# Extreme Values of Multivariate Functions

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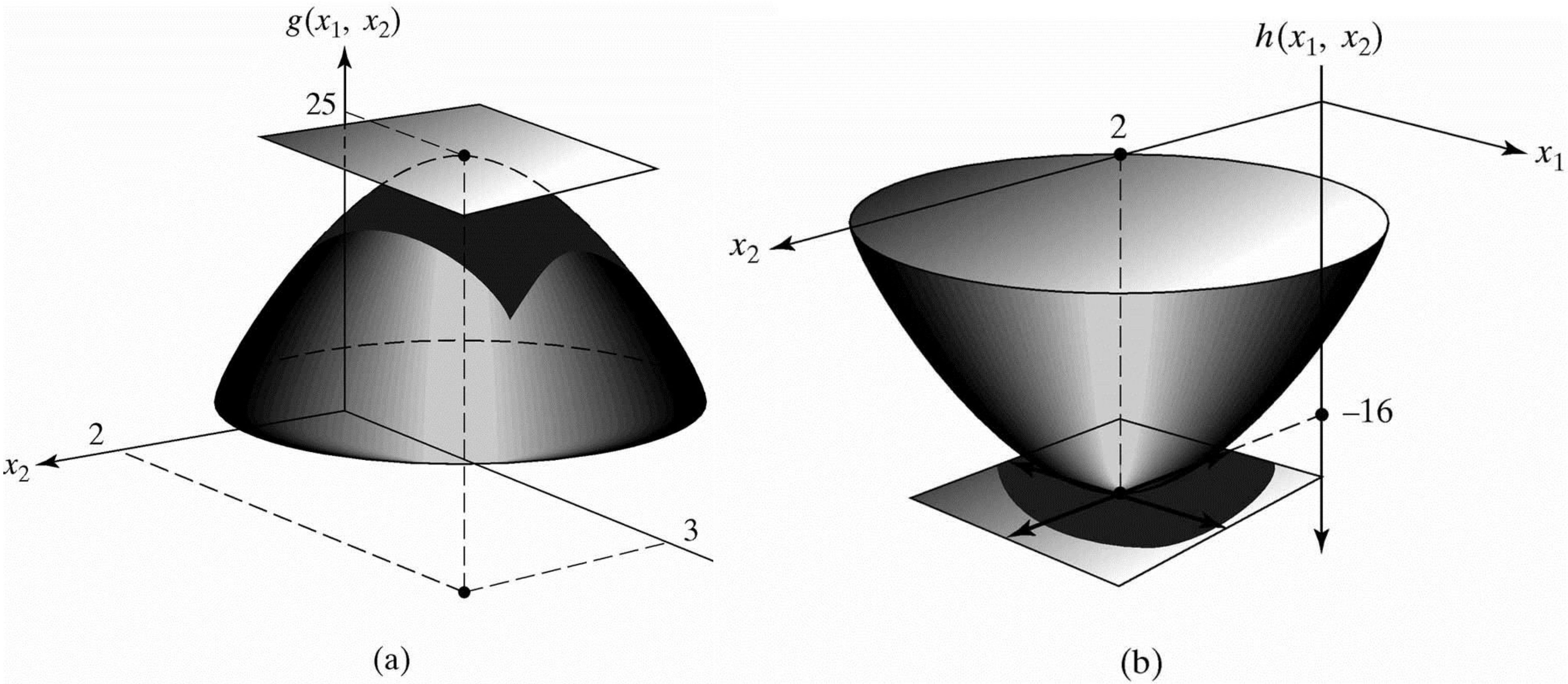
Economics 300

# Extreme values of multivariate functions

- In economics many problems reflect a need to choose among multiple alternatives
  - Consumers decide on consumption bundles
  - Producers choose a set of inputs
  - Policy-makers may choose several instruments to motivate behavior
- We now generalize the univariate techniques

