Extreme Values of Univariate Functions

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Economics 300

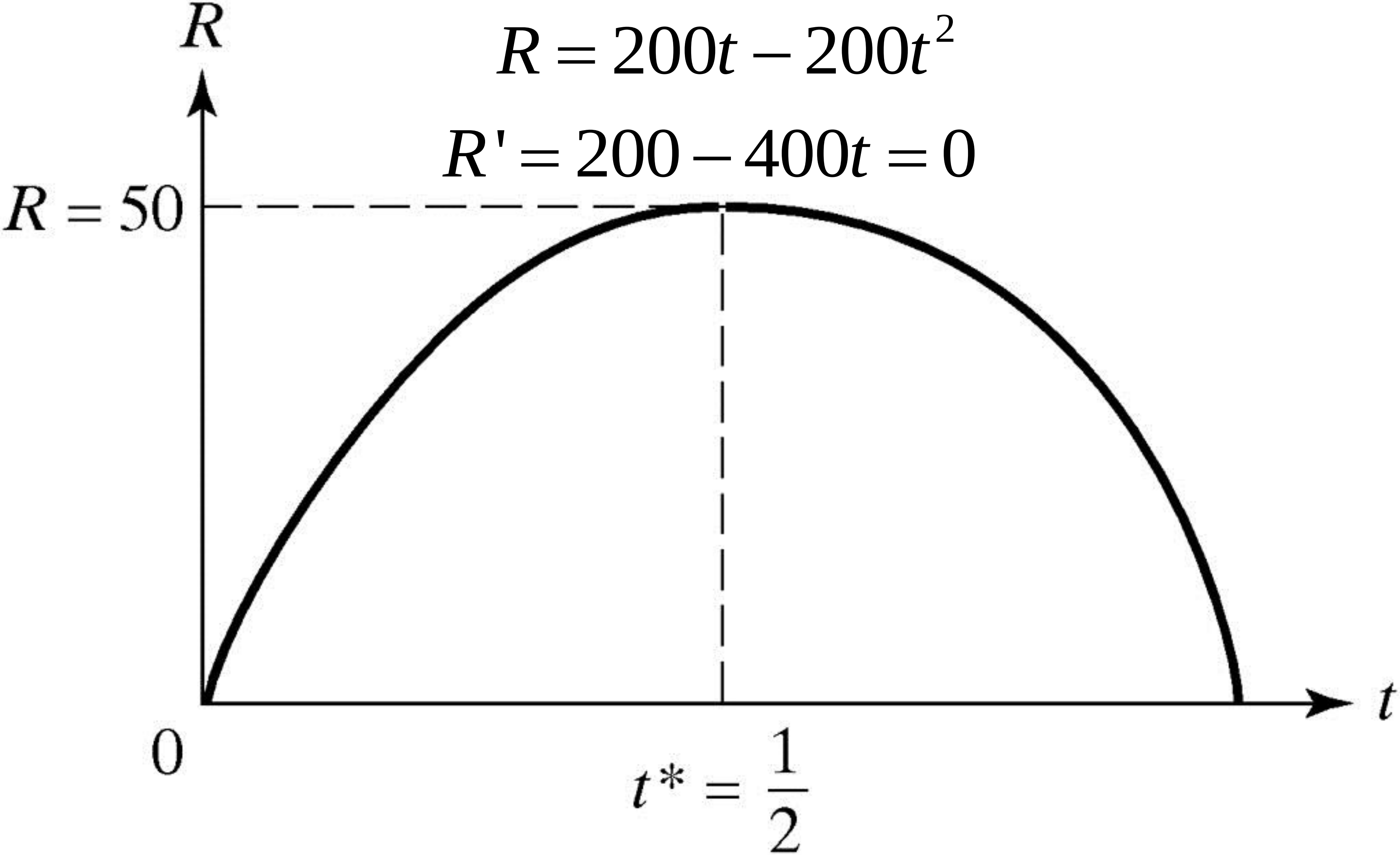
Economic models determine **optimal** outcomes, consumption, interest rates,…

- Calculate **maximum value** to identify
  - Highest profit
  - Highest utility
  - Highest tax revenue
- Calculate **minimum value** to identify
  - Lowest cost
  - Lowest price
  - Lowest risk

