Functions

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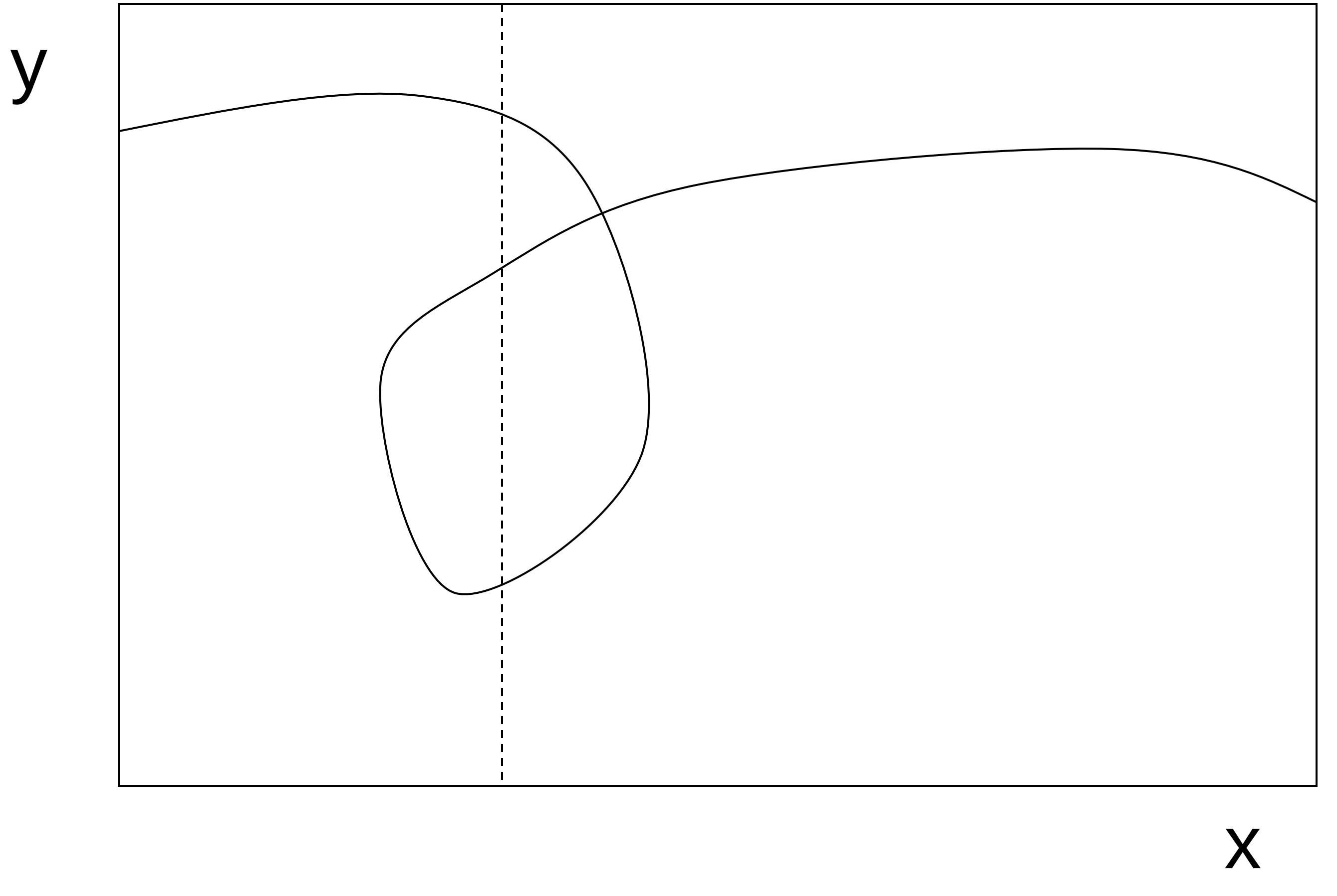
Economics 300

Function

- A mapping from each x in X to some y in Y
  $f: X \rightarrow Y$

  Domain is X; Range is Y
- $y = f(x)$
- Shows how y depends on x
- Each x maps into one y
  - If I know x, I can determine y

Is it a function?

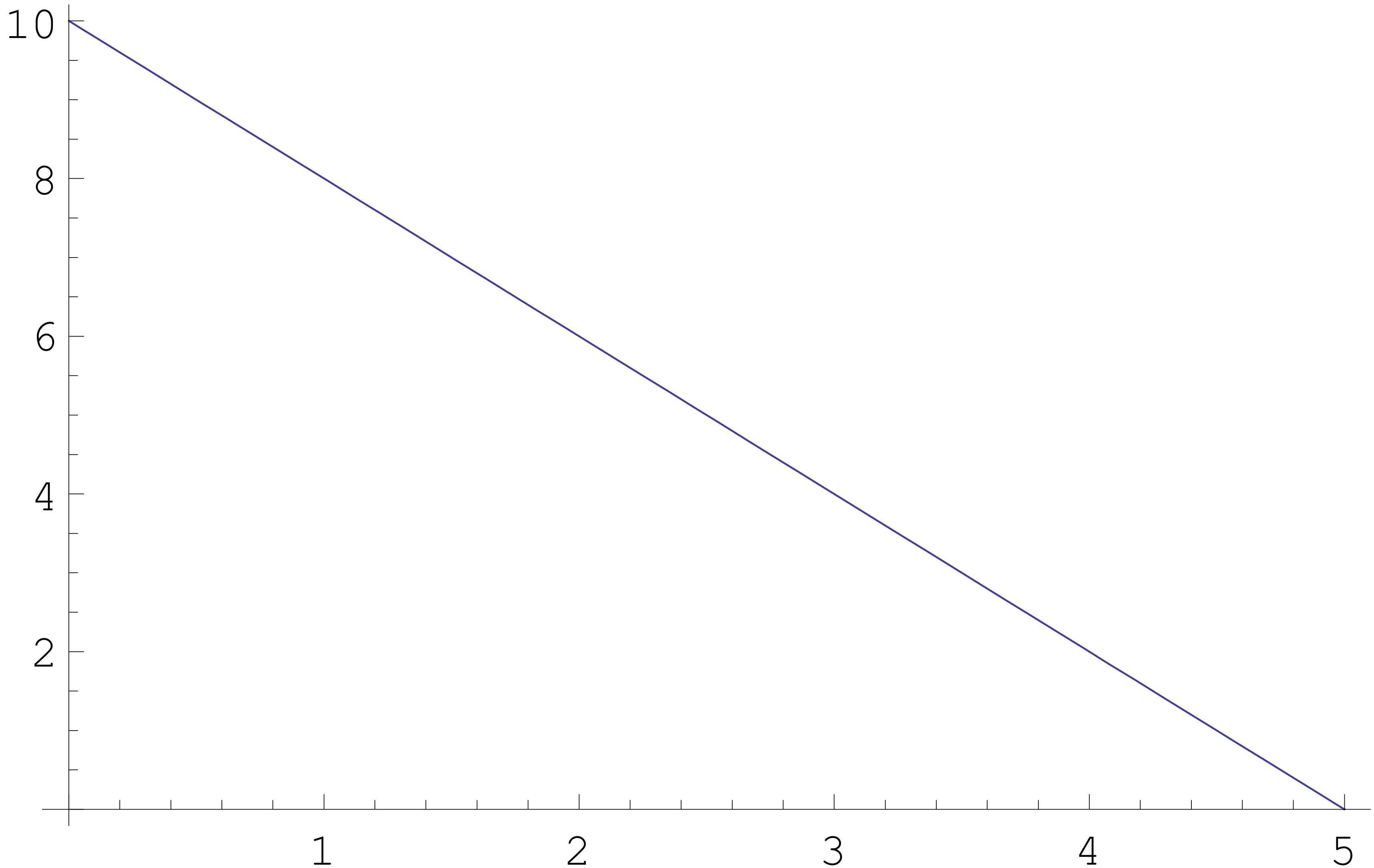


No! Can't say what y is knowing x

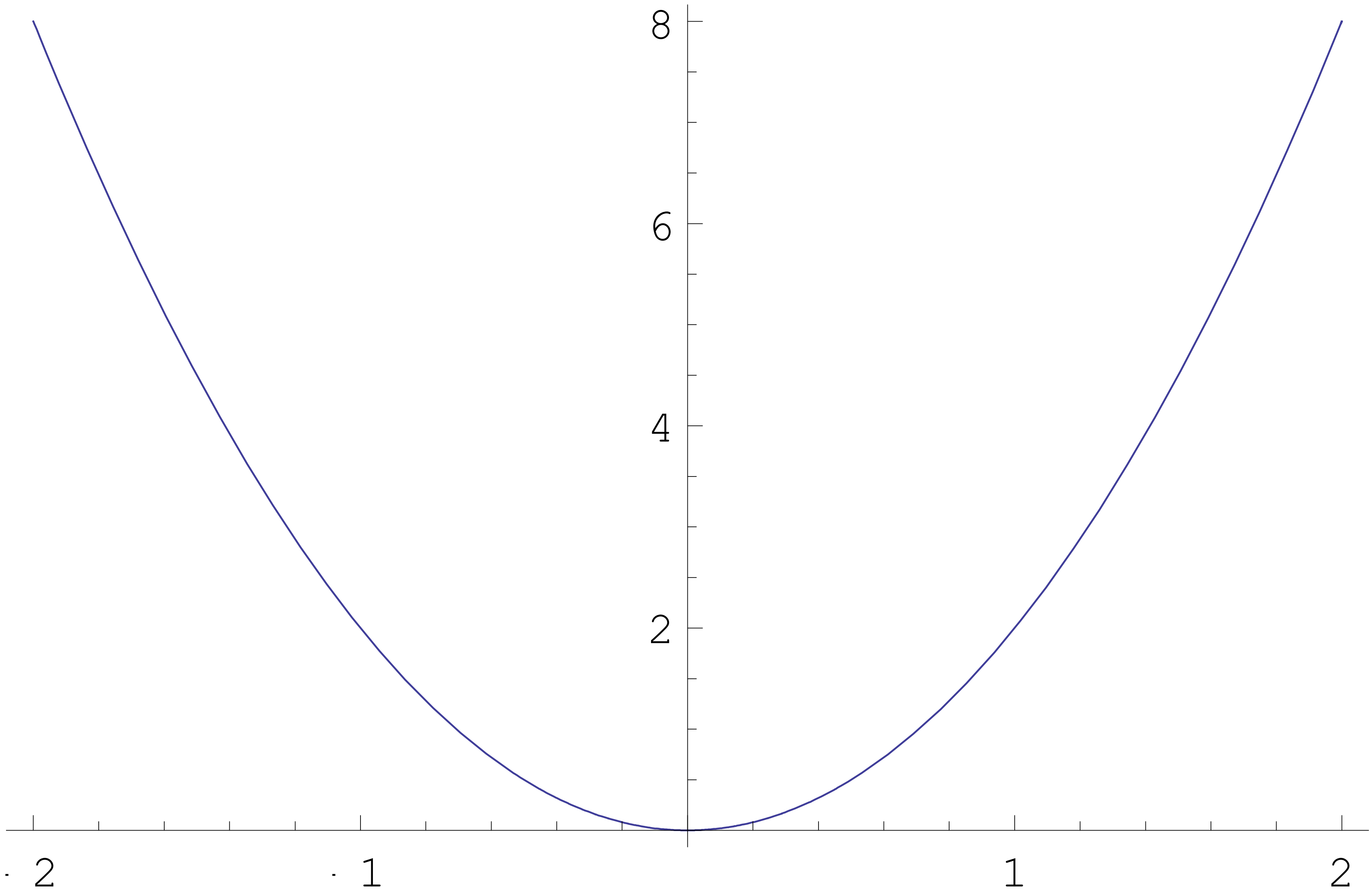
Examples with one variable: univariate functions

What economic relationships might these functions describe?