# Stationary points and tangent planes of bivariate functions

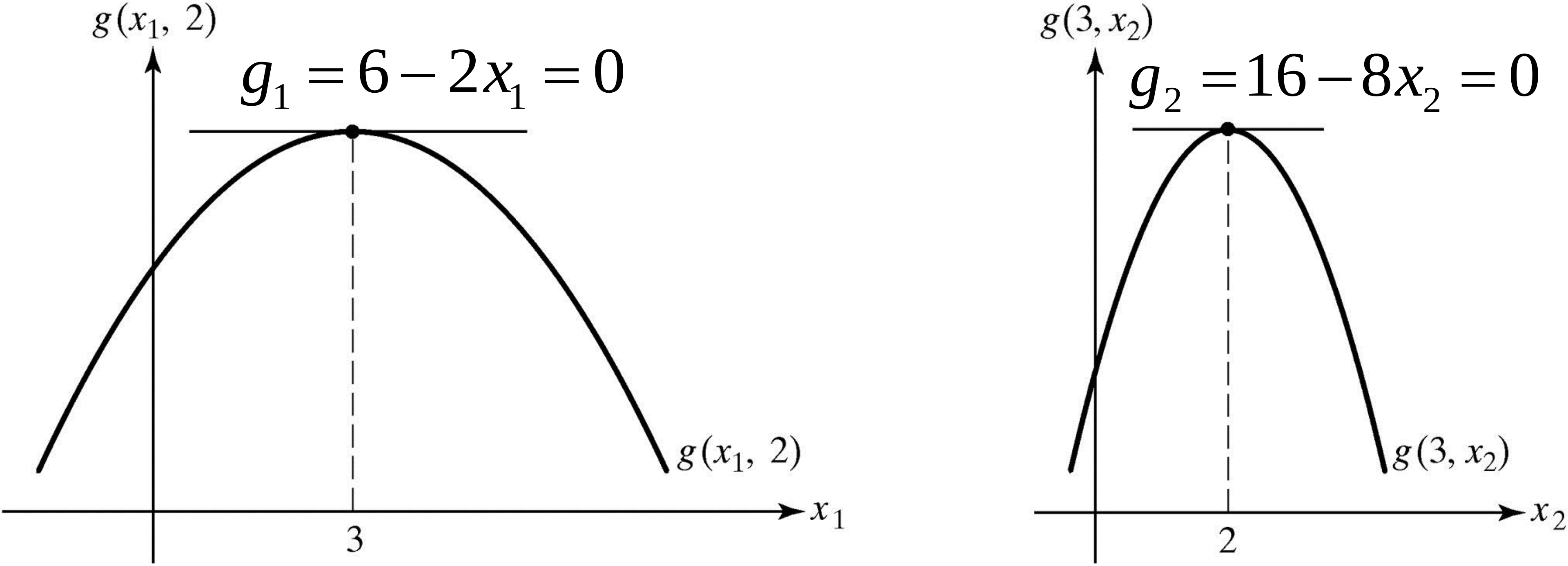



$$g = 6x_1 - x_1^2 + 16x_2 - 4x_2^2$$

$$h = x_1^2 + 4x_2^2 - 2x_1 - 16x_2 + x_1x_2$$

# Slices of a bivariate function

$$g = 6x_1 - x_1^2 + 16x_2 - 4x_2^2$$




(a) (b)

# Multivariate first-order condition

- If $f(x_1, x_2, ..., x_n)$ is differentiable with respect to each of its arguments and reaches a maximum or a minimum at the stationary point, $(x_1^*, ..., x_n^*)$ then each of the partial derivatives evaluated at that point equals zero, i.e.

$$f_1(x_1^*, ..., x_n^*) = 0$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$f_n(x_1^*, ..., x_n^*) = 0$$

# Second-order condition in the bivariate case $f(x_1, x_2)$

**First total differential**

$$y = f(x_1, x_2)$$

$$dy = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$$

i.e.

$$dy = f_1 dx_1 + f_2 dx_2$$

# Second-order condition in the bivariate case $f(x_1, x_2)$

**Second total differential**

$$d^2 y = \frac{\partial[dy]}{\partial x_1} dx_1 + \frac{\partial[dy]}{\partial x_2} dx_2$$

$$= \frac{\partial[f_1 dx_1 + f_2 dx_2]}{\partial x_1} dx_1 + \frac{\partial[f_1 dx_1 + f_2 dx_2]}{\partial x_2} dx_2$$

$$= f_{11} \cdot (dx_1)^2 + 2 f_{12} \cdot (dx_1 \cdot dx_2) + f_{22} \cdot (dx_2)^2$$

# Extreme values and multivariate functions

**Sufficient condition for a local maximum (minimum)**

- If the second total derivative evaluated at a stationary point of a function $f(x_1,x_2)$ is negative (positive) for any $dx_1$ and $dx_2$, then that stationary point represents a local maximum (minimum) of the function

# Extreme values and multivariate functions

**<u>Sufficient Condition for a Local Minimum</u>:**

$$d^2y > 0 \text{ if } f_{11} > 0 \text{ and } f_{22} - \frac{(f_{12})^2}{f_{11}} > 0$$

**<u>Sufficient Condition for a Local Minimum</u>:**

$$d^2y > 0 \text{ if } f_{11} > 0 \text{ and } f_{11} f_{22} > f_{12}^2$$

# Extreme values and multivariate functions

**<u>Sufficient Condition for a Local Maximum</u>:**

$$d^2y < 0 \text{ if } f_{11} < 0 \text{ and } f_{22} - \frac{(f_{12})^2}{f_{11}} < 0$$

**<u>Sufficient Condition for a Local Maximum</u>:**

$$d^2y < 0 \text{ if } f_{11} < 0 \text{ and } f_{11}f_{22} > f_{12}^2$$

# Extreme values of multivariate functions – bivariate case

- Choose $(x_1,x_2)$ to maximize (or to minimize) $f(x_1,x_2)$

# First Order Conditions:

$f_1(x_1,x_2)=0$ and $f_2(x_1,x_2)=0$

stationary points

$$(x_1^*, x_2^*)$$

# Second Order Conditions

Local Minimum if

$$f_{11}(x_1^*, x_2^*) > 0$$

*and*

$$f_{11}(x_1^*, x_2^*)\, f_{22}(x_1^*, x_2^*) > \left( f_{12}(x_1^*, x_2^*) \right)^2$$

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Local Maximum if

$$f_{11}(x_1^*, x_2^*) < 0$$

*and*

$$f_{11}(x_1^*, x_2^*)\, f_{22}(x_1^*, x_2^*) > \left( f_{12}(x_1^*, x_2^*) \right)^2$$

