Lecture Note 4: Refinement

Subgame Perfection Revisited

Sequential Equilibrium

- Sequential Rationality
- Consistency
- Structural Consistency

Subgame Perfection
Perfect Information

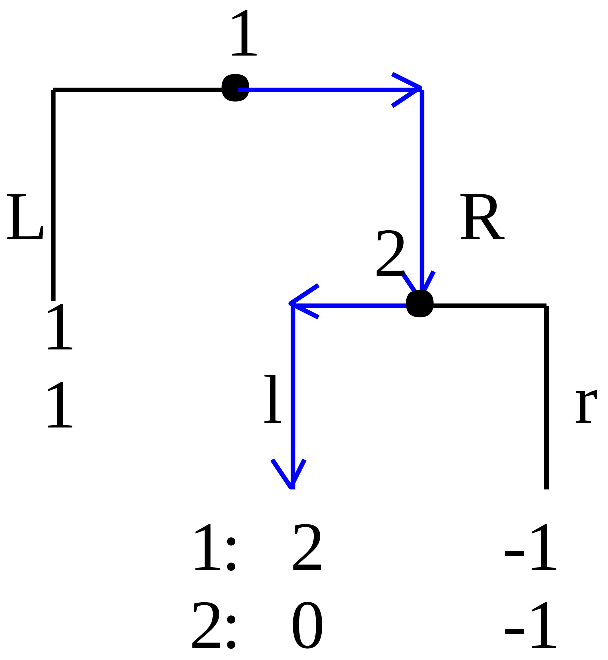
Game 1:

Normal Form

2

		l	r
1	L	$1, \underline{1}$	$\underline{1}, \underline{1}$
	R	$\underline{2}, \underline{0}$	$-1,-1$

Extensive Form



Definition

- A *subgame* consists of the game tree following a singleton information set, provided the resulting subtree does not cut any information sets.
- A Nash equilibrium in the original game is *subgame perfect* if it specifies Nash equilibrium strategies in every subgame.

Subgame Perfection Revisited

Game 1:

- Two NE: (L,r) and (R,l)
- (R,l) relies on an incredible threat by player 2.

- The only proper subgame starts at player 2's decision node. The condition that 2's choice be a NE strategy reduces to a requirement that 2 takes the action that results in the highest payoff. Thus, 2 must play l.

Theorem

Given a finite extensive-form game, there exists a subgame-perfect Nash equilibrium.

Subgame Perfection Imperfect Information

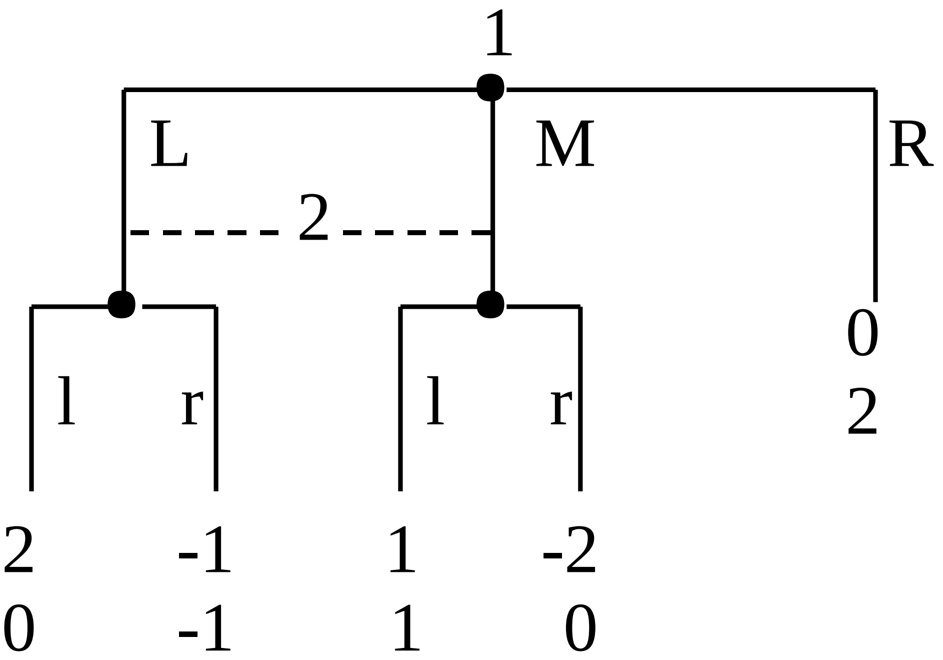
Game 2:

Normal Form

2

		l	r
	L	$\underline{2}$, $\underline{0}$	-1,-1
1	M	1, $\underline{1}$	-2, 0
	R	0, $\underline{2}$	$\underline{0}$, $\underline{2}$

Extensive Form



Subgame Perfection Imperfect Information

Problems:

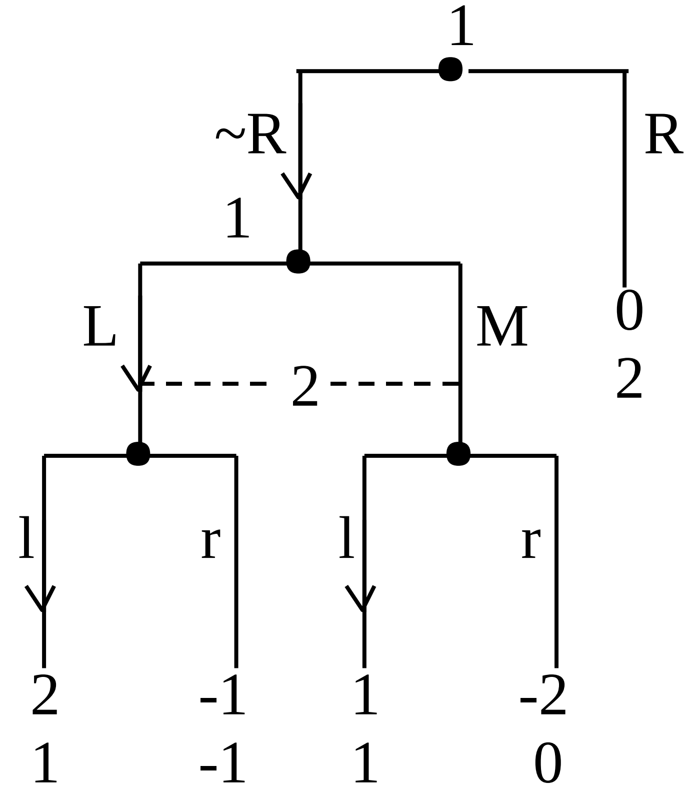
- Can’t prevent use of strictly dominated strategies.
  - In game 2, there are no proper subgames.
  - Both (L,l) and (R,r) are Nash equilibria.
  - (R,r) involves an incredible threat by player 2: if 2 gets the move, then r is a strictly dominated strategy for 2, so no matter what player 1 did it is not in 2's interest to play r. And yet, (R, r) is a SPE.