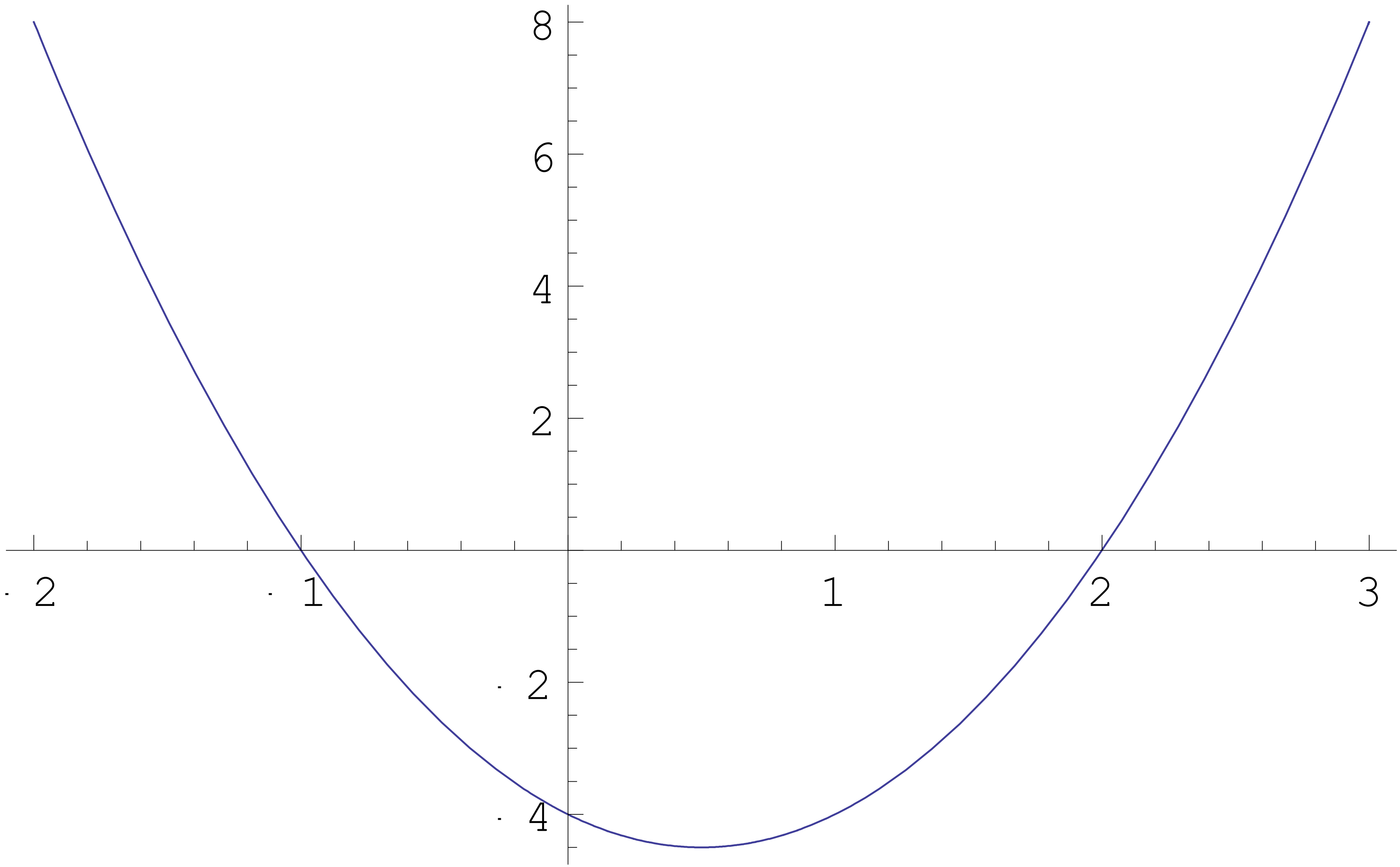
y = 10 - 2 x



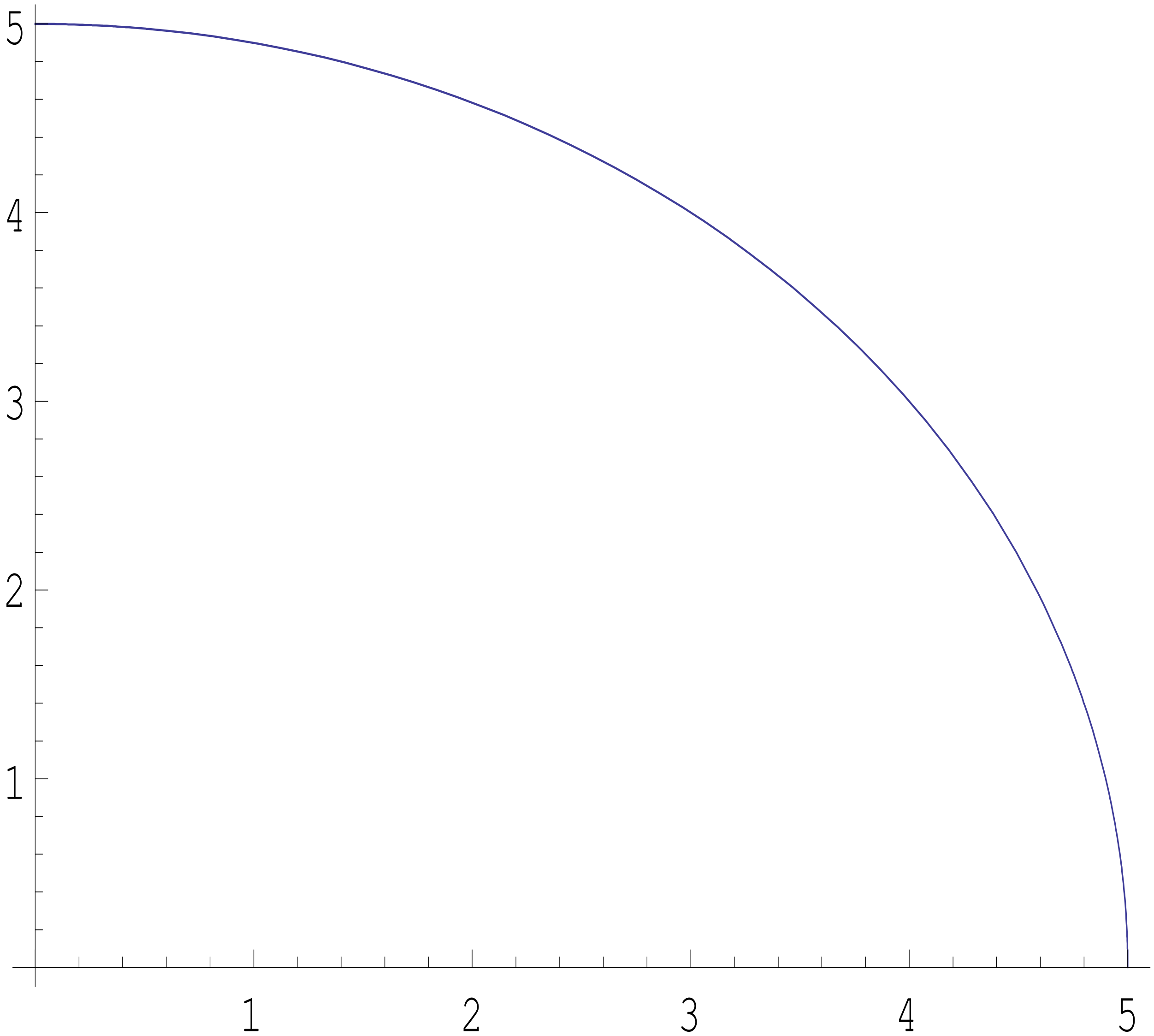
$y = 2x^2$



$$y = 2x^2 - 2x - 4$$



$$y = \sqrt{25 - x^2}$$



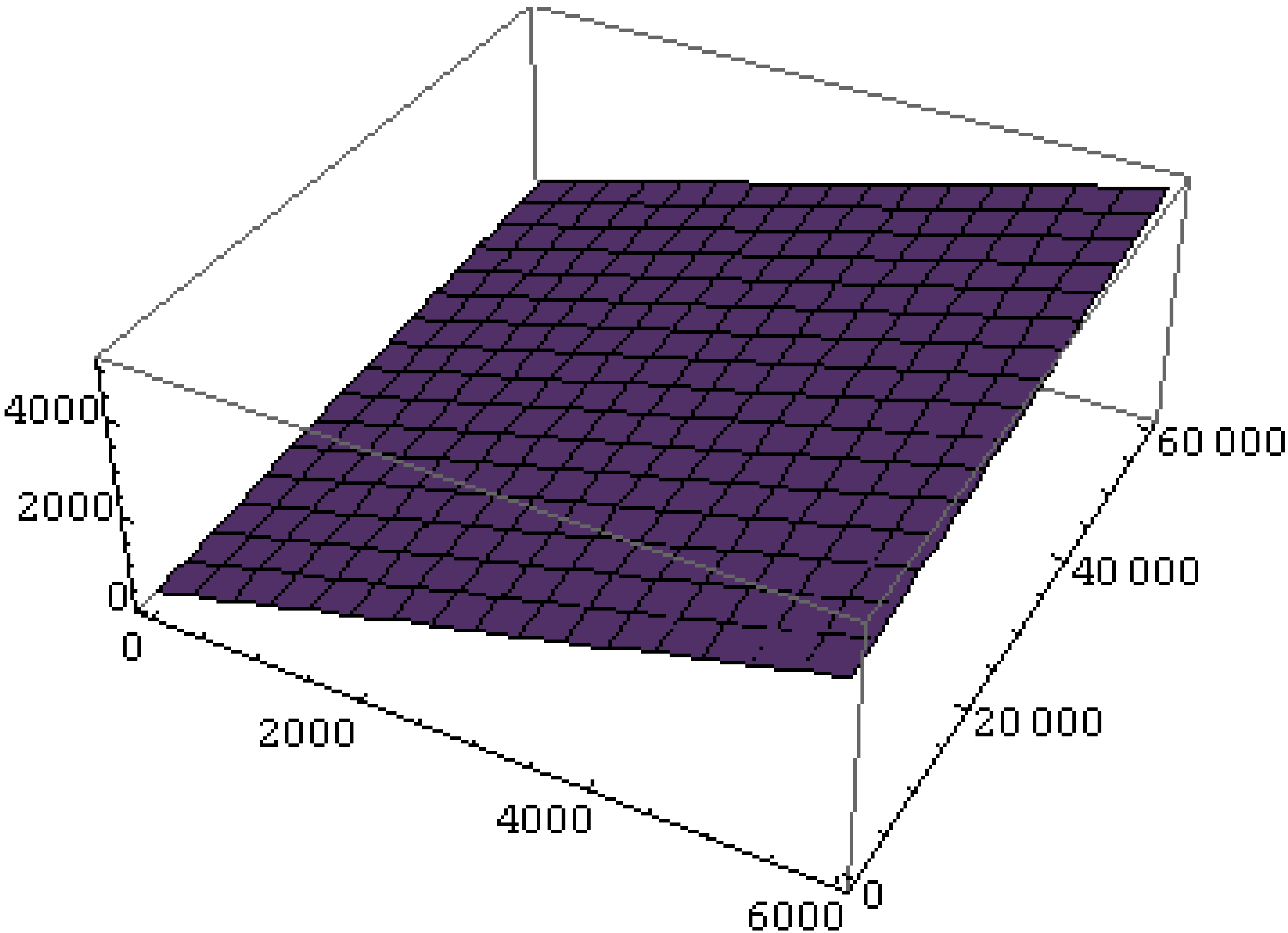
Multivariate functions (many variables)

- z = f(x,y)
- $y = f(x_1, x_2, \ldots, x_n)$

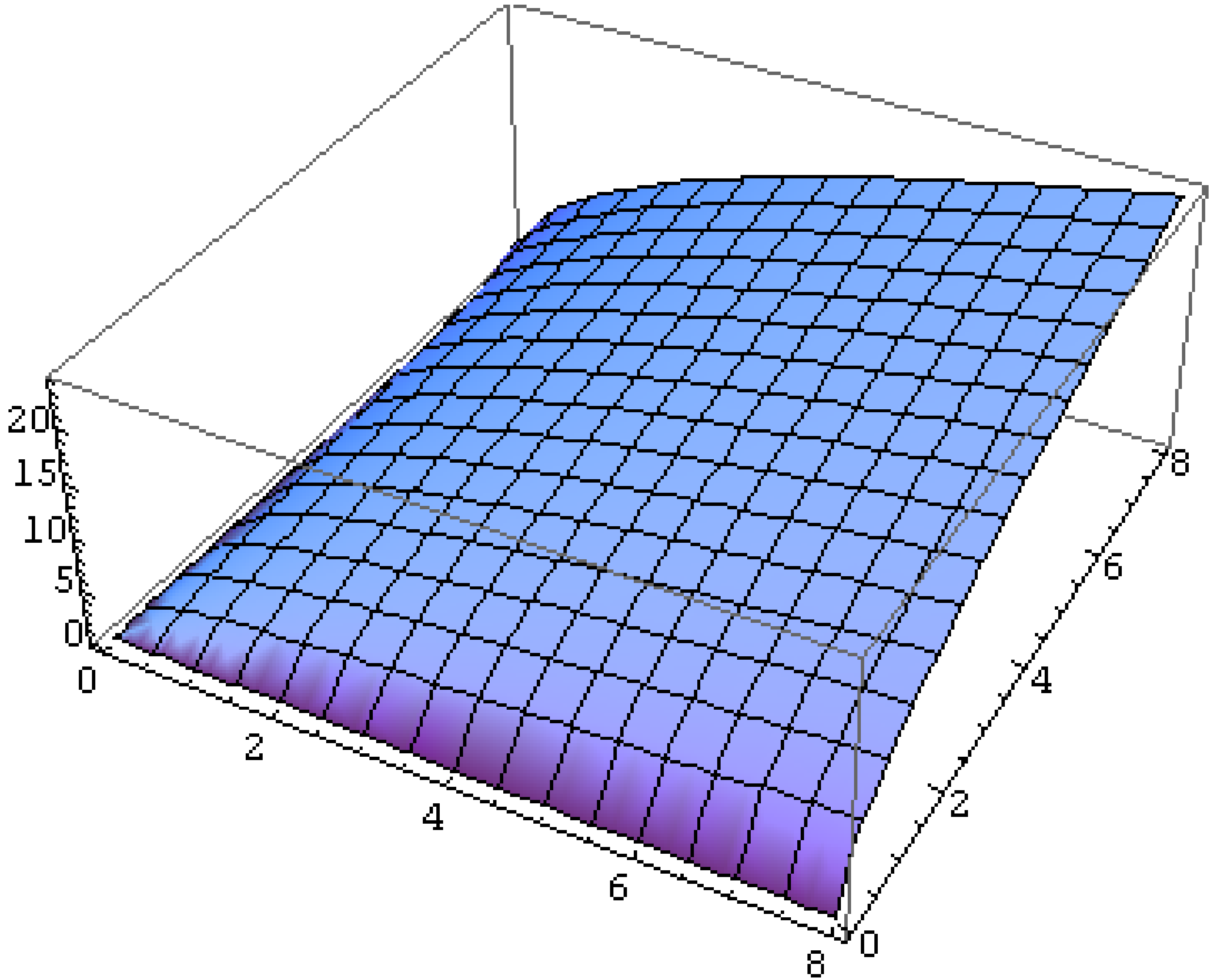
$$y = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$y = \prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \cdots \cdot x_n$$