# Exercises

- Choose $(x_1,x_2)$ to minimize

$$f(x_1,x_2)=4x_1+2x_2^2+x_1^2+x_2$$

$$f(x_1, x_2) = 4x_1 + 2x_2^2 + x_1^2 + x_2$$

$$FOC:$$

$$f_1 = 4 + 2x_1 = 0 \rightarrow \boxed{x_1^* = -2}$$

$$f_2 = 4x_2 + 1 = 0 \rightarrow \boxed{x_2^* = \frac{-1}{4}}$$

$$f_1 = 4 + 2x_1$$

$$f_2 = 4x_2 + 1$$

$SOC:$ We need to find $f_{11}, f_{12}, f_{22}$

If $f_{11} > 0$ and $f_{11}.f_{22} > (f_{12})^2$, then local min

$f_1 = 4 + 2x_1$

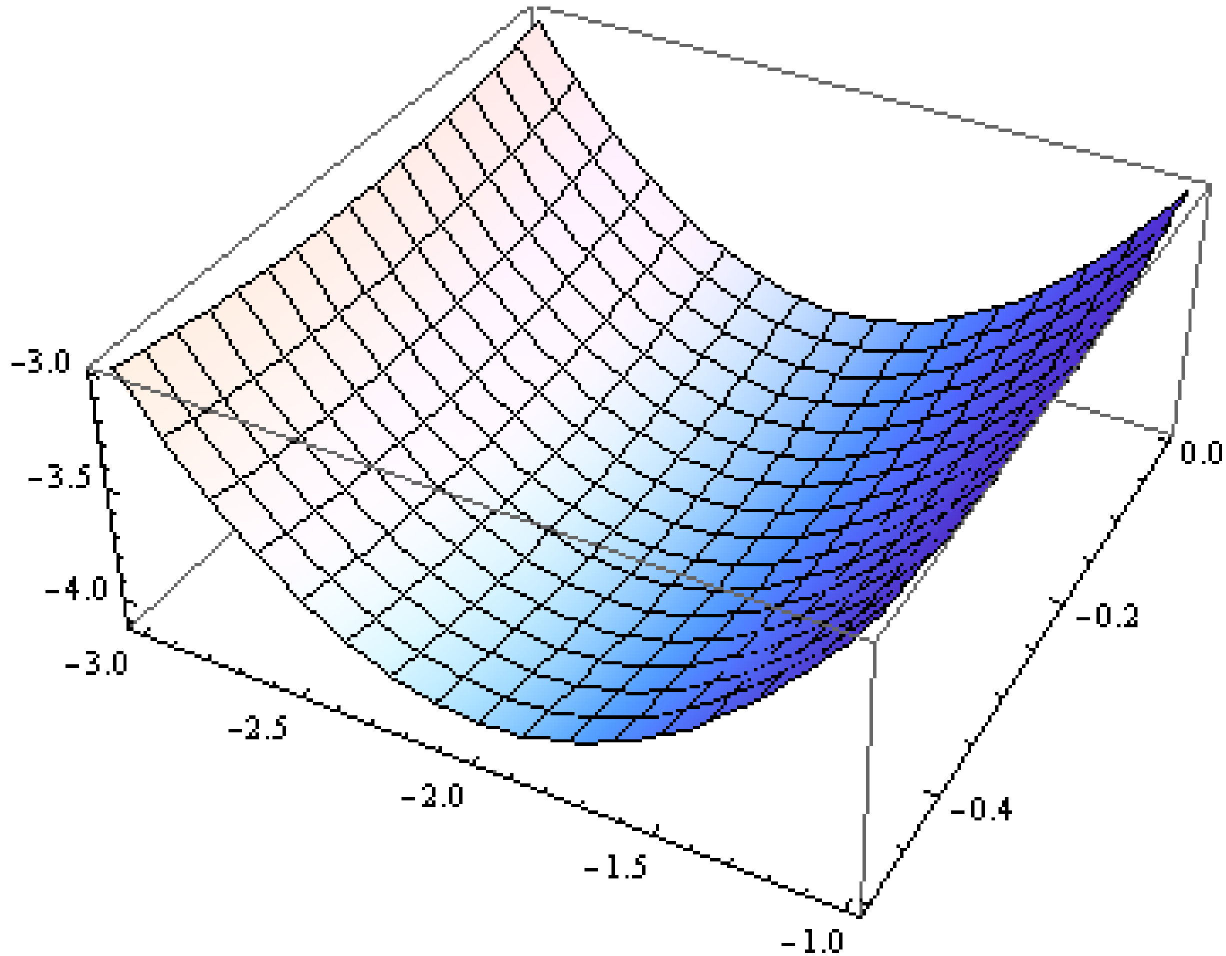
$f_2 = 4x_2 + 1$

$SOC:$

$f_{11} = 2$

$f_{12} = 0$

$f_{22} = 4$




Observe that $f_{11}$

$$f_{11}.f_{22} = 2(4) = 8 > 0 = (f_{12})^2$$

Hence, $(-2, \frac{-1}{4})$ is local minimum.