Subgame Perfection Imperfect Information

Problems:

- The concept is not invariant under inessential transformations of the game tree.

Game 2':



Subgame Perfection Imperfect Information

- In Game 2’, a proper subgame can be defined starting at player 1's second decision node.
- Since r is a dominated strategy for player 2 it cannot be NE behavior in this subgame, hence (R,r) is not subgame perfect.

Sequential Equilibrium

(Kreps and Wilson, *Econometrica* [1982])

Sequential Rationality

- Criterion for equilibrium in extensive games:
  - Every decision must be part of an optimal strategy for the remainder of the game.
  - Games w/ imperfect or incomplete information: At every juncture the player's subsequent strategy must be optimal with respect to some assessment of the probabilities of all uncertain events, including any preceding but unobserved choices made by other players (Savage’s axioms of choice under uncertainty).

Example: Game 2

- Sequential rationality in Game 2: there must exist beliefs that rationalize each action specified by the strategies.
- In game 2 there do not exist beliefs that can rationalize player 2's threat to play r if given the move: l maximizes 2's expected payoff given *any* beliefs about which node has been reached.

The only Nash equilibrium that is sequentially rational is (L,l)

Sequential Equilibrium

- Information possessed by the players in an extensive-form game is represented in terms of information sets.
- An information set h for player i is a set of i's decision nodes among which i cannot distinguish. This implies that the same set of actions must be feasible at every node in an information set.
- Let this set of actions be denoted A(h). Also, let the set of player i's information set be H_i and the set of all information sets be H.
- K&W restrict attention to games of perfect recall.

Sequential Equilibrium

- A behavior strategy for player i is the collection

$$\pi_i \equiv \{\pi_h^i(a)\}_{h \in H_i}$$

where for each $h \in H_i$ and each $a \in A(h)$, $\pi_h^i(a) \geq 0$ and

$$\sum_{a \in A(h)} \pi_h^i(a) = 1.$$

- $\pi_h^i(a)$ is a probability distribution that describes i's behavior at information set h.
- Strategy profile: $\pi = (\pi^1,\ldots,\pi^n)$, and $\pi^{-i} = (\pi^1,\ldots,\pi^{i-1},\pi^{i+1},\ldots,\pi^n)$.

Sequential Equilibrium

- Beliefs:
  - $\mu_h(x)$, where $\mu_h(x) \geq 0$ is the probability player i assesses that a node x $\in$ h has been reached and

$$\sum_{x \in h} \mu_h(x) = 1.$$

  - Let the beliefs throughout the tree be denoted by the collection

$$\mu \equiv \{\mu_h(x)\}_{h \in H}$$

  - Call the beliefs-strategies pair (μ, π) an ***assessment.***

Definition

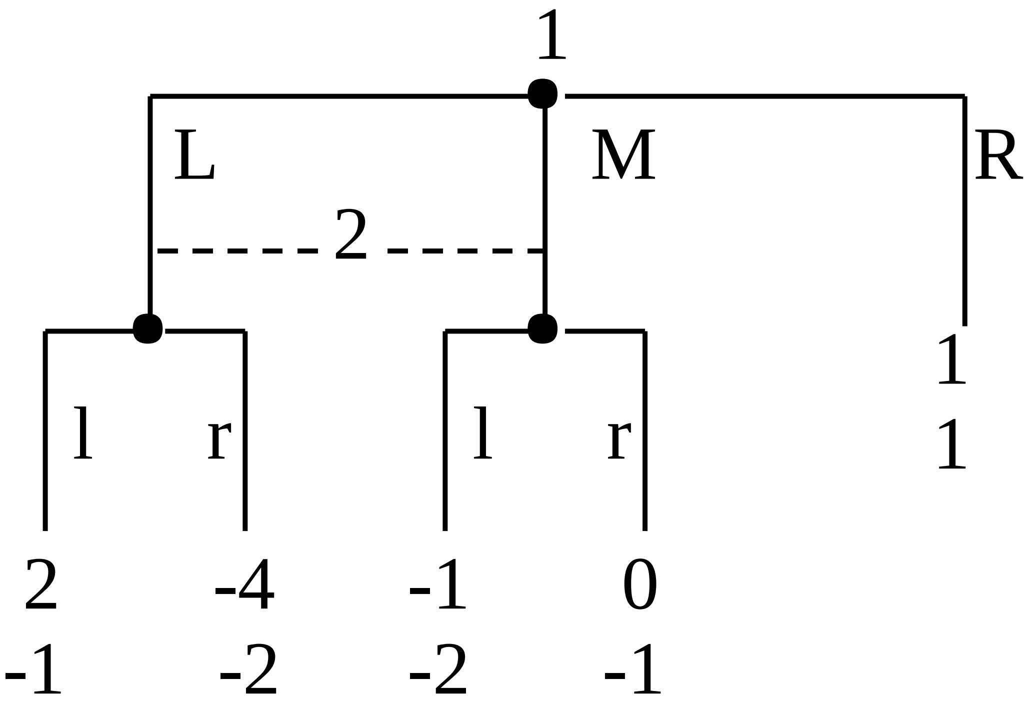
- An assessment (μ,π) is *sequentially rational* if given the beliefs μ no player i prefers at *any* information set h $\in$ H_i to change her strategy π_h^i, given the others' strategies π^{-i} (each player plays best response given her beliefs).