Consumption as a function of income and wealth: C = 300 + .6I + .02 W



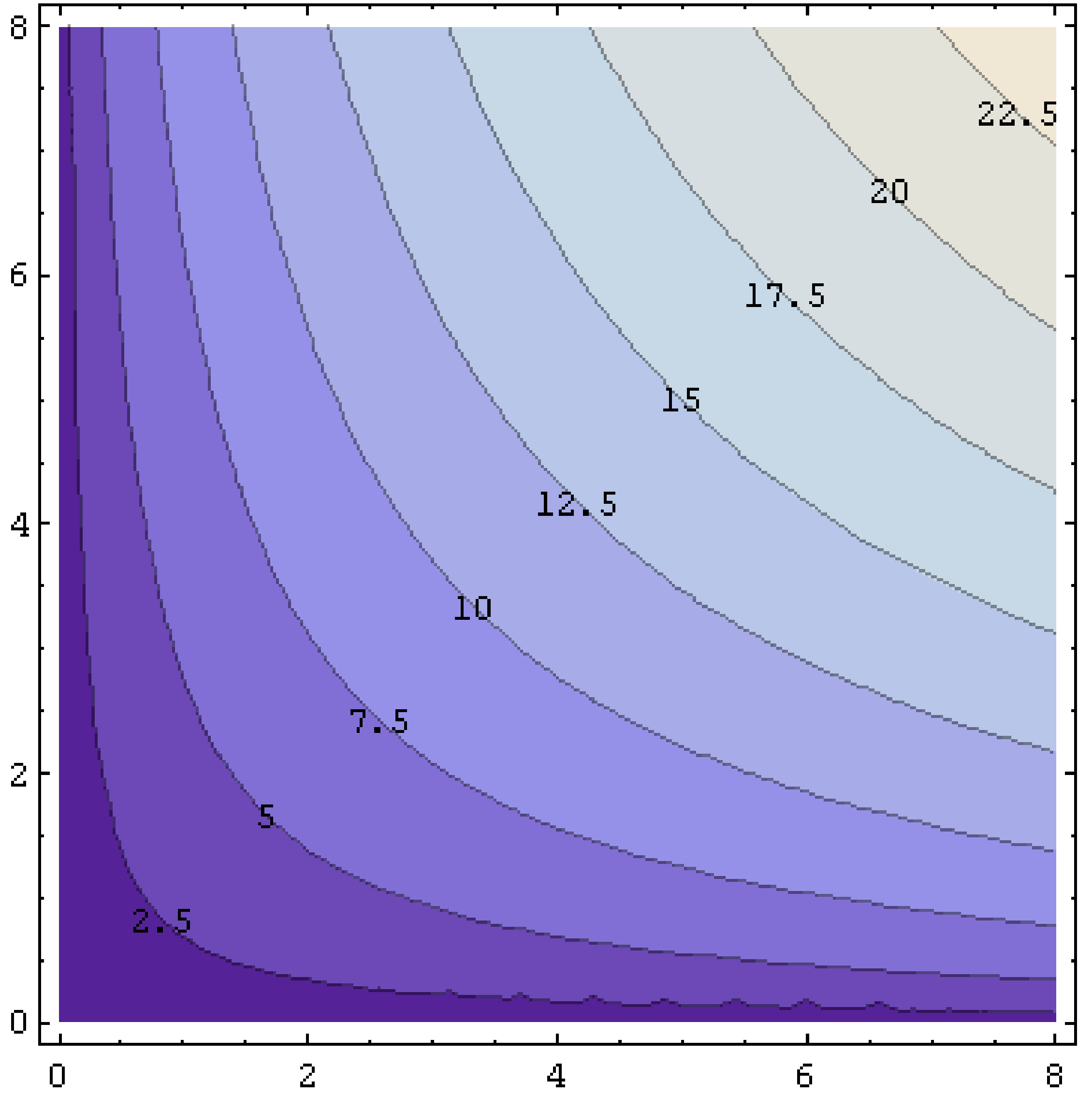
$$y = 3x_1^{1/2}x_2^{1/2}$$



Draw for various y in 2 dimensions

$$y = 3x_1^{1/2} x_2^{1/2}$$

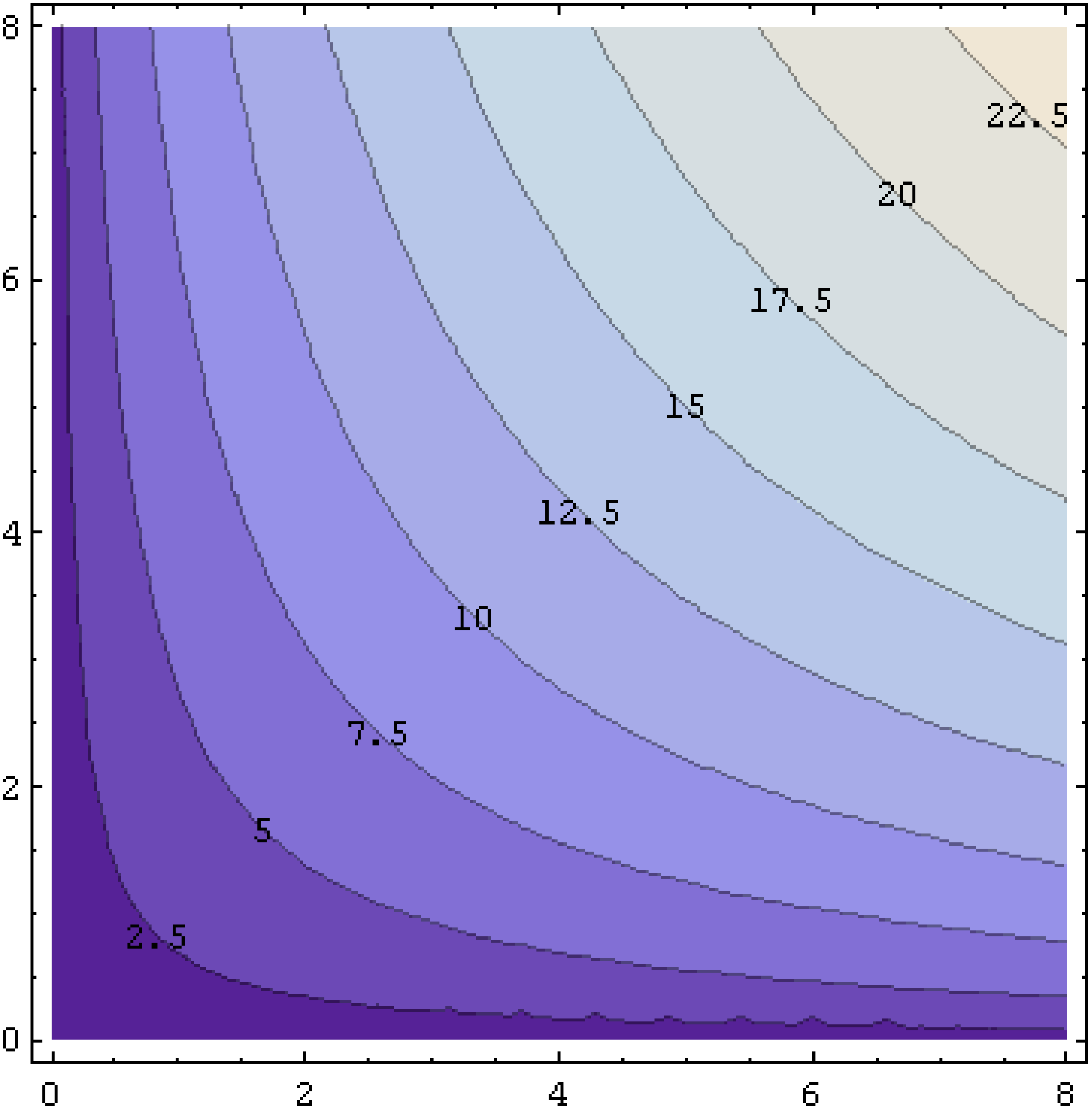
Cobb-Douglas function



With y fixed, $x_2 = f(x_1)$

$$x_2 = y^2 / 9x_1$$

Cobb-Douglas in economics

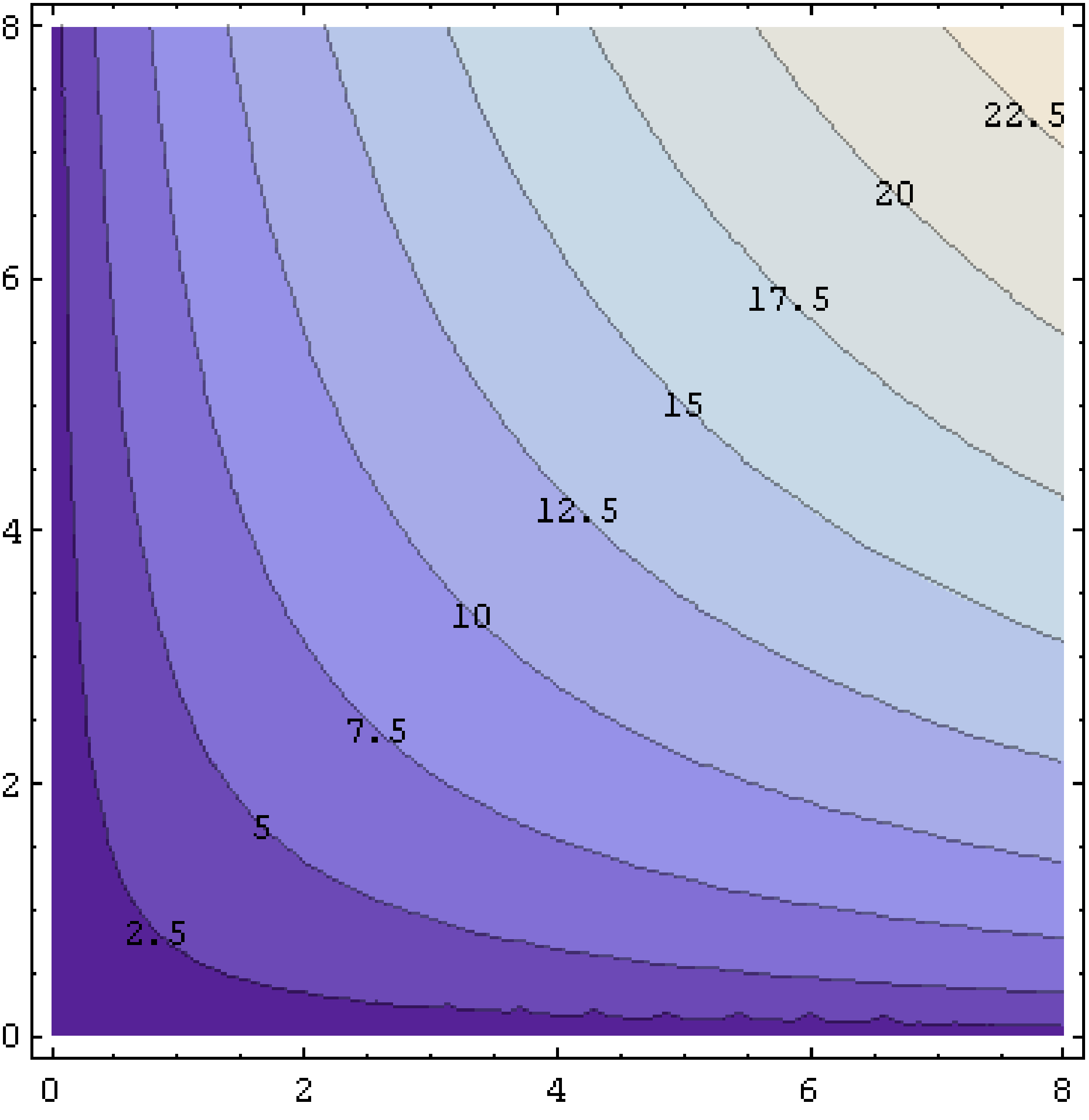


$$y = 3x_1^{1/2}x_2^{1/2}$$

Consumer preferences

- y = utility
- x_1 = pears
- x_2 = cheese
- Indifference curve

Cobb-Douglas in economics



$$y = 3x_1^{1/2} x_2^{1/2}$$

Firm production

- y = output quantity
- x_1 = capital
- x_2 = labor
- Isoquant

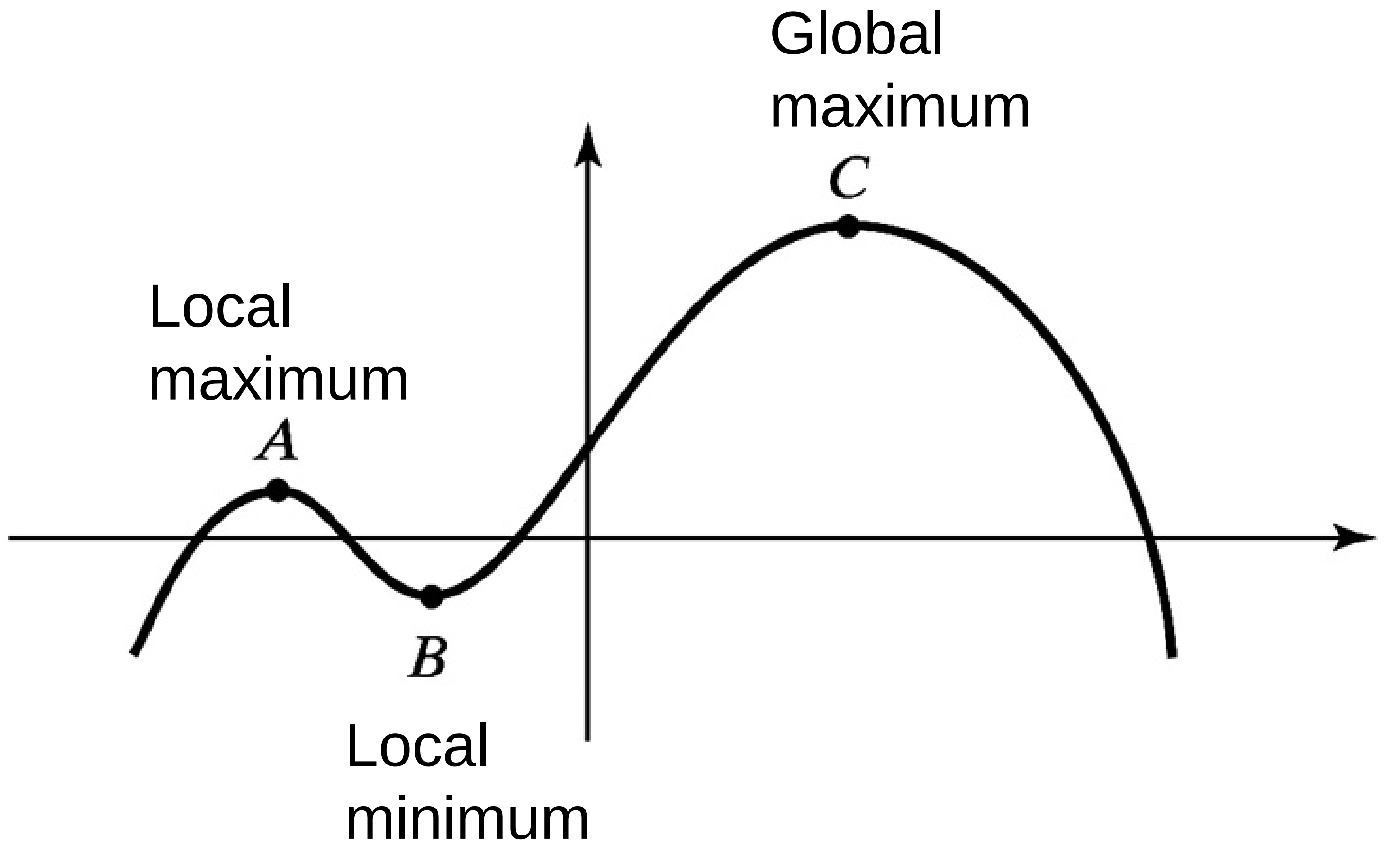
Properties of functions

- Extreme values: maximum and minimum
- Limits and continuity
- Monotonicity: increasing, decreasing
- Concavity and convexity

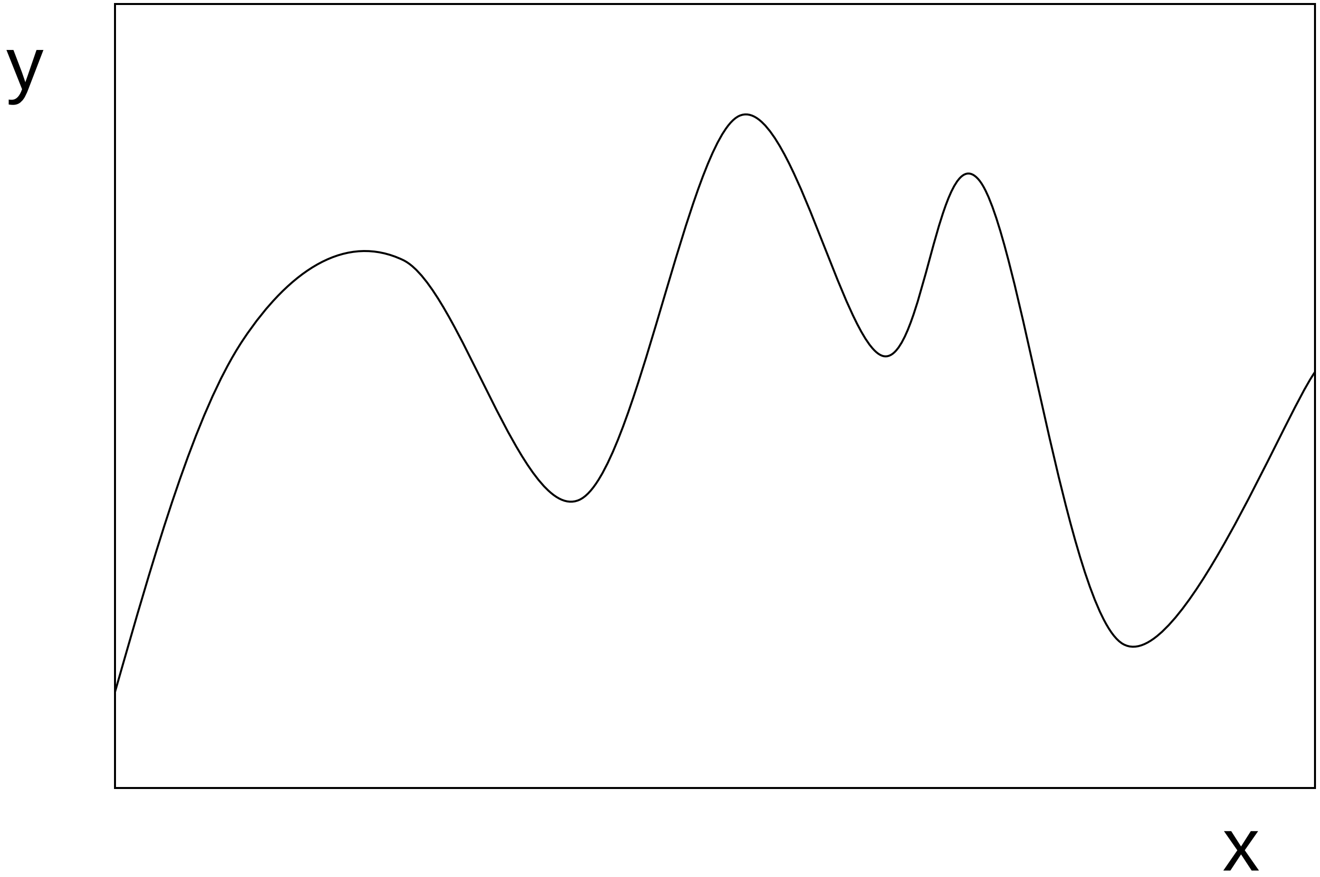
Model of rational behavior

- People have preferences
- People seek to make optimal decisions