# Exercise 2

- Find the local max and local min of

$$f(x_1, x_2) = 8x_1 - 7x_2^2 - x_1^2 + 14x_2$$

$$f(x_1, x_2) = 8x_1 - 7x_2^2 - x_1^2 + 14x_2$$

$FOC:$

$$f_1 = 8 - 2x_1 = 0 \rightarrow \boxed{x_1^* = 4}$$

$$f_2 = -14x_2 + 14 = 0 \rightarrow \boxed{x_2^* = 1}$$

$$f_1 = 8 - 2x_1$$

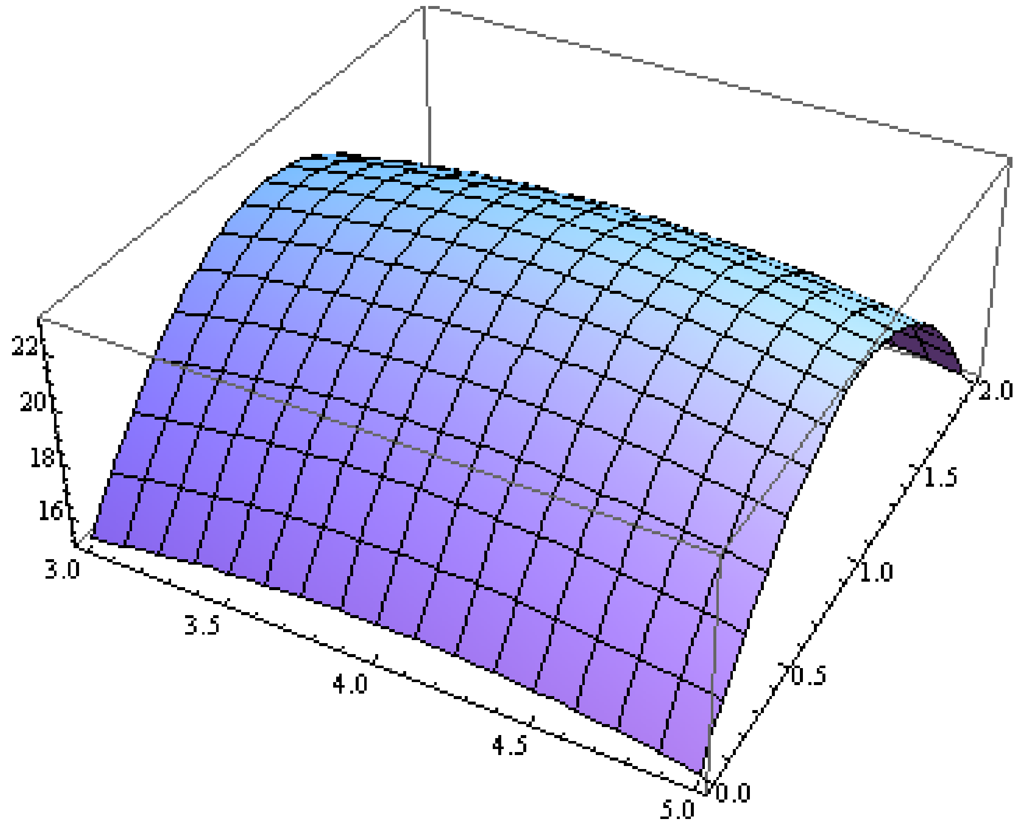
$$f_2 = -14x_2 + 14$$

$SOC:$

$$f_{11} = -2$$

$$f_{12} = 0$$

$$f_{22} = -14$$




Observe that $f_{11} = -2 < 0$ and

$$f_{11}.f_{22} = (-2)(-14) = 28 > 0 = (f_{12})^2$$

Hence, (4,1) is local max.

