}L^{-\alpha}).$$

The first order conditions of the problem are

$$\begin{aligned}
\mathcal{L}_L &= w + \lambda A(1-\alpha)K^{\alpha}L^{-\alpha} = 0, \\
\mathcal{L}_K &= r + \lambda AK^{\alpha-1}L^{1-\alpha} = 0, \\
\mathcal{L}_\lambda &= \bar{Q} - AK^{\alpha}L^{1-\alpha} = 0.
\end{aligned}$$

As under (a), we can combine the first two equations to eliminate λ,

$$-\frac{w}{A(1-\alpha)K^{\alpha}L^{-\alpha}} = -\frac{r}{A\alpha K^{\alpha-1}L^{1-\alpha}}.$$

As this is essentially the same equation as (1), it should be clear that it also leads to

$$L = \frac{(1-\alpha)rK}{\alpha w},$$

which we here also substitute into the third first order condition (which is the constraint). We get

$$K = \frac{\bar{Q}}{A}\left(\frac{\alpha w}{(1-\alpha)r}\right)^{1-\alpha}$$

Combining this optimal level of capital with the previous equation yields the optimal level of labor

$$L = \frac{\bar{Q}}{A}\left(\frac{\alpha w}{(1-\alpha)r}\right)^{-\alpha}.$$