  - Maximize utility
  - Maximize profit
  - Minimize cost
- Use optimization to find extreme values
  - Maxima and minima of functions

Extreme Values

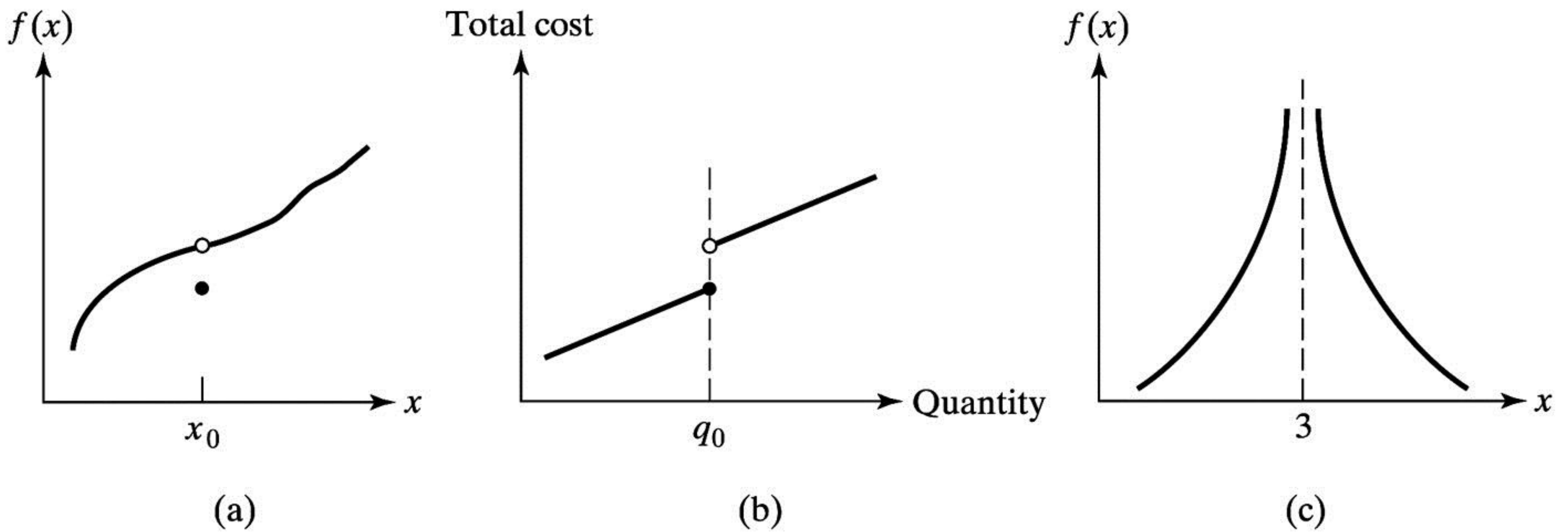


Is it a continuous function?



Yes! Can draw it without lifting pen.

Is it a continuous function?



No! No! No!

Limits

- Left-hand limit
  - Value of function as approach a from left

$$\lim_{x \to a^-} f(x)$$

- Right-hand limit
  - Value of function as approach a from right

$$\lim_{x \to a^+} f(x)$$

- Limit
  - Value of function as approach a in any direction

$$\lim_{x \to a} f(x)$$

  - Requires $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

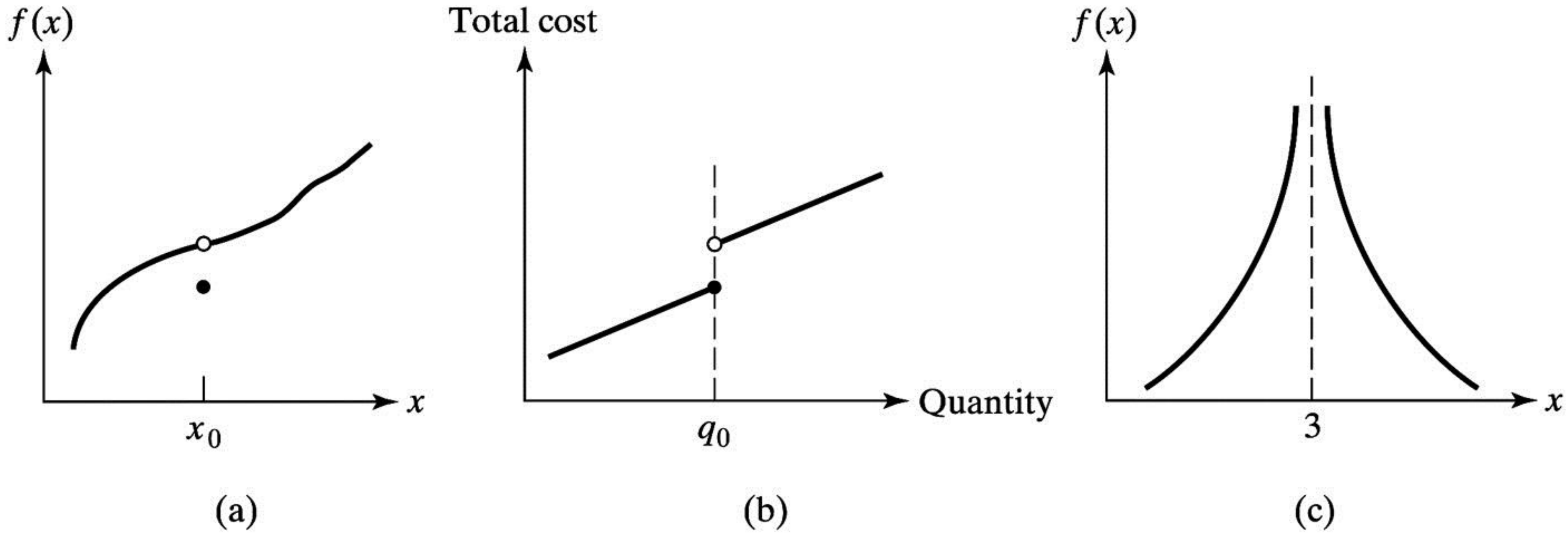
Continuity

- f(x) is continuous at a if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

- f(x) is a continuous function if it is continuous at all points x in domain X

Is it a continuous function?



No!

$$\lim_{x \to a} f(x) \neq f(a)$$

No!