# Exercise 3

- Find the local max and local min of

$$f(x_1, x_2) = -2x_1 + 4x_2^2 + x_1^2 - 16x_2 + x_1 x_2$$

$$f(x_1, x_2) = -2x_1 + 4x_2^2 + x_1^2 - 16x_2 + x_1x_2$$

$$FOC:$$

$$f_1 = -2 + 2x_1 + x_2 = 0$$

$$f_2 = 8x_2 - 16 + x_1 = 0$$

$$f_1 = -2 + 2x_1 + x_2 = 0$$

$$f_2 = 8x_2 - 16 + x_1 = 0$$

$$-2 + 2x_1 + x_2 = 0 \rightarrow x_2 = 2 - 2x_1$$

$$8(2 - 2x_1) - 16 + x_1 = 0 \rightarrow \boxed{x_1^* = 0}$$

$$\boxed{x_2^* = 2}$$

$$f_1 = -2 + 2x_1 + x_2$$

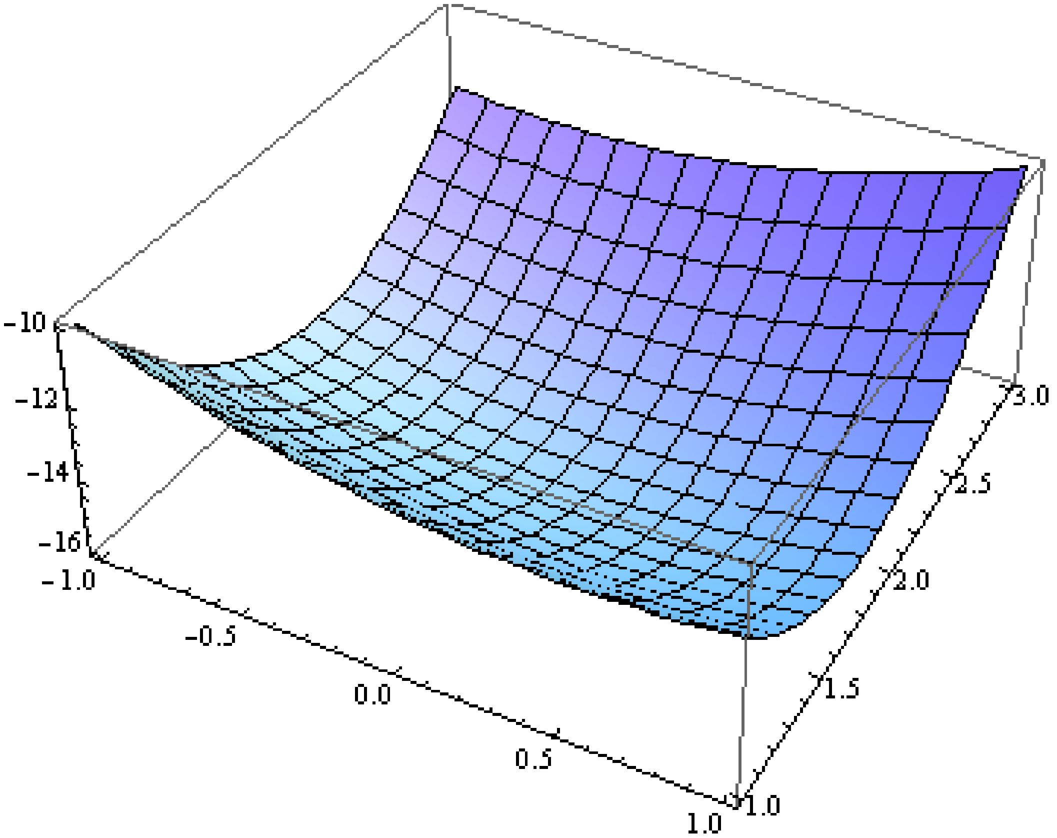
$$f_2 = 8x_2 - 16 + x_1$$

$$SOC:$$

$$f_{11} = 2$$

$$f_{12} = 1$$

$$f_{22} = 8$$




Observe that $f_{11} = 2 > 0$ and

$$f_{11}.f_{22} = (2)(8) = 16 > 1 = (f_{12})^2$$

Hence, $(0, 2)$ is local min.