Tax rate and tax revenue

$$R = 200t - 200t^2$$

$$R' = 200 - 400t = 0$$



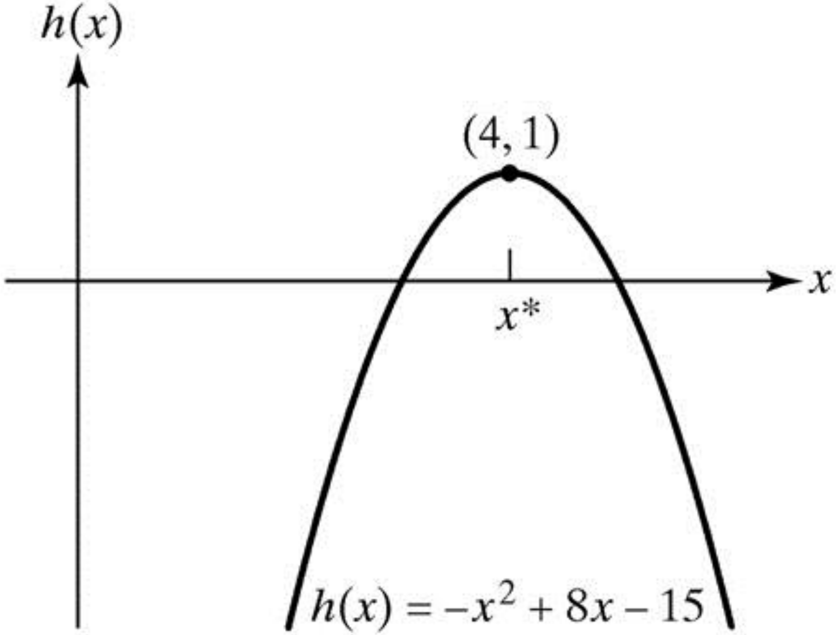
Identifying extreme values

Stationary Point

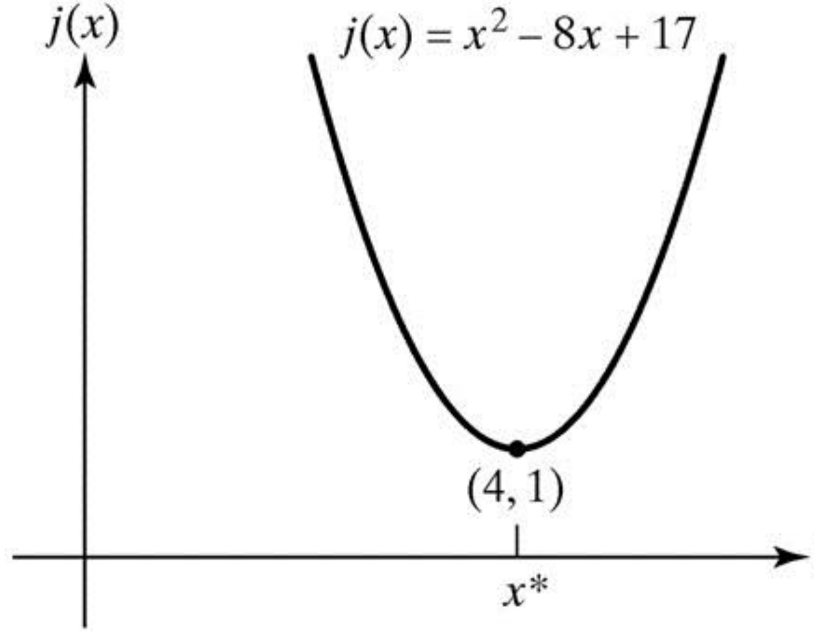
- x^* is a **stationary point** of a differentiable function $f(x)$ if