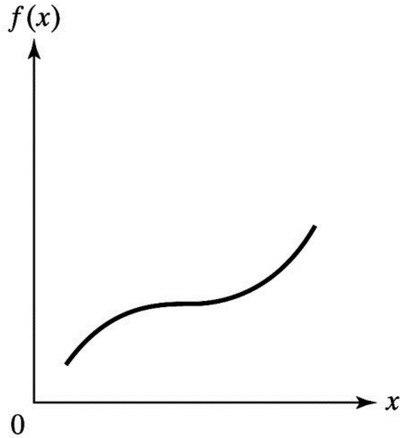
$$\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$$

No! f(3) not defined

Monotonicity

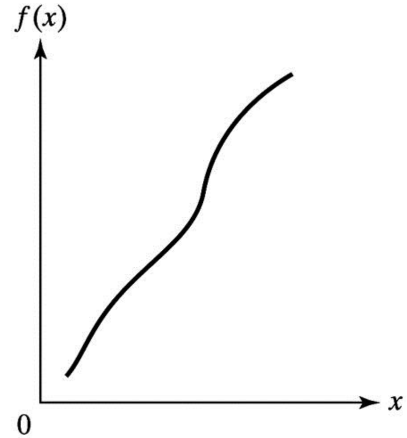
- If for all a,b in X with $a < b$,
  $f(a) \leq f(b)$ then f is *increasing*
  $f(a) \geq f(b)$ then f is *decreasing*
  $f(a) < f(b)$ then f is *strictly increasing*
  $f(a) > f(b)$ then f is *strictly decreasing*
- f is *monotonic* if it is increasing or decreasing
- f is *strictly monotonic* if it is strictly increasing or strictly decreasing

Monotonic functions



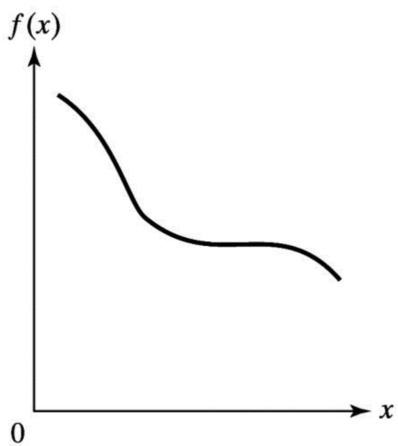
Increasing (not strictly)

(a)



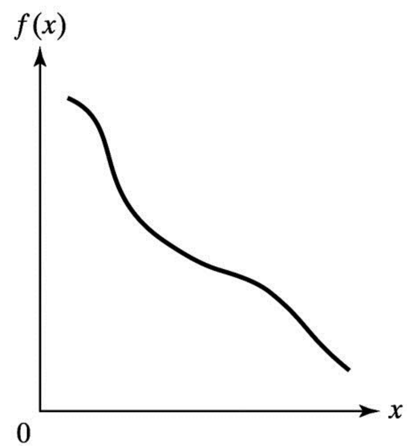
Strictly Increasing

(b)



Decreasing (not strictly)

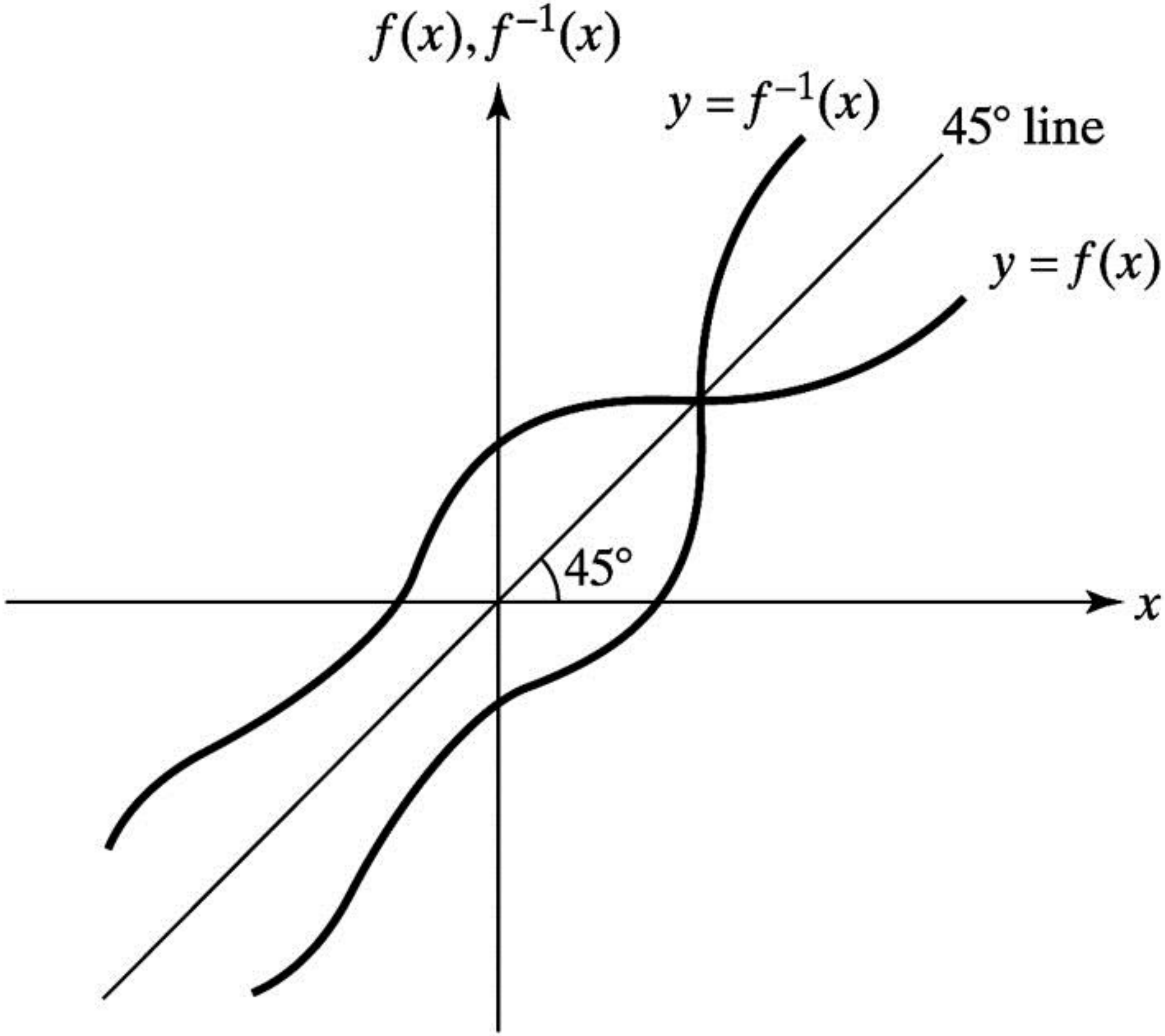
(c)



Strictly Decreasing

(d)

Strictly monotonic functions have inverses



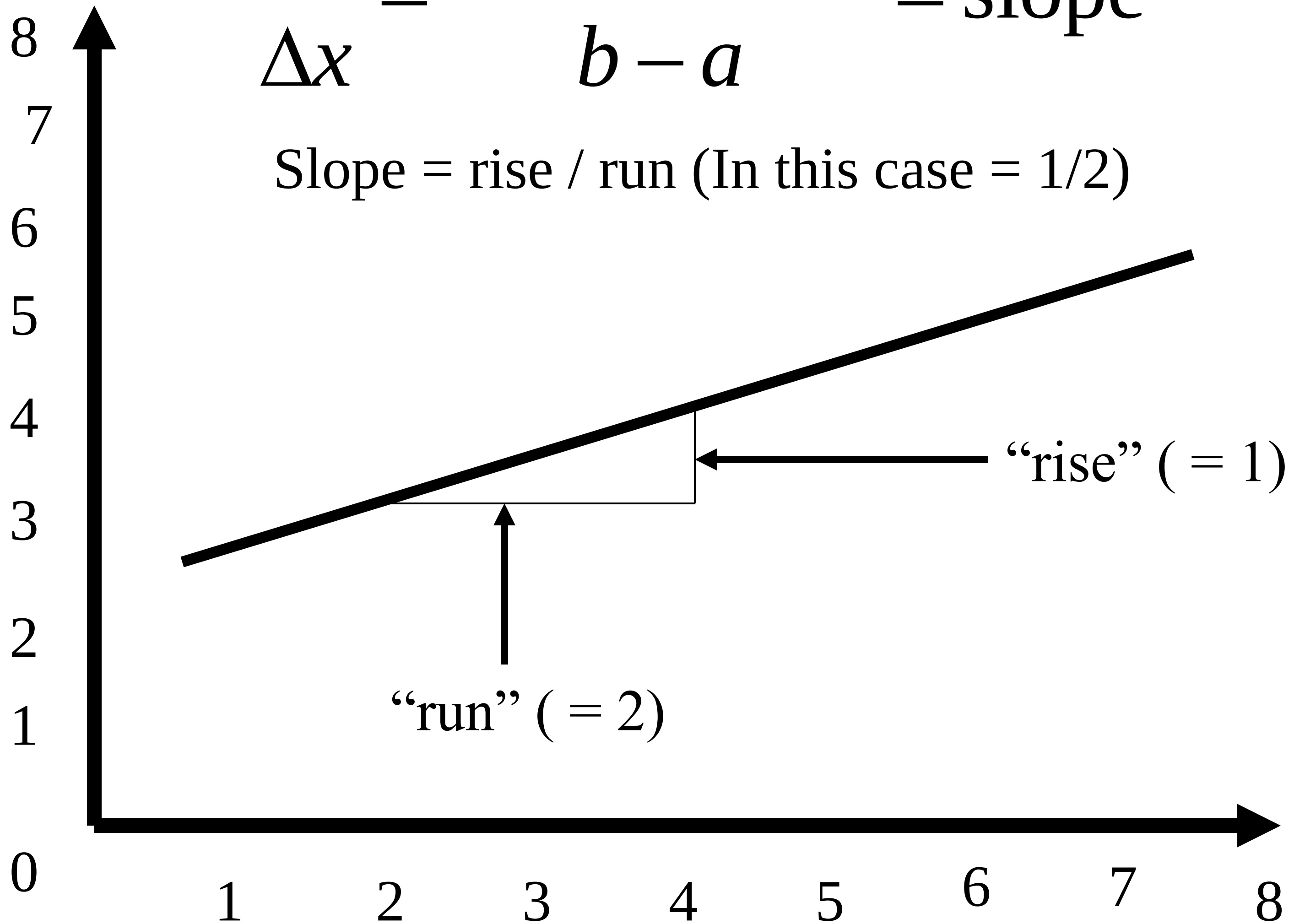
Average rate of change

- How quickly does f(x) change as x goes from a to b?

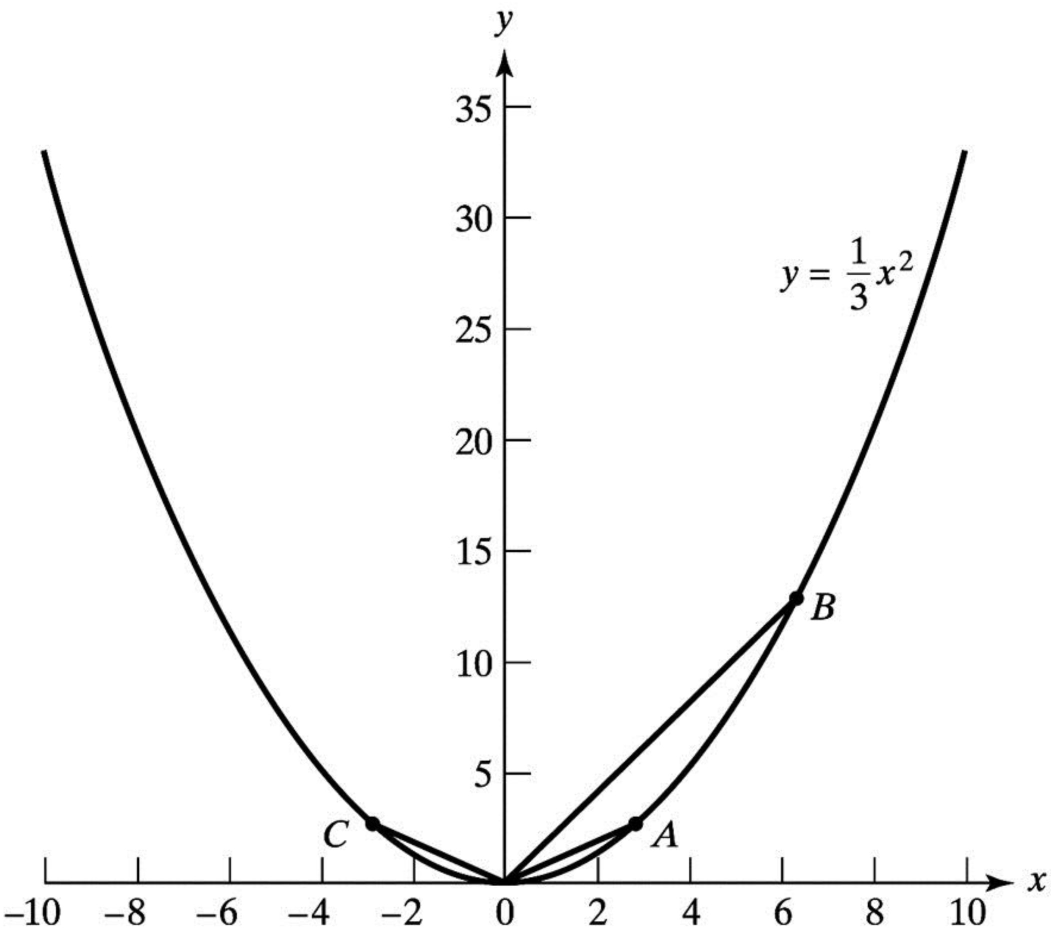
$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Average rate of change is constant for a linear function