# Exercise 4

- Find the local max and local min of

$$f(x_1, x_2) = -x_1 - \frac{1}{8}x_2^2 - \frac{1}{2}x_1^2 + x_2 + x_1 x_2$$

$$f(x_1, x_2) = -x_1 - \frac{1}{8}x_2^2 - \frac{1}{2}x_1^2 + x_2 + x_1 x_2$$

$$FOC:$$

$$f_1 = -1 - x_1 + x_2 = 0$$

$$f_2 = -\frac{1}{4}x_2 + 1 + x_1 = 0$$

$$f_1 = -1 - x_1 + x_2 = 0$$

$$f_2 = -\frac{1}{4} x_2 + 1 + x_1 = 0$$

$$-1 - x_1 + x_2 = 0 \rightarrow x_2 = 1 + x_1$$

$$-\frac{1}{4}(1 + x_1) + 1 + x_1 = 0 \rightarrow 1 + x_1 - 4 - 4x_1 = 0$$

$$\boxed{x_1^* = -1}$$

$$\boxed{x_2^* = 0}$$

$$f_1 = -1 - x_1 + x_2$$

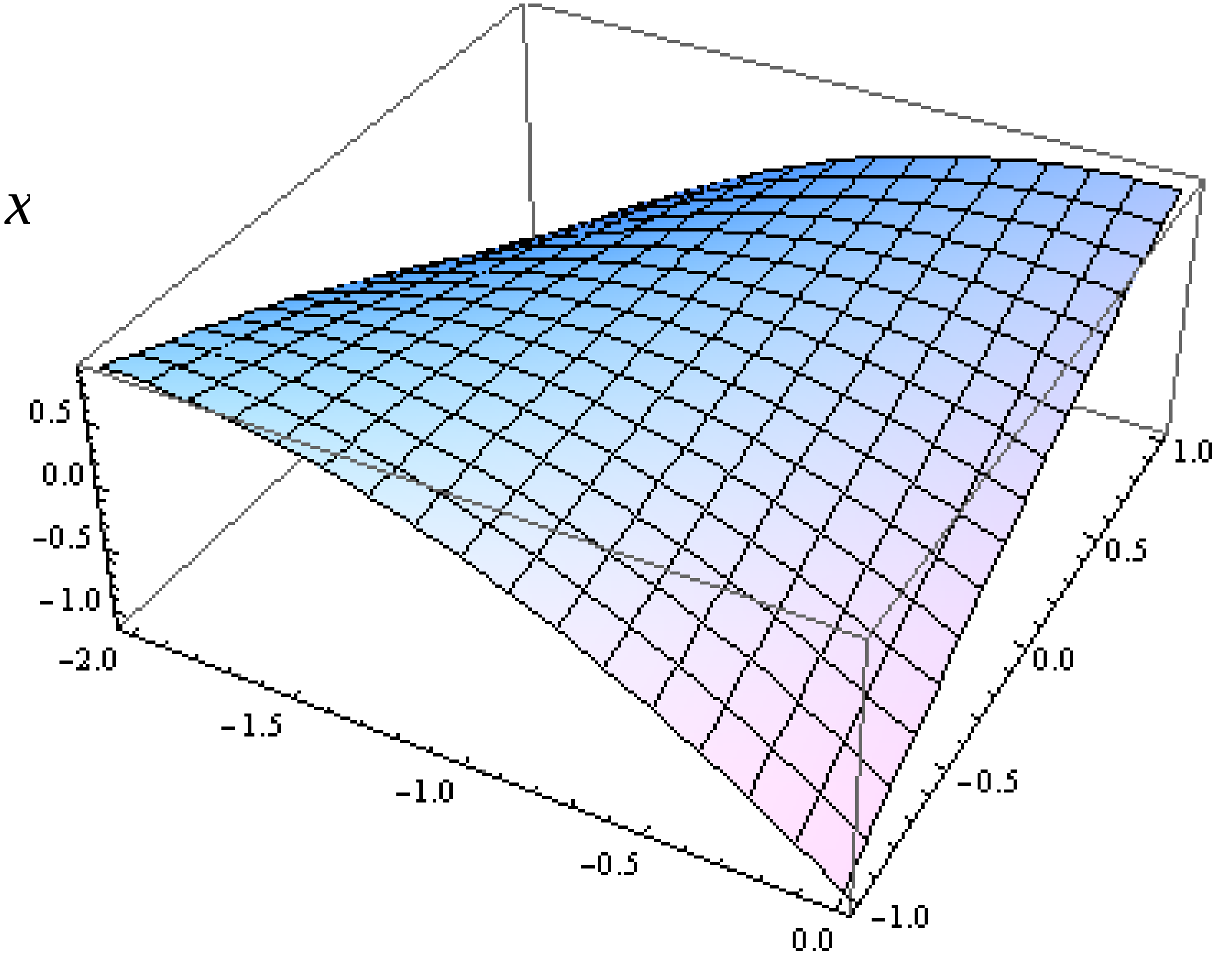
$$f_2 = -\frac{1}{4}x_2 + 1 + x$$

$SOC:$

$$f_{11} = -1$$

$$f_{12} = 1$$

$$f_{22} = -\frac{1}{4}$$




Observe that $f_{11} = -1 < 0$ and

$$f_{11}.f_{22} = (-1)(-\frac{1}{4}) = \frac{1}{4} < 1 = (f_{12})^2$$

Hence, no concl.