$$f'(x^*) = 0$$

Stationary points



(a) A Stationary Point That Is a Maximum



(b) A Stationary Point That Is a Minimum

$$h' = -2x + 8 = 0 \qquad j' = 2x - 8 = 0$$

$$x^* = 4 \qquad x^* = 4$$

$$h'' = -2 < 0 \qquad j'' = 2 > 0$$

Characterizing extreme values

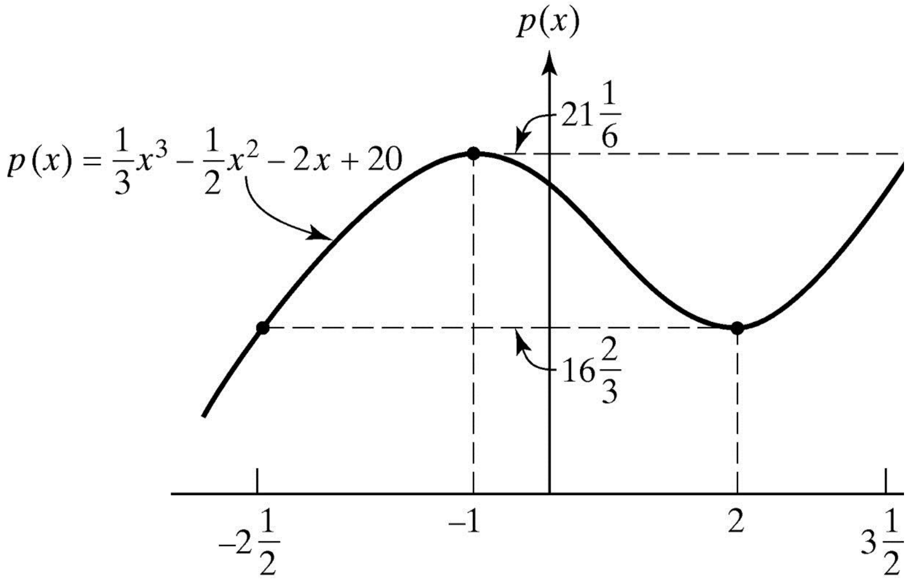
<u>First-Order Condition</u>

- If f(x) is everywhere differentiable and reaches a maximum or minimum at x^*, then $f'(x^*) = 0$ (x^* is stationary)

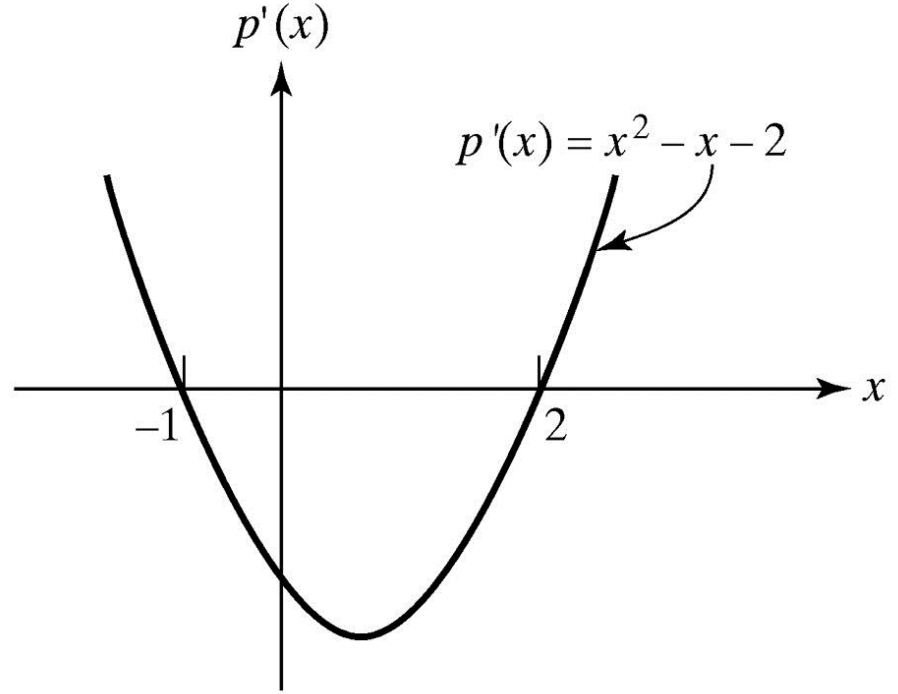
Necessary condition for max or min

- If max or min, then $f'(x^*) = 0$

A function with two extreme points and its derivative



(a)