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \text{slope}$$



Average rate of change is slope of secant line



$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

A: x from 0 to 3

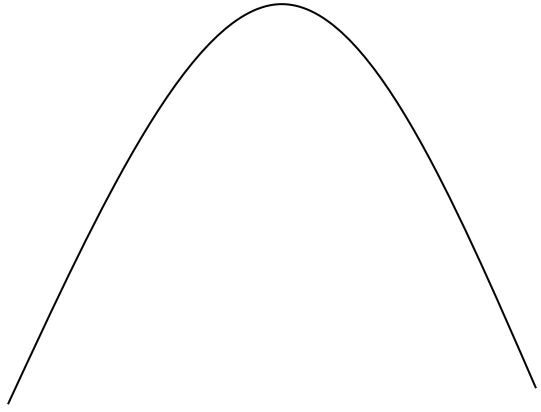
$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{3}3^2 - \frac{1}{3}0^2}{3 - 0} = 1$$

B: x from 0 to 6

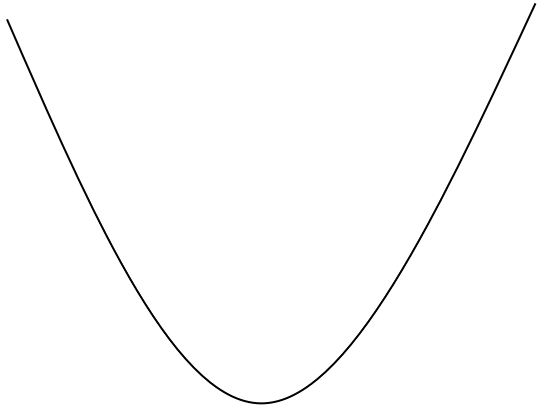
$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{3}6^2 - \frac{1}{3}0^2}{6 - 0} = 2$$

Convexity

- Concave

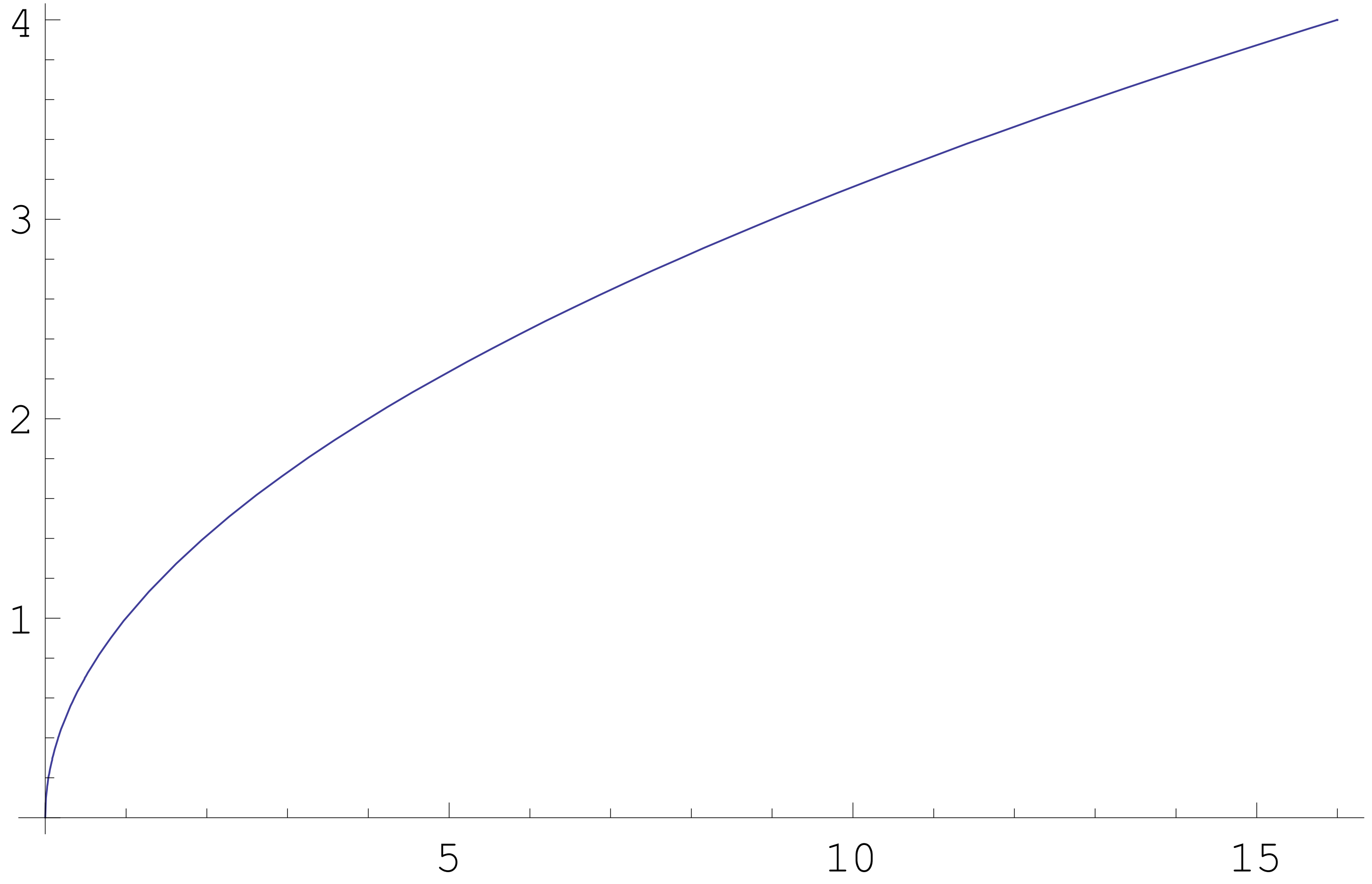


- Convex



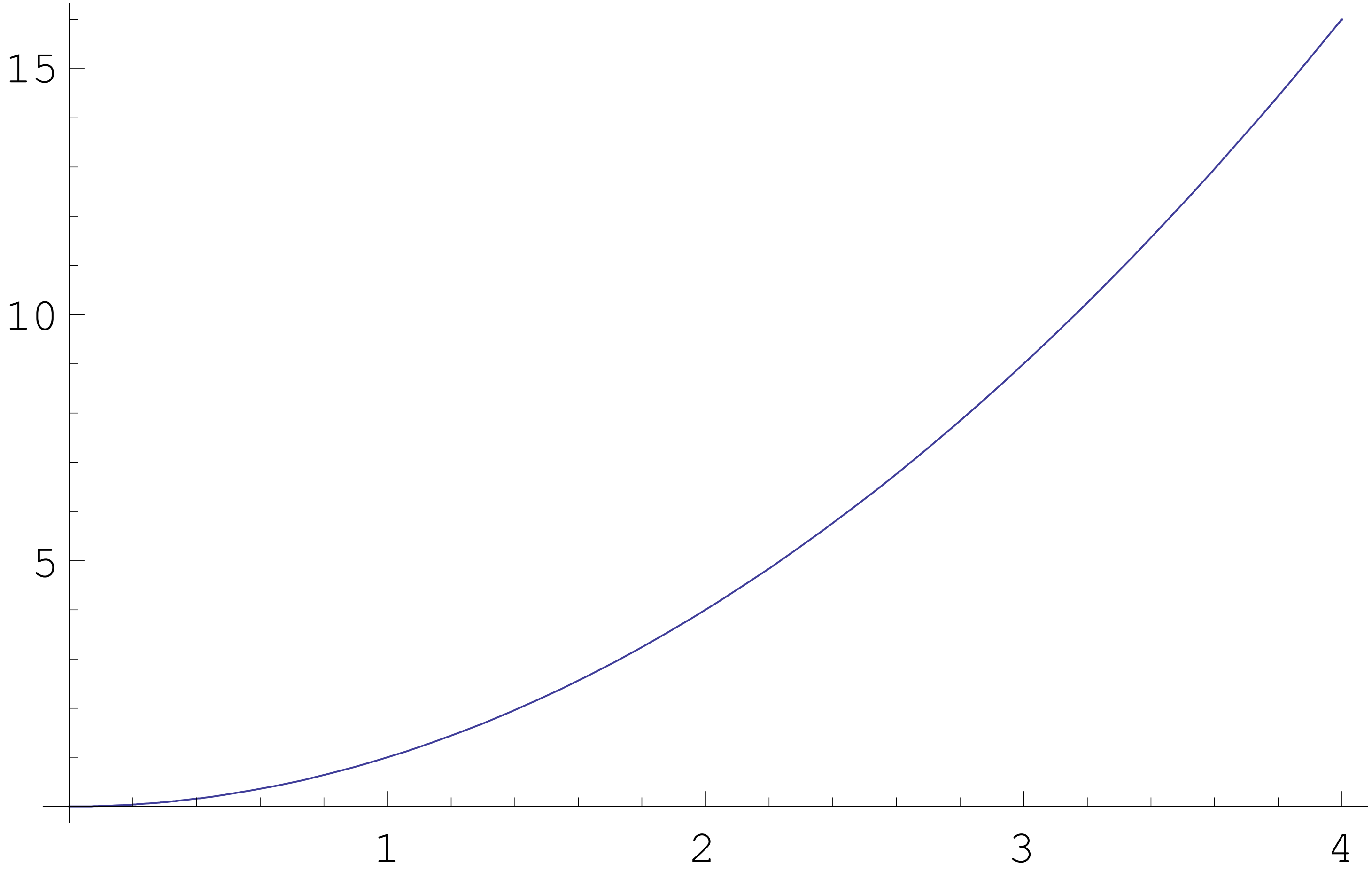
Why economists care?

- Diminishing marginal utility



Why economists care?

- Increasing marginal costs

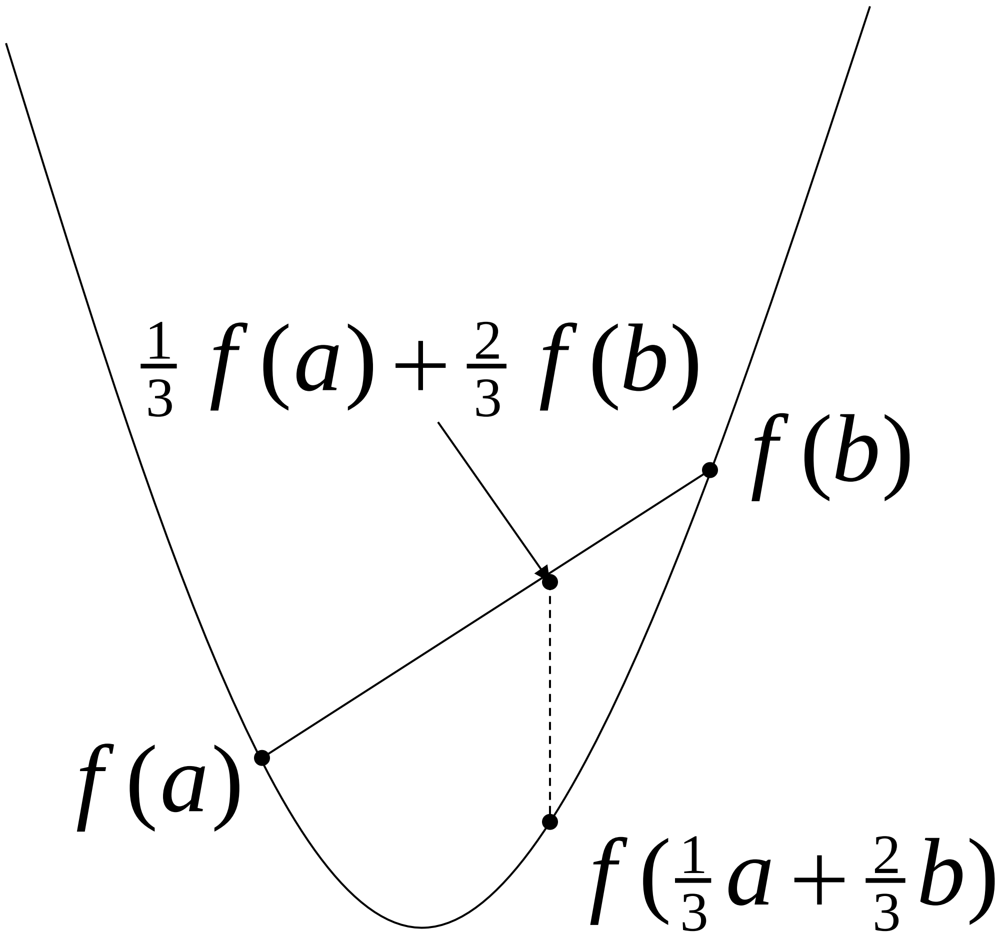


Equivalent definitions of convex

- f is convex if average rate of change is increasing
- f is convex if f is at or below all secant lines
- f is convex if for all a,b in X and α in (0,1),

$$f(\alpha a+(1-\alpha)b)\leq \alpha f(a)+(1-\alpha)f(b)$$

A convex function



Strictly convex

- f is strictly convex if average rate of change is strictly increasing
- f is strictly convex if f is below all secant lines
- f is strictly convex if for all $a \neq b$ in X and α in (0,1),