# Exercise 6

- Find the local max and local min of

$$f(x_1, x_2) = -\frac{1}{2}x_2^2 - \frac{1}{3}x_1^3 + x_2$$

$$f(x_1, x_2) = -\frac{1}{2}x_2^2 - \frac{1}{3}x_1^3 + x_2$$

$FOC:$

$$f_1 = -x_1^2 = 0 \rightarrow \boxed{x_1^* = 0}$$

$$f_2 = -x_2 + 1 = 0 \rightarrow \boxed{x_2^* = 1}$$

$$f_1 = -x_1^2$$

$$f_2 = -x_2 + 1$$

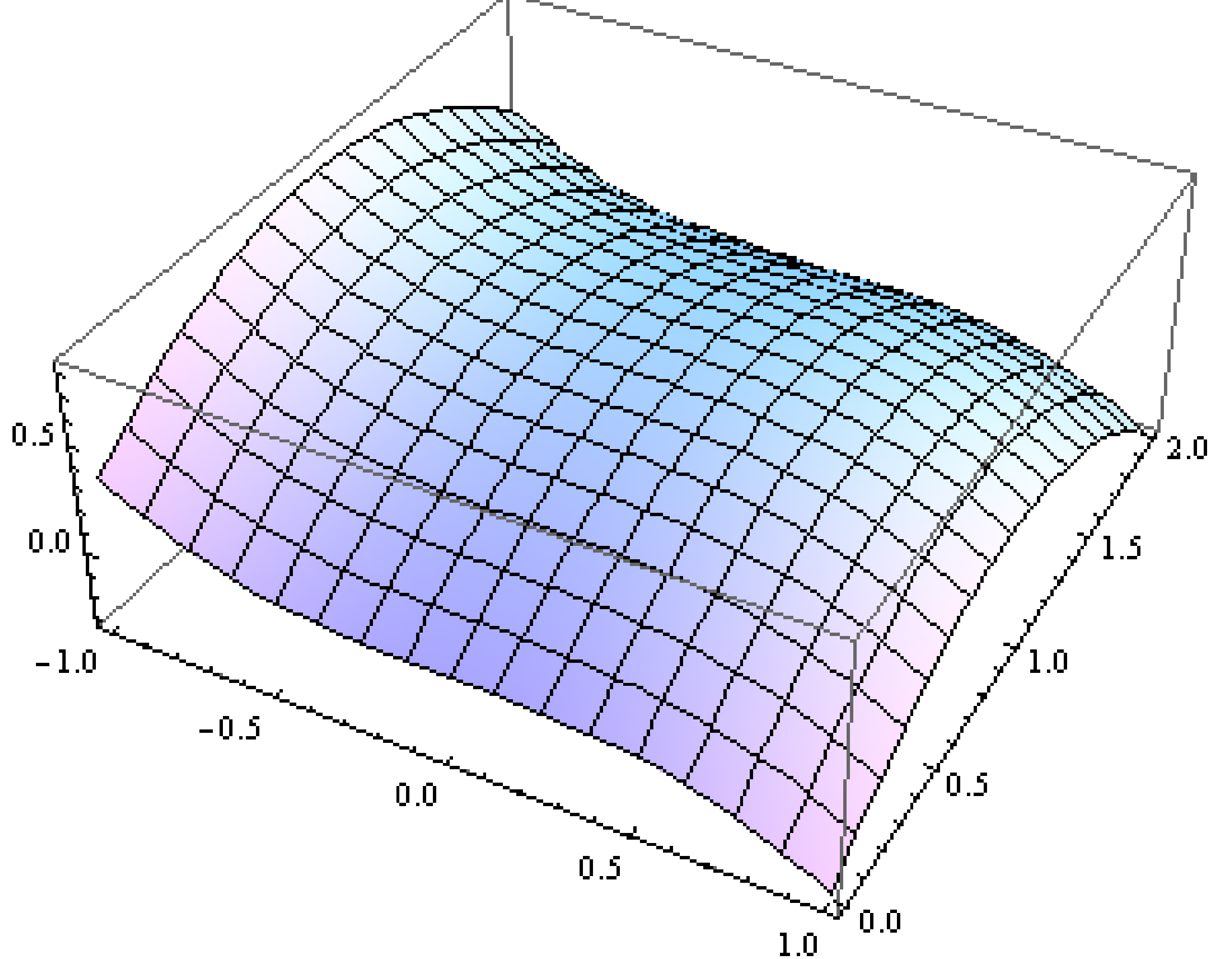
$$SOC:$$

$$f_{11} = -2x_1$$

$$f_{12} = 0$$

$$f_{22} = -1$$

At (0,1)




$SOC:$

$f_{11} = -2x_1 = 0$

$f_{12} = 0$

$f_{22} = -1$

Observe that $f_{11} = 0$ and

$f_{11}.f_{22} = (0)(-1) = 0 = 0 = (f_{12})^2$

Hence, no conl.

# Exercise 7

- Find the local max and local min of

$$f(x_1, x_2) = x_1 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3 + x_2$$

$$f(x_1, x_2) = x_1 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3 + x_2$$

$FOC:$

$$f_1 = 1 - x_1^2 = 0 \rightarrow \boxed{x_1^* = -1} \text{ or } \boxed{x_1^* = 1}$$

$$f_2 = -x_2 + 1 = 0 \rightarrow \boxed{x_2^* = 1}$$

Two stationary points (-1,1) and (1,1)