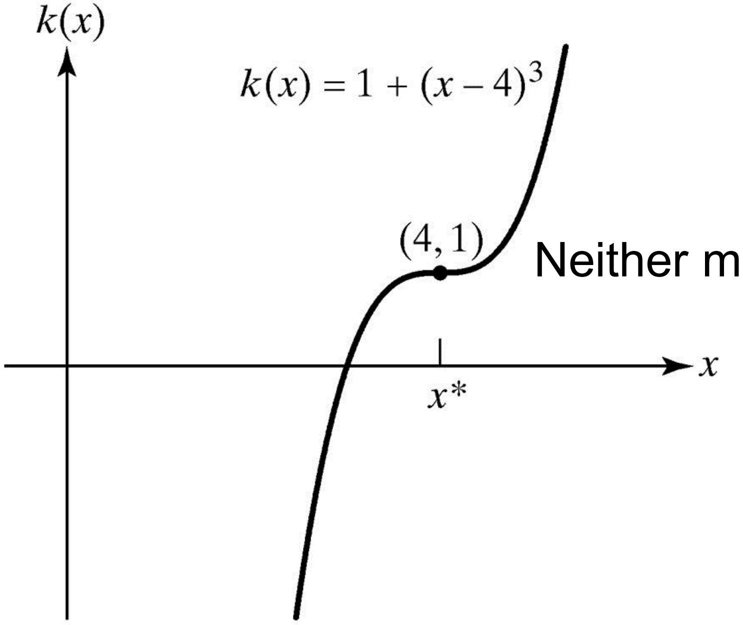
(b)

$$p''(x) = 2x - 1$$

$$p''(-1) < 0$$

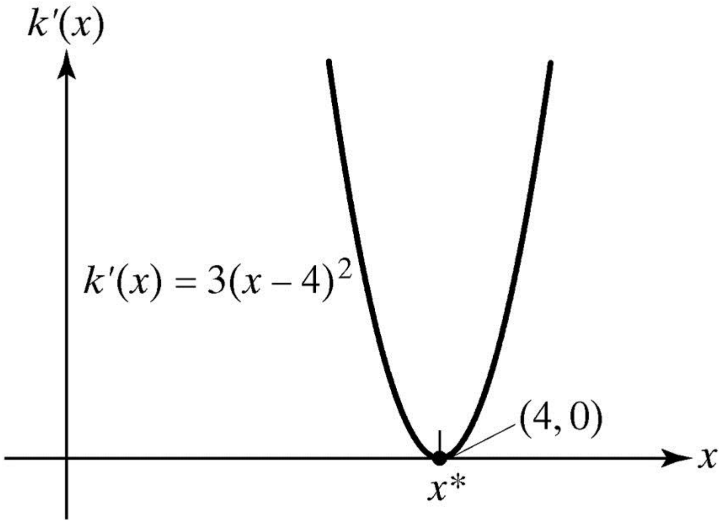
$$p''(2) > 0$$

A function with an inflection point and its derivative



(a)

Neither max nor min, but k’(x)=0



(b)