Sequential Rationality

Effects:

- First, it eliminates dominated strategies from consideration off the equilibrium path.
- Second, it elevates beliefs to the importance of strategies. This second effect provides a language—the language of beliefs—for discussing the merits of competing sequentially rational equilibria.

Game 3



Game 3

Two sequentially rational Nash equilibria:

- (L,l): forces the belief at player 2's information set to be that player 1 played L with probability one.
- (R,r): player 2's information set is not reached in equilibrium.
- If 2 believes that 1 played M with probability $p \geq 1/2$ then r is a best response.

Game 3

Problem:

- (R, r) is still implausible.
- Player 2 is using *implausible beliefs* rather than *incredible actions* to threaten player 1, and the threat succeeds in making 1 play R.

Game 3: Forward Induction

- Note that the reason 2's beliefs are implausible is that 1 gets a payoff of 1 from playing the equilibrium strategy R.
- This is superior to any payoff 1 could receive by playing M. And yet when 2's information set is reached 2 believes with probability at least 1/2 that M has been played.
- Instead, 2 might reason that 1 would not deviate unless there was something to be gained, so L must have been played with probability close to one, in which case l rather than r is the BR.

Definitions

- A *sequential equilibrium* is an assessment (μ,π) that is both *sequentially rational* and *consistent*.

- A strategy profile π is *totally mixed* if it assigns strictly positive probability to each action a $\in$ A(h) for each information set h $\in$ H.

Consistency

Definition:

- An assessment (μ,π) is *consistent* if there exists a sequence of totally mixed strategies π_n and corresponding beliefs μ_n derived from Bayes' rule such that

$$\lim_{n \to \infty}(\mu_n, \pi_n) = (\mu, \pi).$$

Theorem

For every finite extensive-form game there exists at least one sequential equilibrium. Also, if (μ,π) is a sequential equilibrium then π is a subgame-perfect Nash equilibrium.

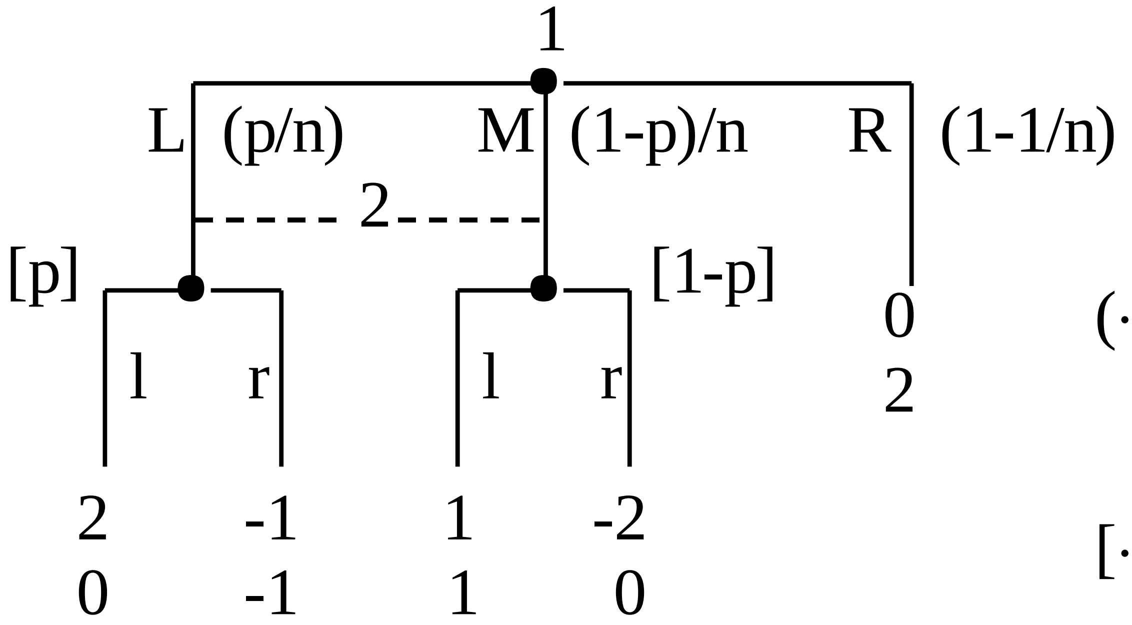
Consistency

Limitations:

- Given the Nash equilibrium strategy vector π = (R,r), *any* belief over the nodes in player 2's information set is consistent.

Game 2
Dominated Strategies

- Consider the totally mixed strategy specified below:



(·) denotes sequence of trembles (π_n),

[·] denotes sequence of beliefs [μ_n].