$$f(\alpha a + (1-\alpha)b) < \alpha f(a) + (1-\alpha) f(b)$$

Concave

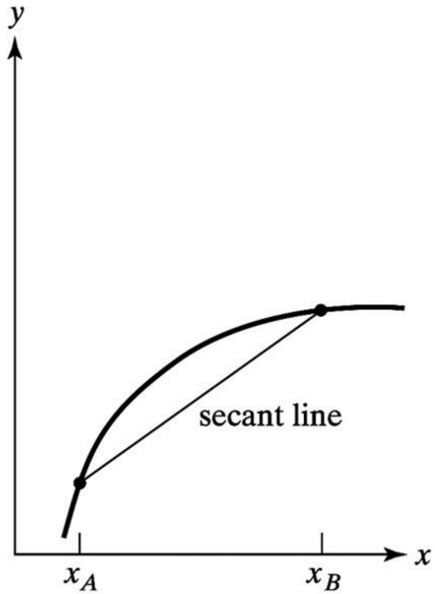
- f is concave if average rate of change is decreasing
- f is concave if f is at or above all secant lines
- f is concave if for all a,b in X and α in (0,1),

$$f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$$

Strictly concave

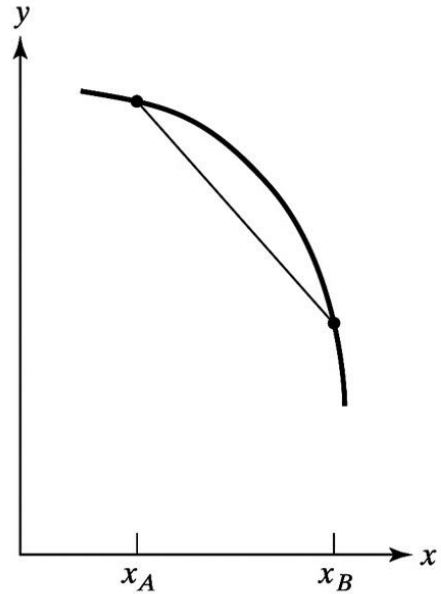
- f is strictly concave if average rate of change is strictly decreasing
- f is strictly concave if f is above all secant lines
- f is strictly concave if for all a≠b in X and α in (0,1),

$$f(\alpha a+(1-\alpha)b)>\alpha f(a)+(1-\alpha)f(b)$$



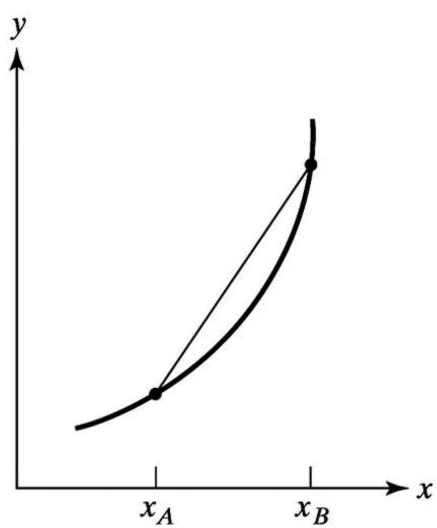
Strictly Concave

(a)



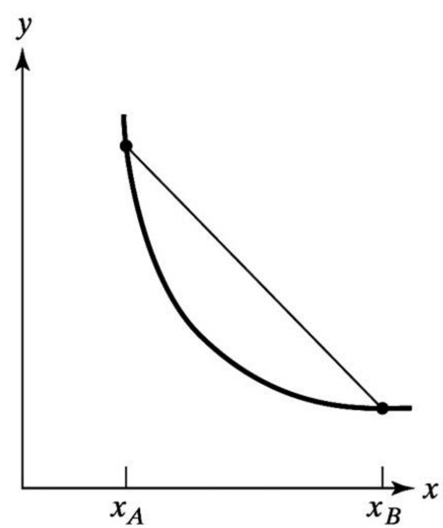
Strictly Concave

(b)



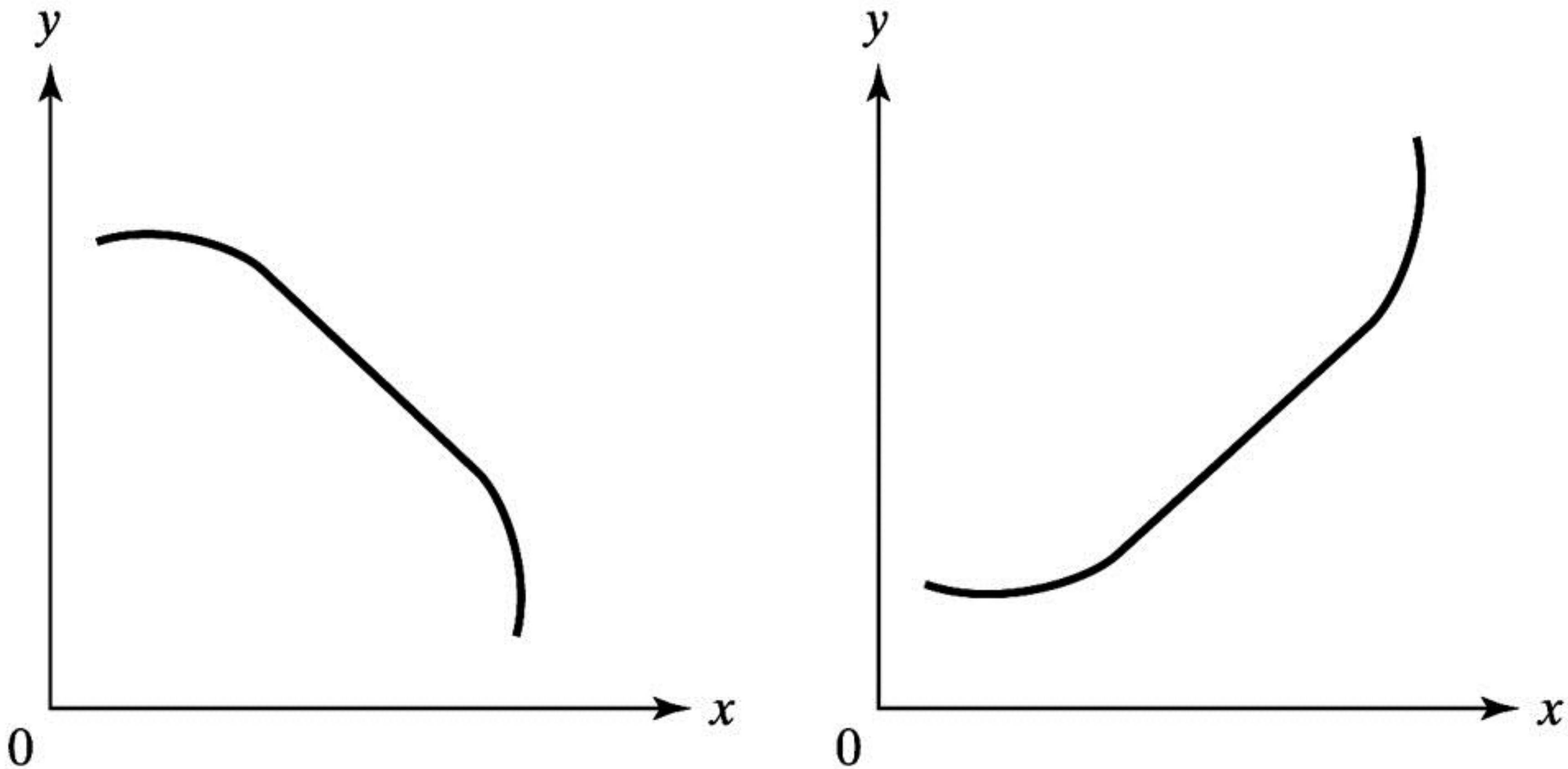
Strictly Convex

(c)



Strictly Convex

(d)



Concave

(a)

Convex

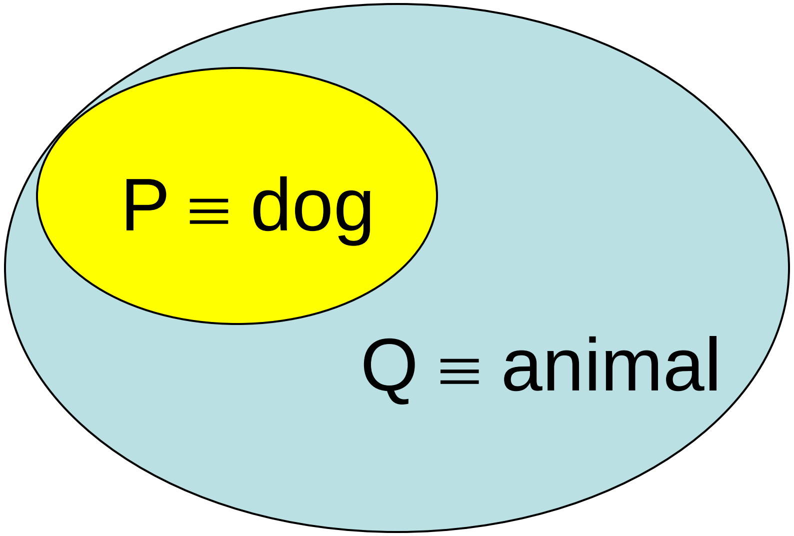
(b)

Some Logic

Necessary and sufficient conditions

All are equivalent

- If P then Q
- $P \Rightarrow Q$ (P implies Q)
- P only if Q
- P is sufficient for Q
- Q is necessary for P



Necessary and sufficient conditions

All are equivalent

- If P then Q & if Q then P
- $P \Rightarrow Q$ & $Q \Rightarrow P$
- $P \equiv Q$ (P equivalent to Q)
- $P \Leftrightarrow Q$ (P if and only if Q)
- P is necessary and sufficient for Q
- Q is necessary and sufficient for P

$P \equiv$ f has constant rate of change

$Q \equiv$ f is linear

Is a line convex or concave?

- Linear $\Rightarrow$ convex
- Linear $\Rightarrow$ concave
- Therefore, linear $\Rightarrow$ convex & concave
- Does convex & concave $\Rightarrow$ linear?
- Concave $\Rightarrow f(\alpha a+(1-\alpha)b)\geq \alpha f(a)+(1-\alpha)f(b)$
- Convex $\Rightarrow$ $f(\alpha a+(1-\alpha)b)\leq \alpha f(a)+(1-\alpha)f(b)$
- Both $\Rightarrow$ $f(\alpha a+(1-\alpha)b)= \alpha f(a)+(1-\alpha)f(b)$
- Therefore, linear $\Leftrightarrow$ convex & concave

Useful functions

- Power functions

$$f(x) = kx^p$$

- Polynomial functions

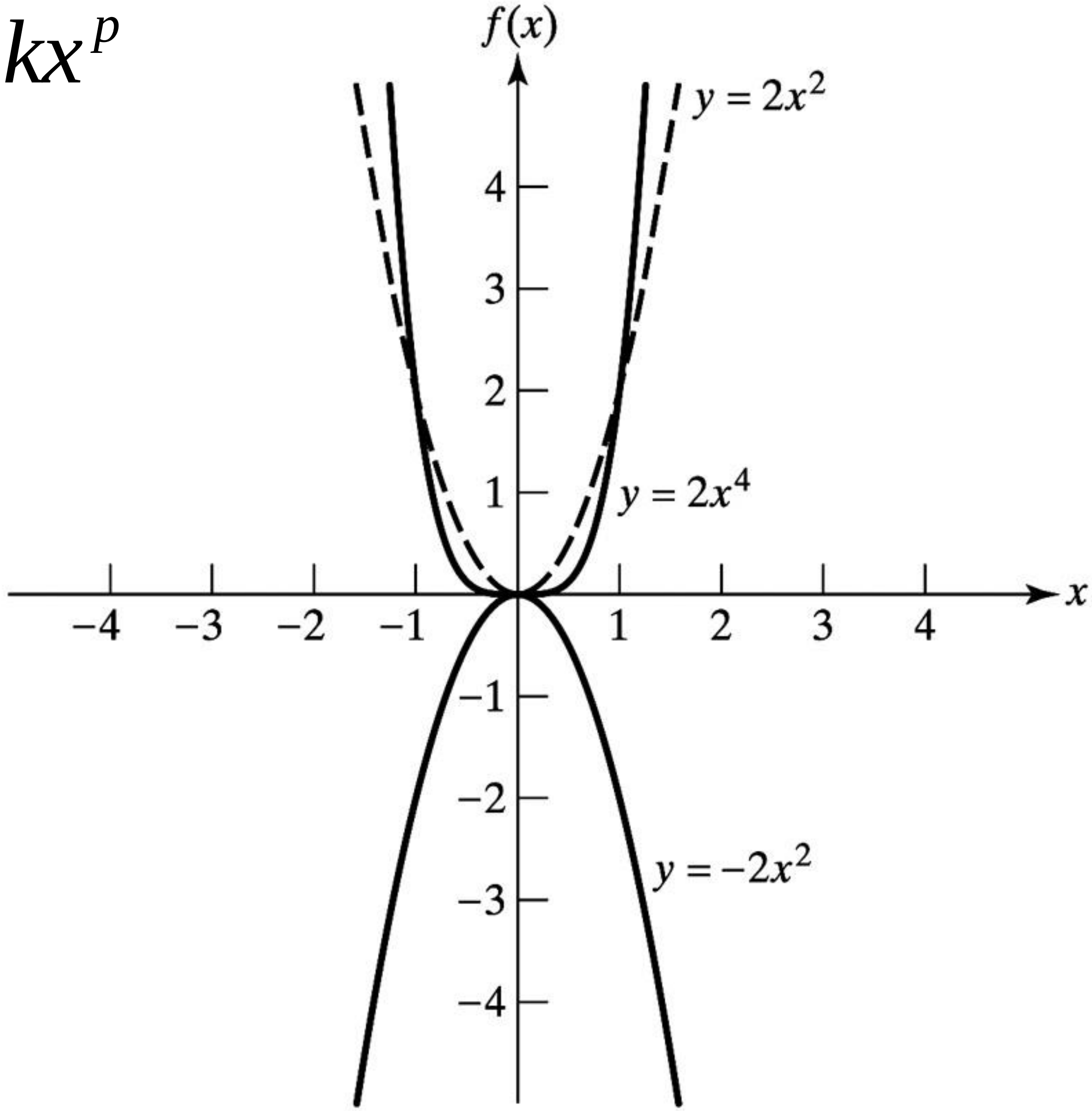
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