$$k'' = 6(x-4)$$

$$k''(4) = 0$$

Characterizing extreme values

Second-Order Condition

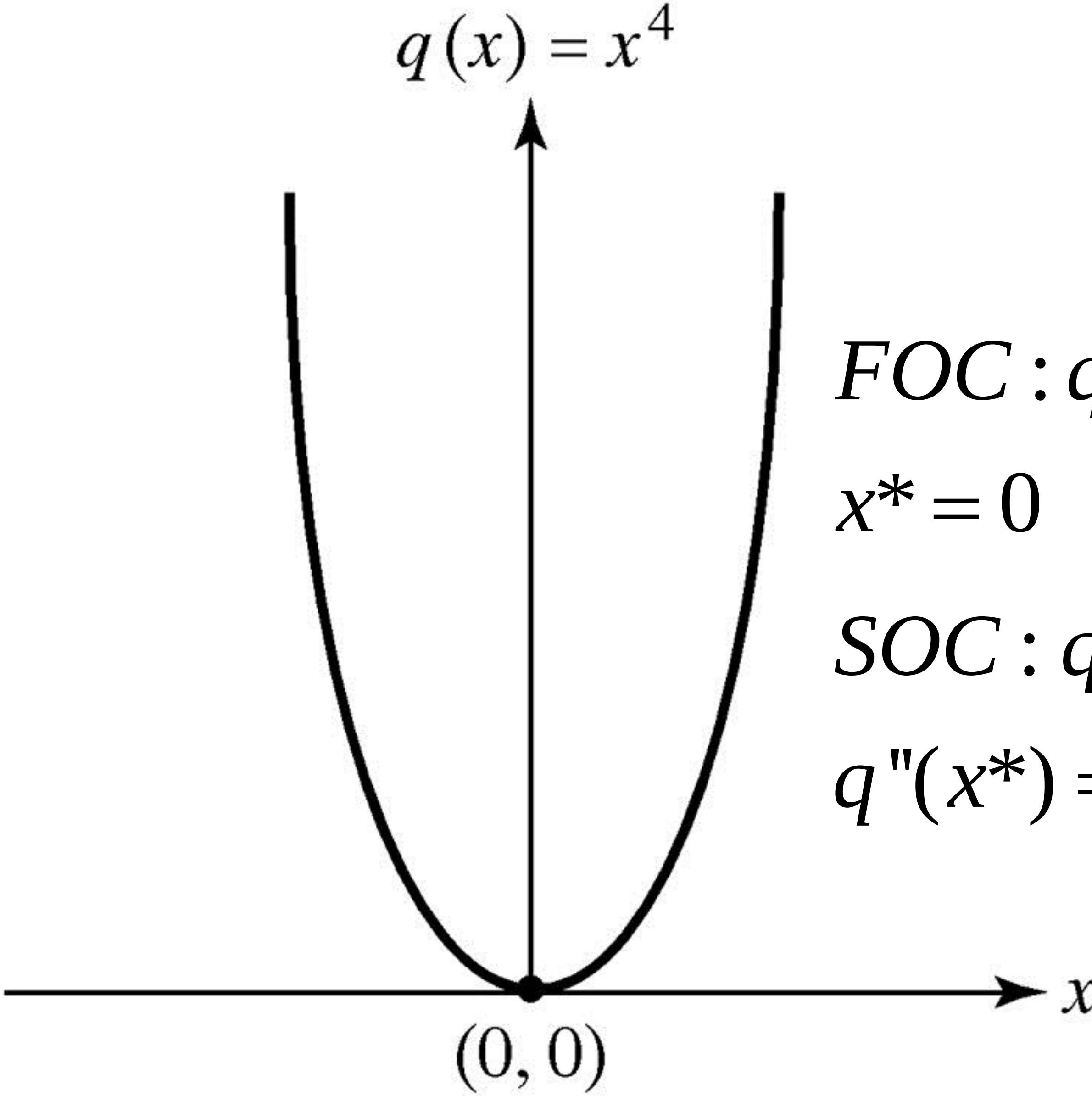
- If the second derivative of a differentiable function $f(x)$ is negative when evaluated at a stationary point ($f''(x^*) < 0$) then x^* is a local maximum.

- If $f''(x^*) > 0$ then x^* is a local minimum.

Sufficient condition for max or min

- Local max if $f'(x^*) = 0$ & $f''(x^*) < 0$
- Local min if $f'(x^*) = 0$ & $f''(x^*) > 0$

Failure of the second-order condition



$$FOC : q'(x) = 4x^3 = 0$$

$$x^* = 0$$

$$SOC : q''(x) = 12x^2$$

$$q''(x^*) = 0$$

Exercise

- Consider a monopolist's linear demand function P=12 – 2Q where P is the price of the good and Q is the quantity.
- The monopolist's total cost function is

$$TC = \frac{1}{3}Q^3 - 5Q^2 + 17Q + 25$$

- Find the output level at which monopolist should produce in order to maximize her profits

Exercise - solution

- Profit = TR – TC = P × Q – TC

$$\text{Profit} = (12-2Q)Q - \left[\frac{1}{3}Q^3 - 5Q^2 + 17Q + 25\right]$$

$$= 12Q - 2Q^2 - \frac{1}{3}Q^3 + 5Q^2 - 17Q - 25$$

$FOC:$

$$\pi'(Q) = 12 - 4Q - Q^2 + 10Q - 17 = 0$$

$$Q^2 - 6Q + 5 = 0 \Rightarrow (Q-5)(Q-1) = 0$$

$Q^* = 1$ and $Q^* = 5$ are stationary points.

SOC:

$$\pi''(Q) = -4 - 2Q + 10 = 6 - 2Q$$

$$\pi''(1) = 4 > 0 \Rightarrow \text{ local min.}$$

$$\pi''(5) = -4 < 0 \Rightarrow \text{ local max. at Q*=5}$$