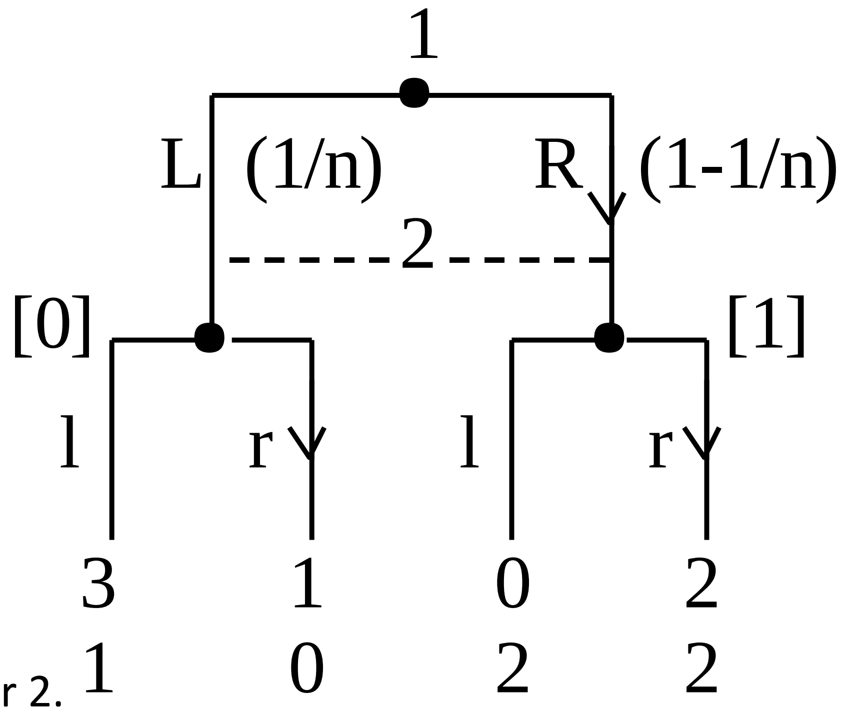
Game 2

- Strategy r is not sequentially rational for *any* possible belief
- This is how sequential equilibrium prevents strictly dominated strategies from being used as threats off the equilibrium path: they are not sequentially rational for any beliefs

Game 4

Weakly Dominated Strategies

		2	
		l	r
1	L	$\underline{3}$, $\underline{1}$	1, 0
	R	0, $\underline{2}$	$\underline{2}$, $\underline{2}$



- (L,l) and (R,r) are Nash equilibria.
- r is a weakly dominated strategy for player 2.

Weakly Dominated Strategies

- (R,r) is a sequential equilibrium strategy because the belief {[0],[1]} at 2's information set yields a consistent assessment.
- Sequential equilibrium has (essentially) no cutting power on weakly dominated strategies.

Structural Consistency

- *Definition:*

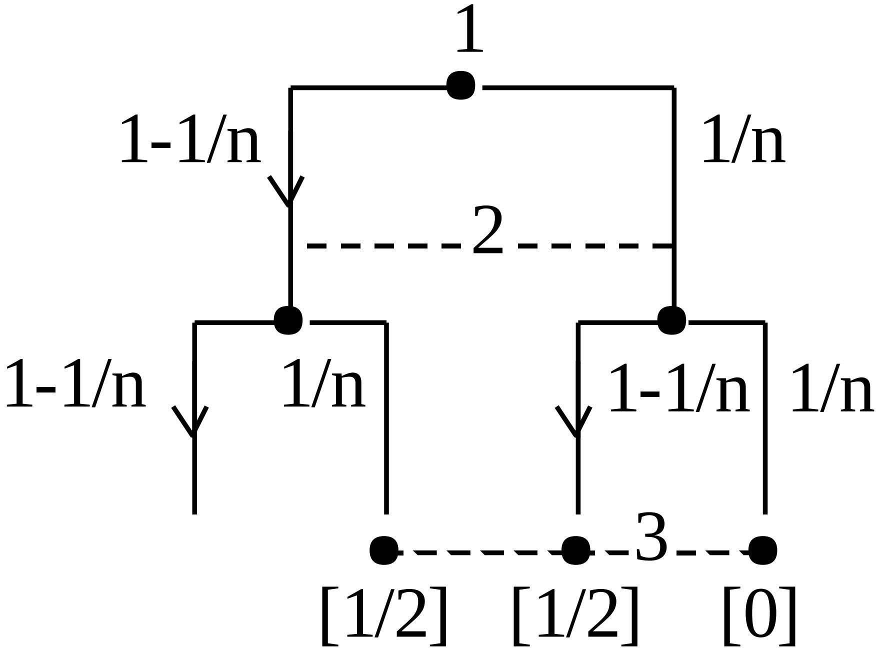
  An assessment (μ,π) is *structurally consistent* if for each h $\in$ H there exists a strategy π' that reaches h with positive probability and yields the beliefs $\mu_h(x)$ over nodes x $\in$ h via Bayes' rule.

Structural Consistency

- If π itself reaches h with positive probability, then π' = π will suffice.
- The definition applies to information sets that are off the equilibrium path defined by π—information sets π reaches with probability zero.
- π' is an alternative hypothesis about how the game has been played that can be adopted when events reject the original hypothesis π.

Structural Consistency

- Consistency does not imply Structural Consistency (Ramey, 1985).



Player 3's information set is unreached in equilibrium.

Structural Consistency vs. Consistency

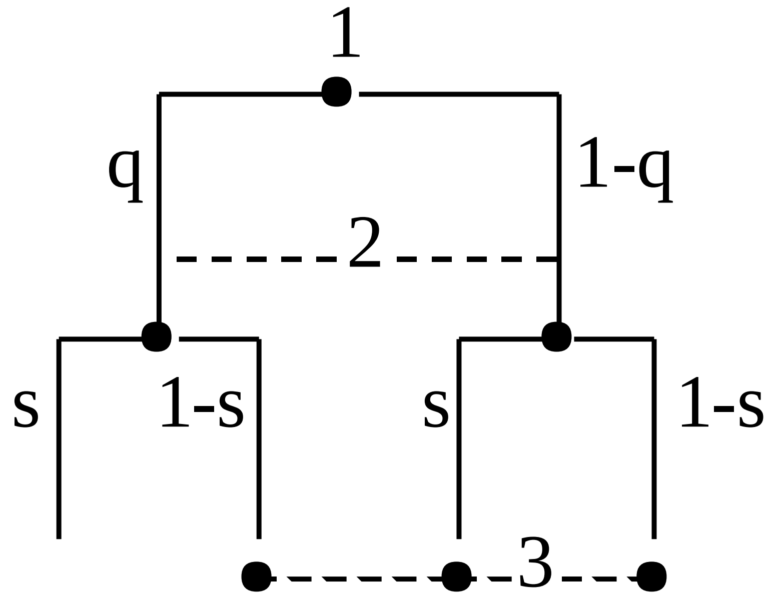
To show consistency:

- Let π_n be defined as above. Then μ_n at player 3's information set is of the form

  $(p_n, p_n, 1\text{-}2p_n)$ and $p_n \to 1/2$ as $n \to \infty$.