$$f_1 = 1 - x_1^2$$

$$f_2 = -x_2 + 1$$

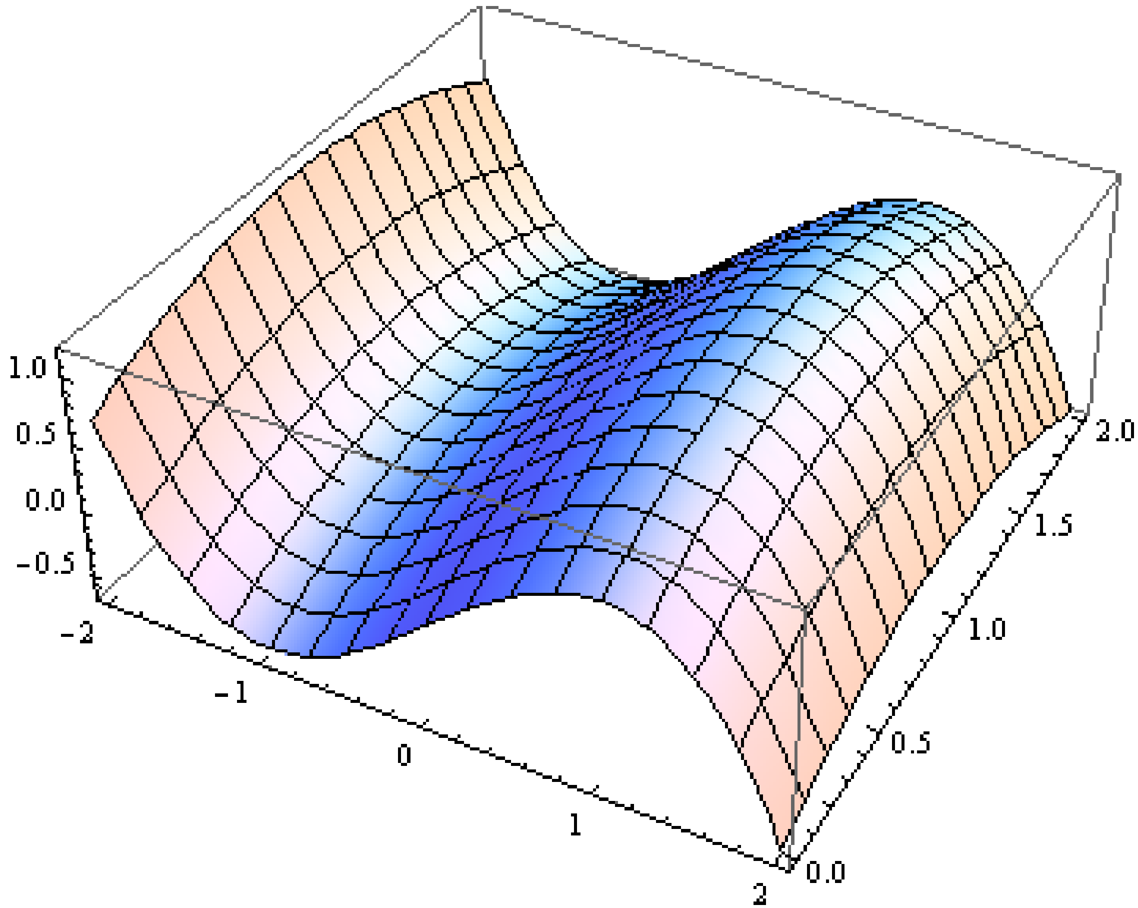
$$SOC:$$

$$f_{11} = -2x_1$$

$$f_{12} = 0$$

$$f_{22} = -1$$

At (1,1)




$SOC:$

$f_{11} = -2x_1 = -2$

$f_{12} = 0$

$f_{22} = -1$

Observe that $f_{11} = -2 < 0$ and

$f_{11} . f_{22} = (-2)(-1) = 2 > 0 = (f_{12})^2$

Hence, (1,1) is local max.

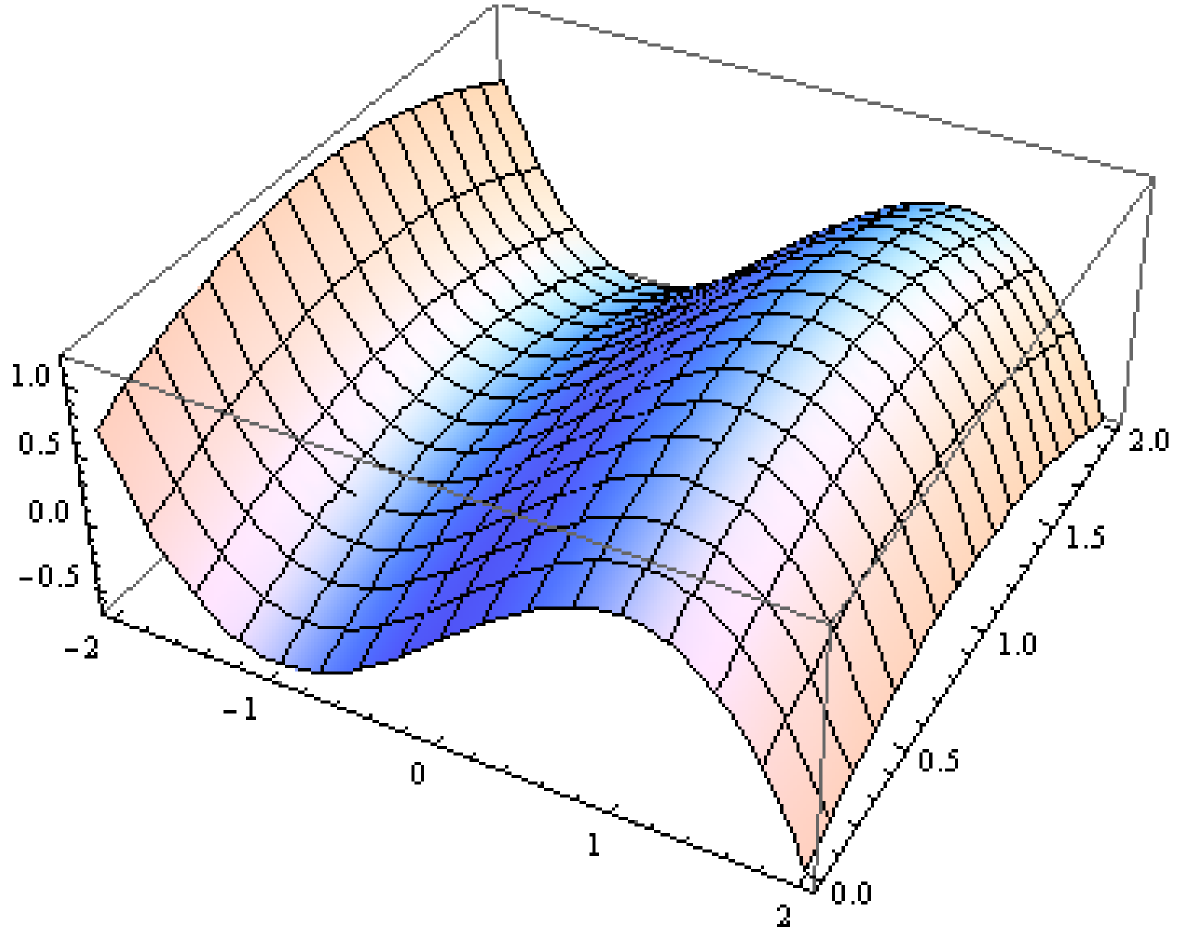
At $(-1,1)$

$SOC:$

$f_{11} = -2x_1 = -2(-1) = 2$

$f_{12} = 0$

$f_{22} = -1$




Observe that $f_{11} = 2 > 0$ and

$f_{11} . f_{22} = (2)(-1) = -2 < 0 = (f_{12})^2$

Hence, at (-1,1) no concl.