- Exponential functions

$$f(x) = kb^x$$

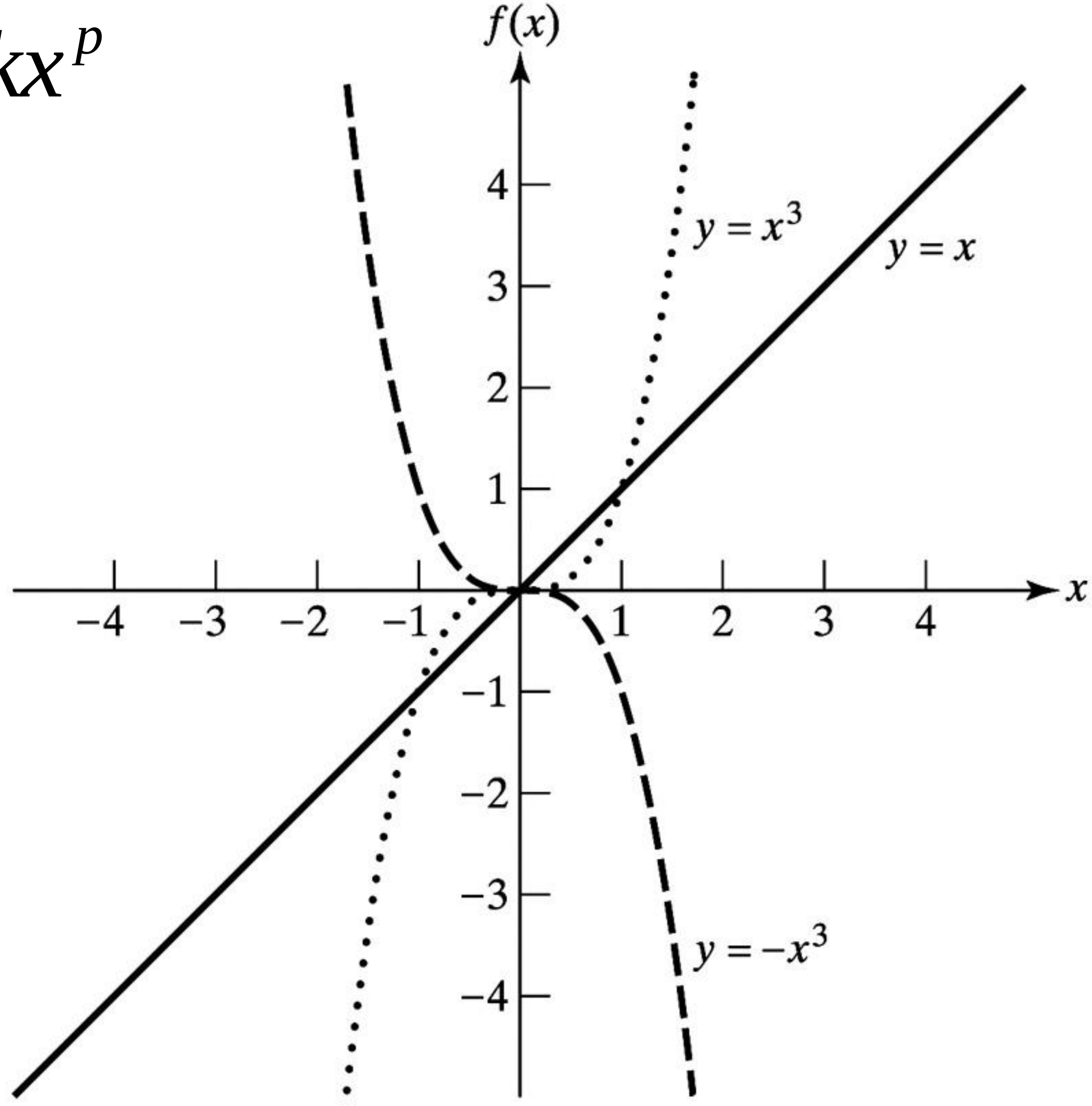
Power function (even exponent)

$$f(x) = kx^p$$



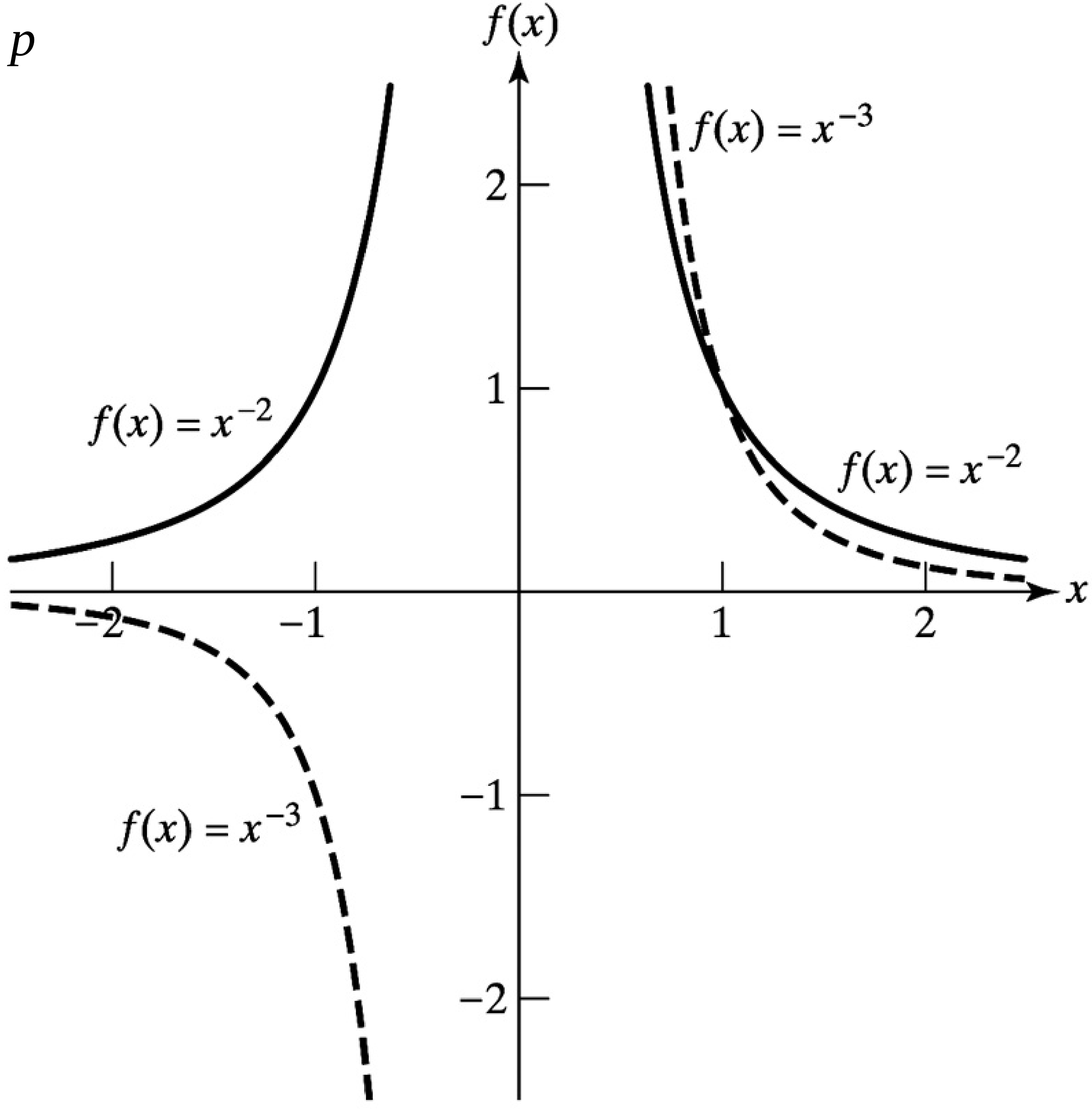
Power function (odd exponent)

$$f(x) = kx^p$$



Power function (negative exponent)

$$f(x) = kx^p$$



Rules of Exponents

Rule	Example
• $x^0 = 1$	• $2^0 = 1$
• $x^1 = x$	• $2^1 = 2$
• $x^{-1} = 1 / x$	• $2^{-1} = 1 / 2$
• $(x^a)^b = (x^b)^a = x^{ab}$	• $(2^1)^3 = (2^3)^1 = 8$
• $x^a x^b = x^{a+b}$	• $2^2 2^3 = 2^5 = 32$
• $x^a / x^b = x^{a-b}$	• $2^3 / 2^2 = 2^1 = 2$
• $x^a y^a = (xy)^a$	• $2^2 3^2 = 6^2 = 36$
• $x^a / y^a = (x/y)^a$	• $4^2 / 2^2 = (4/2)^2 = 4$
• $x^{1/a} = \sqrt[a]{x}$	• $9^{1/2} = \sqrt{9} = 3$

Polynomial functions

- Linear (1st order)

$$f(x) = a_0 + a_1 x$$

- Quadratic (2nd order)

$$f(x) = a_0 + a_1 x + a_2 x^2$$

- Cubic (3rd order)

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

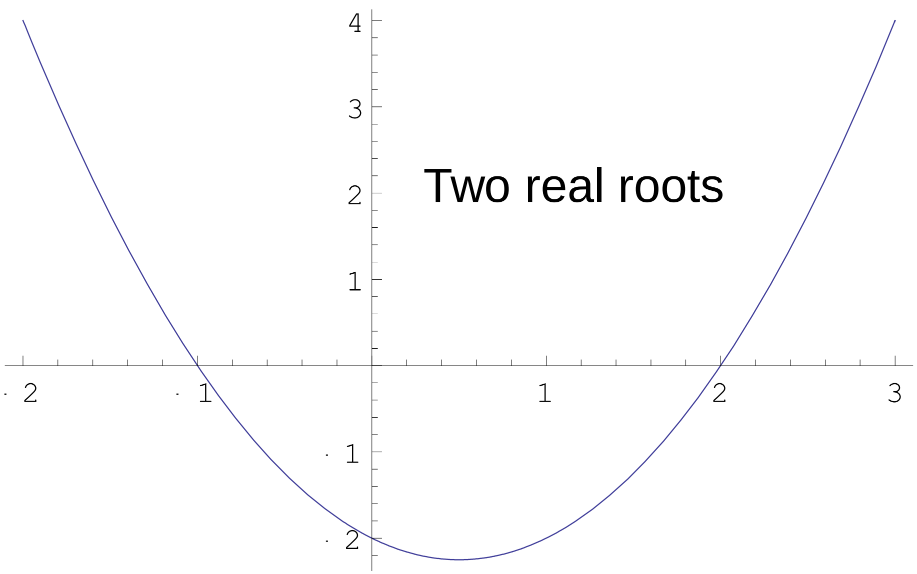
- General (nth order)

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

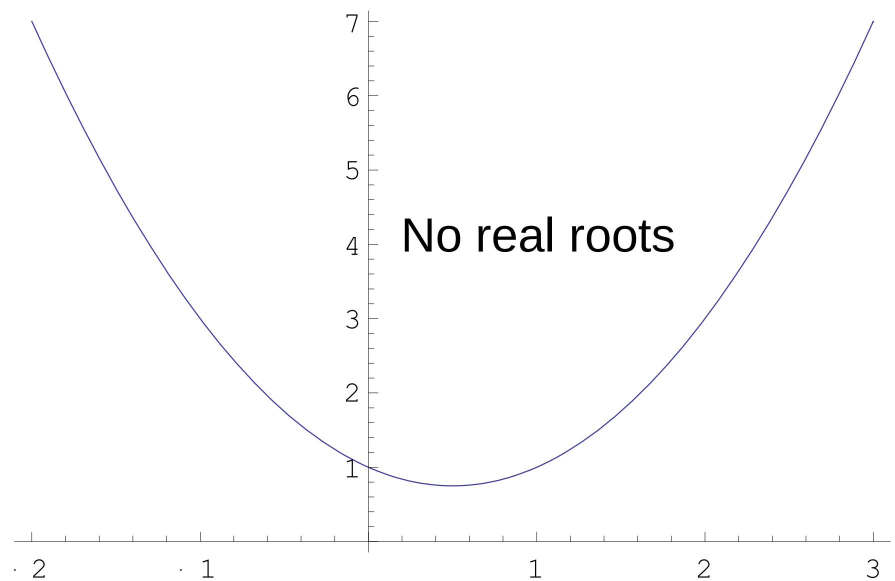
Roots of a polynomial equation

- Linear: ax + b = 0 ⇒ x = -b/a (one root)
- Quadratic:
  $ax^2 + bx + c = 0 \Rightarrow$ $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x^2 - x - 2 = 0$



$x^2 - x + 1 = 0$



Exponential function $f(x) = kb^x$