Structural Consistency vs. Consistency

To show structural consistency:



$$\left[\frac{q(1-s)}{1-qs},\frac{(1-q)s}{1-qs},\frac{(1-q)(1-s)}{1-qs}\right],$$

Structural Consistency vs. Consistency

To show structural consistency:

- Let player 1 play (q,1-q) and player 2 play (s,1-s).
- Then any strategy π' can be represented by the pair (q,s) $\in$ $[0,1]^2$. But the beliefs at 3's information set are then

$$\left[\frac{q(1-s)}{1-qs},\frac{(1-q)s}{1-qs},\frac{(1-q)(1-s)}{1-qs}\right],$$

and the only way to get zero probability on the third node is q=1

or s=1.

Lecture Note 4: Refinement

- Trembling-Hand Perfection
- Strategic Stability
  - Backward Induction and Invariance
  - Admissibility and Iterated Dominance
  - Stable Equilibrium

Trembling-Hand Perfection
(Selten [1975])

Definition:

- The behavioral strategy π is a *trembling-hand-perfect equilibrium* if there exists a sequence of totally mixed strategies π_n such that

  (1) $\pi_n \rightarrow \pi$, and

  (2) for each i *and each n*, π^i is a best response to π_n^{-i}.

Theorem

For every finite extensive-form game there exists at least one trembling-hand-perfect equilibrium. Also, if π *is a trembling-hand-perfect equilibrium then there exists beliefs* μ *such that* (μ,π) *is a sequential equilibrium.*

Trembling-Hand-Perfection Alternative Definition

- Let μ be the limit of the sequence μ_n. In terms of assessments, (μ,π) is trembling-hand perfect if there exists a sequence of totally mixed strategies π_n and associated beliefs μ_n such that

(1a) $(\mu_n,\pi_n) \rightarrow (\mu,\pi)$,

(1b) (μ,π) is sequentially rational, and

(2) for each i and n, π^i is a best response to π_n^{-i}.

Trembling-Hand-Perfection Alternative Definition

- (1a) and (1b) are consistency and sequential rationality, the defining properties of a sequential equilibrium. This proves the second sentence of the Theorem.
- From (2): sequential rationality is required only *at* the limit for a sequential equilibrium, but *also on the way to* the limit for a trembling-hand-perfect equilibrium.

Alternative Definition

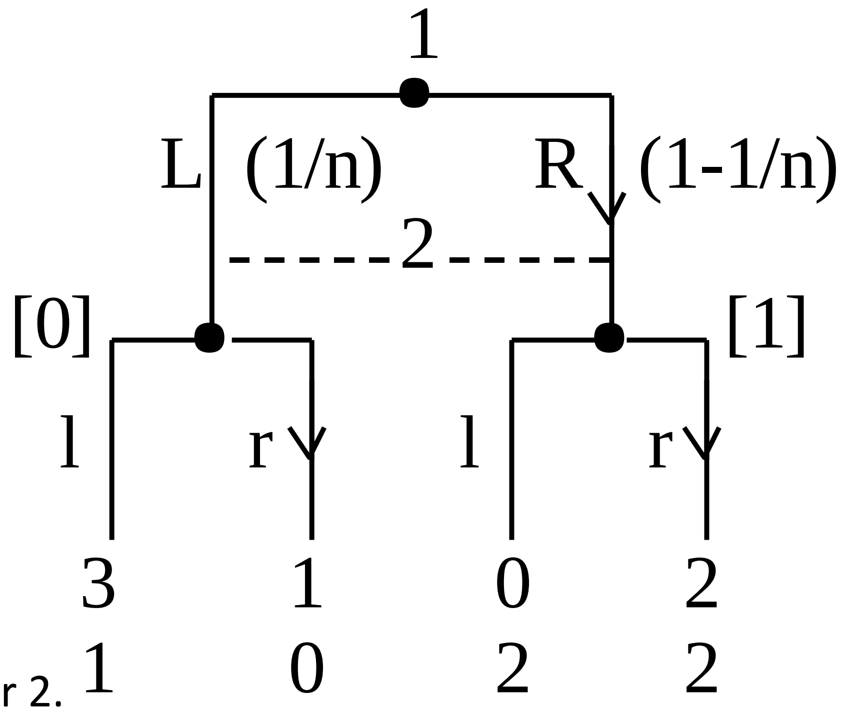
- Advantages: It eliminates weakly dominated strategies.

- Disadvantages: Condition (2) is harder to verify than (1a) and (1b).

Game 4 (from before)

Weakly Dominated Strategies

		2	
		l	r
1	L	$\underline{3}$, $\underline{1}$	1, 0
	R	0, $\underline{2}$	$\underline{2}$, $\underline{2}$



- (L,l) and (R,r) are Nash equilibria.
- r is a weakly dominated strategy for player 2.

