Game Theory

Professor Peter Cramton

Economics 300

Definition

Game theory is the study of mathematical models of conflict and cooperation between *intelligent and rational* decision makers.

- *Rational:* each individual maximizes her expected utility
- *Intelligent:* individual understands situation, including fact that others are intelligent rational decision makers

Game Theory

- Game theory lets us study multi-person decision problems
- Use game theory to model:
  - Trading process (auctions, bargaining, markets)
  - Competition among firms
  - Sporting events
  - Voting
  - Military decisions
  - Competition or collusion among countries in choosing tariffs, trade policies, environmental standards, etc.

What is a game?

- A set of rules specifying:
  - Players
  - Alternatives (actions players choose from)
  - Order of play
  - Outcomes and payoffs

Normal form of a game

- Players: $I = \{1, 2, \ldots, N\}$
- Action sets: $A_1, A_2, \ldots, A_N$
- Players simultaneously choose their actions $a_1, a_2, \ldots, a_N$
- Payoffs are realized $U_i(a_1, a_2, \ldots, a_N)$ for each player i

Example: The Prisoner’s Dilemma

- Two suspects are arrested by the police
- If both stay mum they get a light 1-year sentence
- If one finks and other stays mum, the finker goes free and the other gets 9 years
- If both fink on the other, they both get 6 years
- What should the prisoner’s do?

Example: The Prisoner's Dilemma

- Players: *Prisoner 1* and *Prisoner 2*
- Actions: $A_1 = A_2 = \{\text{Mum, Fink}\}$
- Payoffs:

$$u_1(Mum, Mum) = -1 = u_2(Mum, Mum)$$

$$u_1(Fink, Fink) = -6 = u_2(Fink, Fink)$$

$$u_1(Mum, Fink) = -9 = u_2(Fink, Mum)$$

$$u_1(Fink, Mum) = 0 = u_2(Mum, Fink)$$

Example: The Prisoner's Dilemma

		P2	
		Mum	Fink
P1	Mum	-1, -1	-9, 0
	Fink	0, -9	-6, -6

Solution Concepts

	Mum	Fink
Mum	-1, -1	-9, 0
Fink	0, -9	-6, -6

- **Best Response Mapping:** Each player wants to make an optimal (i.e., the best) action for himself, given actions of others.
- **Example: Prisoner's Dilemma**

$$BR_1(Mum) = Fink \text{ (since } 0 > -1)$$

$$BR_1(Fink) = Fink \text{ (since } -6 > -9)$$

Fink is the dominant strategy for prisoner 1.

Definition: A dominant strategy is the best choice for a player regardless of what the others are doing (i.e., Best Response is always the same).

If each player has a dominant strategy, then we call this strategy profile a dominant strategy equilibrium (DSE).

The Prisoner's Dilemma

	Mum	Fink
Mum	-1, -1	-9, 0
Fink	0, -9	-6, -6

$$BR_1(Mum) = Fink \text{ (since } 0 > -1)$$

$$BR_1(Fink) = Fink \text{ (since } -6 > -9)$$

$$BR_2(Mum) = Fink \text{ (since } 0 > -1)$$

$$BR_2(Fink) = Fink \text{ (since } -6 > -9)$$

Hence, (Fink, Fink) is a unique DSE

Example Two

P2

		L	M	R
P1	U	1, 0	1, 2	3, 1
	D	0, -9	0, 3	1, 2

$$\left.\begin{array}{l} BR_1(L) = U \\ BR_1(M) = U \\ BR_1(R) = U \end{array}\right\} \Rightarrow U \text{ is dominant strategy for P1}$$

$$\left.\begin{array}{l} BR_2(U) = M \;\; (2 > 0 \text{ and } 2 > 1) \\ BR_2(D) = M \;\; (3 > -9 \text{ and } 3 > 2) \end{array}\right\} \Rightarrow \text{ M is dominant strategy for P2}$$

$$\Rightarrow (U, M) \text{ is DSE.}$$

Example:

		P2 L	P2 M	P2 R
P1	U	1, 0	1, 2	0, 1
P1	D	0, 3	0, 1	2, 0

$BR_1(L) = U$ $\quad BR_2(U) = M$

$BR_1(M) = U$ $\quad BR_2(D) = L$

$BR_1(R) = D$

$\Rightarrow$ No dominant strategy

$\Rightarrow$ No DSE

Solution Concept 2: Iterated Elimination of Never a Best Response Strategies

- Rational players do not play strategies that are never a best response
- So remove strategies that are never a best response

Example:

		P2: L	P2: M	P2: R
P1	U	1, 0	1, 2	0, 1
	D	0, 3	0, 1	2, 0

$BR_1(L) = U$ $BR_2(U) = M$

$BR_1(M) = U$ $BR_2(D) = L$

$BR_1(R) = D$

$\Rightarrow$ For P2, R is never a best response, eliminate R

$BR_1(L) = U$

$BR_1(M) = U$

$\Rightarrow$ For P1, D is never a best response, eliminate D

$BR_2(U) = M$

$\Rightarrow$ Hence, (U,M) is the Iterated Elimination of Never a Best Response Strategies

Example: Battle of the Sexes

		W: Football	W: Ballet
M	Football	2, 1	0, 0
	Ballet	0, 0	1, 2

$$BR_M(B) = B \ (1 > 0) \qquad BR_W(B) = B \ (2 > 0)$$

$$BR_M(F) = F \ (2 > 0) \qquad BR_W(F) = F \ (1 > 0)$$

$\Rightarrow$ No dominant strategy & No IENS

Solution Concept 3: Nash Equilibrium (Pure Strategies)

Nash Equilibrium is a set of mutual best responses: Each player is playing a best response to what the others are doing.