Game 4

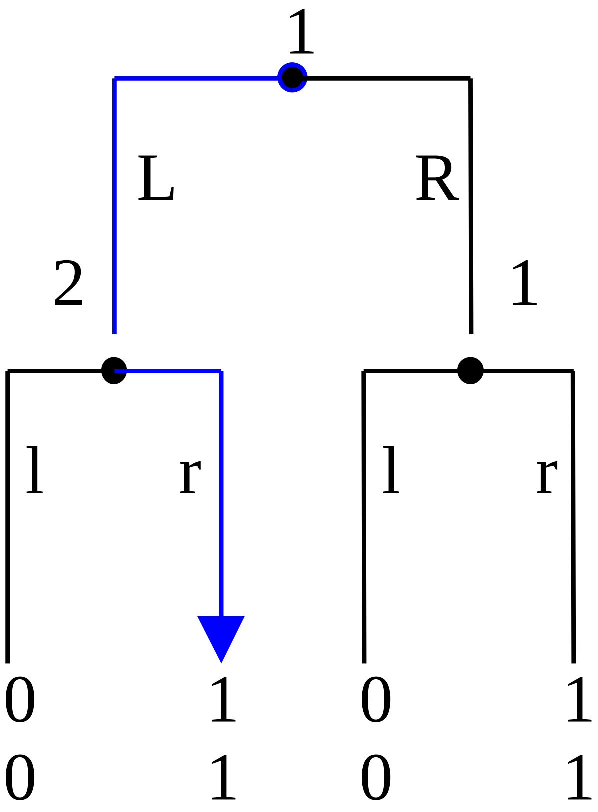
- Recall that r is only a best response for player 2 if she believes that player 1 has played R with probability one.
- If 2 assesses any positive probability that 1 played L then l, not r, is the best response.
- Thus, r fails (2).

Perfection in Normal Form Games

Is every extensive-form perfect equilibrium, normal-form perfect?

No.

Example



	l	r
Ll	0	1
Lr	0	1
Rl	0	0
Rr	1	1

Perfection in Normal Form Games

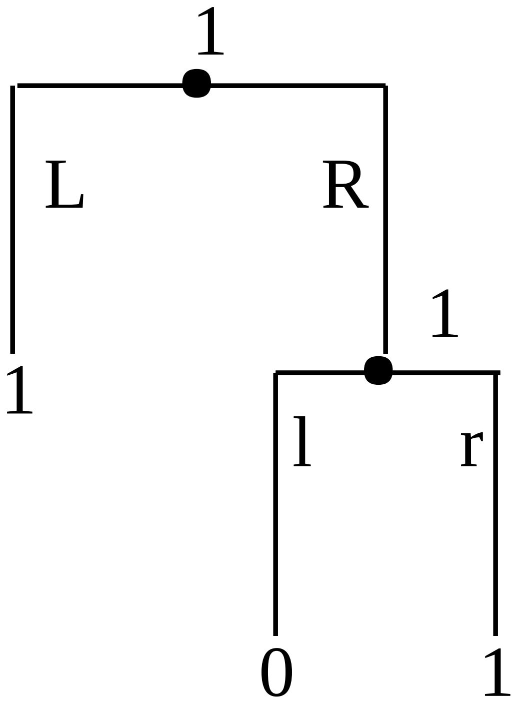
- (Rr,r) is the unique normal-form perfect equilibrium, but (Lr,r) is an extensive-form perfect equilibrium.
- Rr weakly dominates all of 1's other strategies in the normal form, but in the extensive form Lr does just as well Rr, since 1 is just as worried about her own tremble after moving R as 2's tremble after moving L.

Perfection in Normal Form Games

Is every normal-form perfect equilibrium an extensive form perfect equilibrium?

No.

Example



L	$\underline{1}$
Rl	0
Rr	$\underline{1}$

Example

- L is the unique extensive-form perfect equilibrium, but Rr is a normal-form perfect equilibrium as well.
- Rr is worse than L in the extensive form, because 1 has to be concerned about a tremble after moving R. But L and Rr are equivalent pure strategies in the normal form.

Strategic Stability

(Kohlberg and Mertens, *Econometrica* [1986])

- Which Nash equilibria are strategically stable; that is, once a particular strategy profile is specified, is it the case that no player can benefit from unilaterally deviating from his prescribed strategy?
- Does every game have a strategically stable equilibrium?

Definitions

- The *agent normal form* of a game tree is the normal form of the game between *agents*, obtained by letting each information set be manned by a different agent where we give any agent of the same player that players' payoff.
- A *behavior strategy* describes what to do at each information set (i.e., a (mixed) strategy for each of a player's agents).

Definitions

- A *sequential equilibrium* of an extensive form game is an n-tuple of behavior strategies which is a limit of the sequence (σ_m) of completely mixed behavior strategies such that every agent maximizes his payoff given strategy and beliefs at each information set implied by the limit of (σ_m).
- An ε*-perfect equilibrium* of a normal form game (Selten) is a totally mixed strategy vector, such that any pure strategy which is not a best reply has weight less than ε.

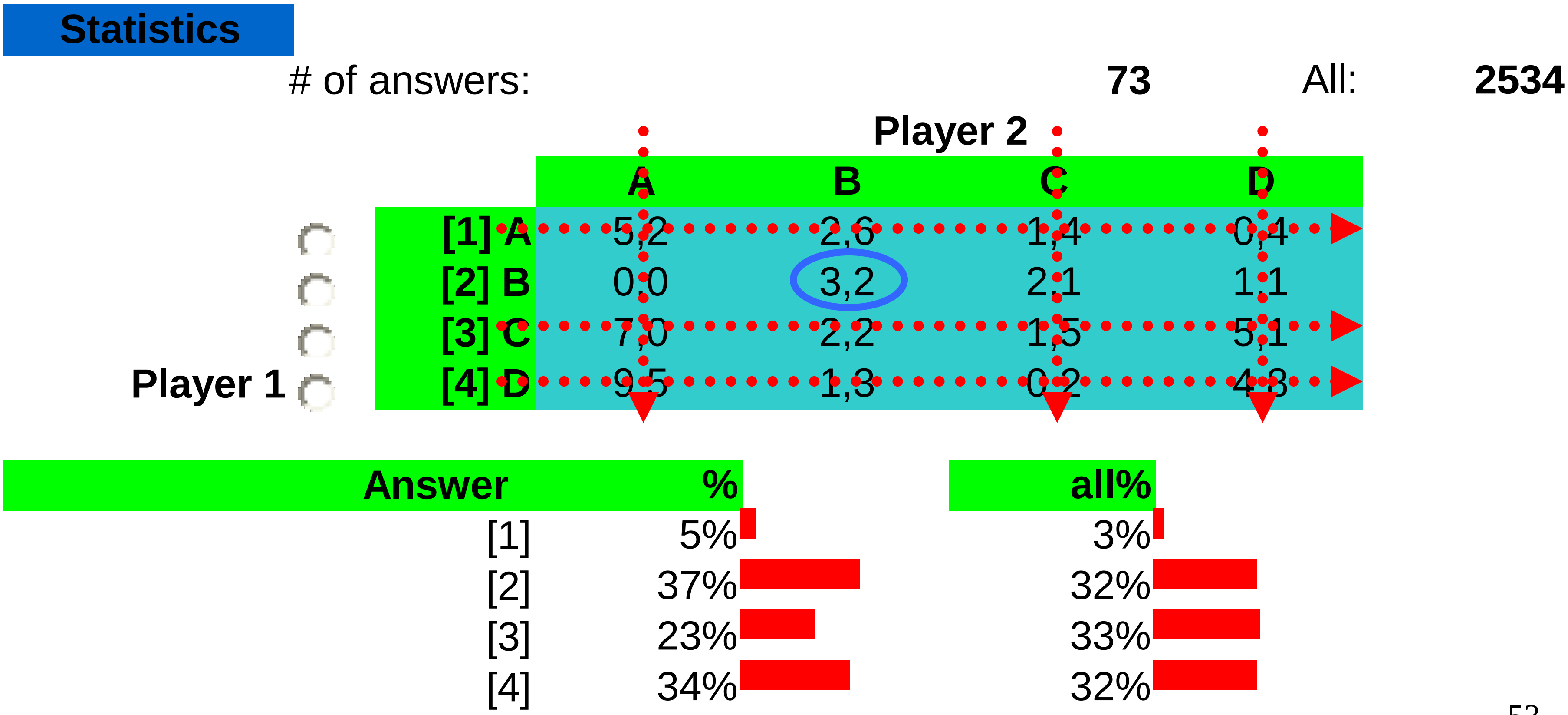
More Definitions

- An ε-*proper equilibrium* of a normal form game is a totally mixed strategy vector, such that whenever some pure strategy s_1 is a worse reply than some other pure strategy s_2, the weight on s_1 is smaller than ε times the weight on s_2.
- A *perfect (proper) equilibrium of a normal form game* is a limit ($\varepsilon \to 0$) of ε-perfect (proper) equilibria.
- A *perfect (proper) equilibrium of a tree* is a perfect (proper) equilibrium of its agent normal form.

Successive Elimination

You are player 1 in a two-person game with the following payoff matrix:

What will you play?



Statistics

\# of answers: **73** All: **2534**

Player 2

Player 1	**A**	**B**	**C**	**D**
○ **[1] A**	5,2	2,6	1,4	0,4
○ **[2] B**	0,0	3,2	2,1	1,1
○ **[3] C**	7,0	2,2	1,5	5,1
○ **[4] D**	9,5	1,3	0,2	4,3

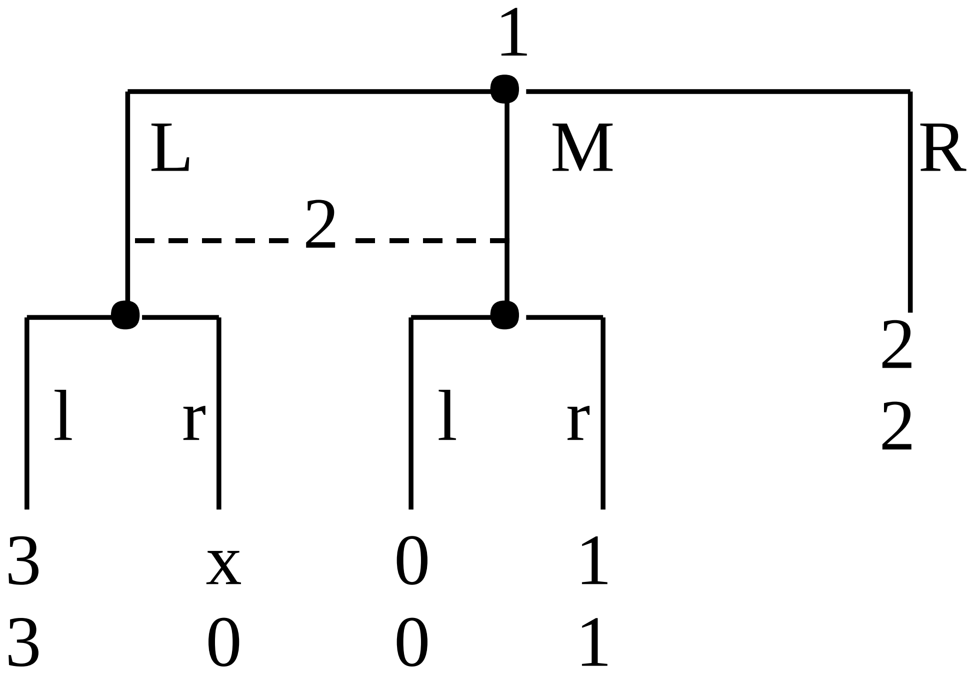
Answer	**%**	**all%**
[1]	5%	3%
[2]	37%	32%
[3]	23%	33%
[4]	34%	32%

Backwards Induction Rationality and Invariance

- A necessary condition for stability is sequential rationality (backwards induction): the players act optimally at every point in the game tree.
- Sequential, perfect, and proper equilibria satisfy this condition.
- Problem: the set of sequential equilibria is not invariant to inessential transformations of the game.