With two players:

(a, b) is NE if $BR_1(b)=a$ and $BR_2(a)=b$

Example: Battle of the Sexes

		W: Football	W: Ballet
M	Football	2, 1	0, 0
M	Ballet	0, 0	1, 2

$$BR_M(B) = B \ (1 > 0) \qquad BR_W(B) = B \ (2 > 0)$$

$$BR_M(F) = F \ (2 > 0) \qquad BR_W(F) = F \ (1 > 0)$$

$\Rightarrow$ (F,F) and (B,B) are NE

Solution Concept 3:
Nash Eq. in Pure Strategies

P2

		L	M	R
P1	a	1, 0	2, 1	3, 2
	b	2, 1	0, 0	2, 0

$$BR_1(L) = b \ (2 > 1)$$

$$BR_1(M) = a \ (2 > 0)$$

$$BR_1(R) = a \ (3 > 2)$$

$$BR_2(a) = R$$

$$BR_2(b) = L$$

$\Rightarrow$ (a, R) and (b, L) are Nash Eq.

Nash Eq. in pure strategies don't always exist

Example: Matching Pennies

		P2: H	P2: T
P1	H	-1, 1	1, -1
P1	T	1, -1	-1, 1

$$BR_1(H) = T$$

$$BR_1(T) = H$$

$$BR_2(H) = H$$

$$BR_2(T) = T$$

$\Rightarrow$ No Nash Eq. in pure strategies

Solution Concept:
Nash Eq in Mixed Strategies

Example: Matching Pennies

P2

		H (r)	T (1-r)
P1	H (q)	-1, 1	1, -1
	T (1-q)	1, -1	-1, 1

A <u>mixed strategy</u> is a probability distribution over a player's pure strategies.

Solution Concept: Nash Eq in Mixed Strategies

Example: Matching Pennies

P2

P1	H (r)	T (1-r)
H (q)	-1, 1	1, -1
T (1-q)	1, -1	-1, 1

P1 plays H with probability q

T with probability 1-q

P2 plays H with probability r

T with probability 1-r, where r, q $\in$ [0,1]

Why mixed strategies?

- Mathematical point of view
  - Needed if pure strategy equilibria do not exist
  - May coexist with pure strategy NE
- Practical point of view
  - Be unpredictable
    - Tennis service
    - Penalty kick in football
    - Poker
    - War

Defining the expected payoffs from mixed strategy play for P1

P1 (rows), P2 (columns)

	H (r)	T (1-r)	EU_1
H	-1, 1	1, -1	r(-1)+(1-r)1
T	1, -1	-1, 1	r(1)+(1-r)(-1)

$$EU_1\left(H,\left(r,1-r\right)\right)=r\left(-1\right)+\left(1-r\right)1=1-2r$$

$$EU_1\left(T,\left(r,1-r\right)\right)=r\left(1\right)+\left(1-r\right)\left(-1\right)=2r-1$$

P1 will only randomize if indifferent!

$$1-2r=2r-1$$

$$\boxed{r=1/2}$$

Defining the expected payoffs from mixed strategy play for P2

		P2 H	P2 T
P1	H (q)	-1, 1	1, -1
	T (1-q)	1, -1	-1, 1

EU_2: q(1)+(1-q)(-1)   q(-1)+(1-q)(1)

$$EU_2\left((q,1-q),H\right)=q\left(1\right)+\left(1-q\right)(-1)=2q-1$$

$$EU_2\left((q,1-q),T\right)=q\left(-1\right)+\left(1-q\right)\left(1\right)=1-2q$$

P2 will only randomize if indifferent!

$$2q-1=1-2q$$

$$\boxed{q=1/2}$$

Each player randomizes to make other indifferent

P2

		H (1/2)	T (1/2)	EU_1
P1	H 1/2	-1, 1	1, -1	0
				=
	T 1/2	1, -1	-1, 1	0
	EU_2:	0	= 0	

Nash Existence Theorem (Nash, 1950)

- **Theorem**: Every finite game has at least one Nash equilibrium (when mixed strategies are permitted).

- Remark: If, in a mixed-strategy equilibrium, player *i* places positive probability on each of two strategies then player *i* must be indifferent between these two strategies – i.e., they yield player *i* the same expected payoff.

Mixed strategy in battle of the sexes

		P2: F (r)	P2: B (1-r)	EU_1
P1	F	2, 1	0, 0	r(2)+(1-r)0
	B	0, 0	1, 2	r(0)+(1-r)(1)

$$EU_1\left(F,\left(r,1-r\right)\right)=r\left(2\right)+\left(1-r\right)0=2r$$

$$EU_1\left(B,\left(r,1-r\right)\right)=r\left(0\right)+\left(1-r\right)\left(1\right)=1-r$$

P1 will only randomize if indifferent!

$$2r=1-r$$

$$\boxed{r=1/3}$$

Mixed strategy in battle of the sexes

		P2 F (r)	P2 B (1-r)
P1	F (q)	2, 1	0, 0
	B (1-q)	0, 0	1, 2

EU_2: q(1)+(1-q)(0) q(0)+(1-q)(2)

$$EU_2\left((q,1-q),F\right)=q\left(1\right)+\left(1-q\right)(0)=q$$

$$EU_2\left((q,1-q),B\right)=q\left(0\right)+\left(1-q\right)\left(2\right)=2-2q$$

P2 will only randomize if indifferent!

$$q=2-2q$$

$$\boxed{q=2/3}$$

Each player randomizes to make other indifferent

P2

P1	F (1/3)	B (2/3)	EU_1
F (2/3)	2, 1	0, 0	2/3
			=
B (1/3)	0, 0	1, 2	2/3
EU_2:	2/3	=	2/3