Game 5

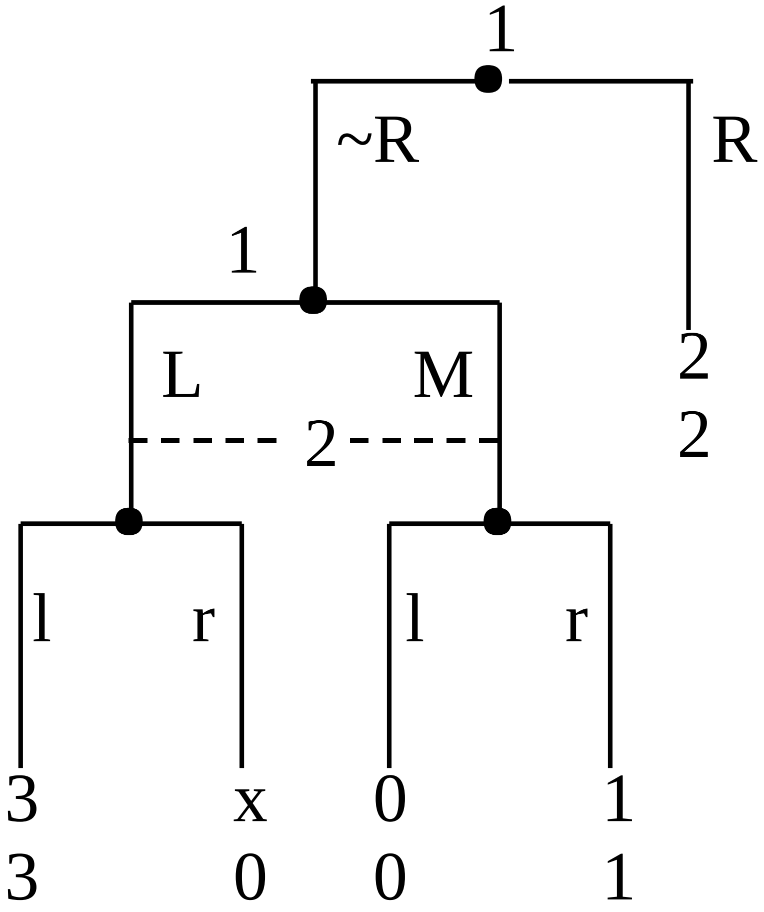
- $1 < x \leq 2$.



(R,r) is a sequential equilibrium

Game 5'

- $1 < x \leq 2$.



(R,r) is **not** a sequential equilibrium

Invariance

Requirement 1:

- It seems natural to require that a strategically stable equilibrium of a game tree must be sequential in any other game tree having the same normal form. Is such a requirement too strong to guarantee existence?

Proposition 0

A proper equilibrium of a normal form is sequential in any tree with that normal form.

Proof

- Let x = lim x_ε be a proper equilibrium where the x_ε are ε-proper equilibria.
- Given a tree, let σ_ε be behavioral strategies equivalent to x_ε and let μ_ε be the vector of conditional probabilities they imply on information sets, and let $\sigma_\varepsilon \to \sigma$ and $\mu_\varepsilon \to \mu$.
- We have to show that σ is such that each agent maximizes her payoff given μ and the strategies of the others.

Proof

By Contradiction:

- There is some player, say 1, and a last information set for her, say J, s.t. σ^1 assigns positive probability to a move in J, say L, whose expected payoff (given μ and σ) is less than that of another move, say R.
- Clearly, the same is true for μ_ε and σ_ε, provided $\varepsilon > 0$ is sufficiently small.

Proof

- It follows that every normal form strategy of 1 that does not avoid J and chooses L in J has a smaller expected payoff, given x_ε, than a modification of that strategy that chooses R and then continues as in σ^1.
- Since x_ε is ε-proper, x_ε^1 assigns the first strategy less than ε times the probability of the second strategy.

Proof

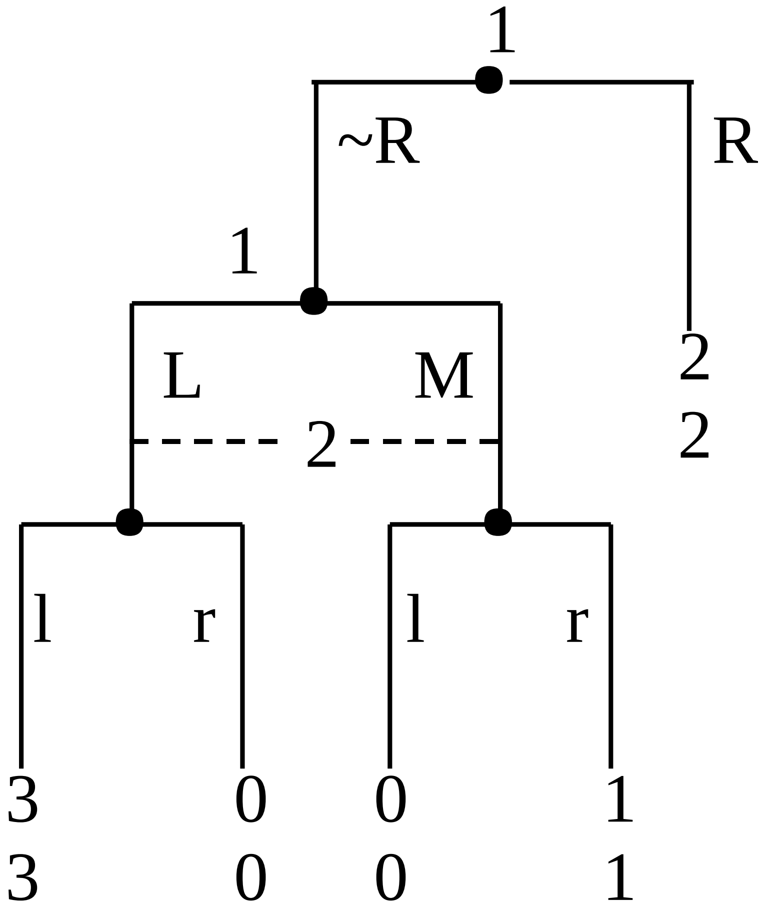
- It follows that σ_{ε}^{1} assigns to L a probability of at most kε, where k is the number of normal-form strategies of 1.
- Letting $\varepsilon \to 0$, we see that σ^{1} assigns to L zero probability, a contradiction.■

Invariance

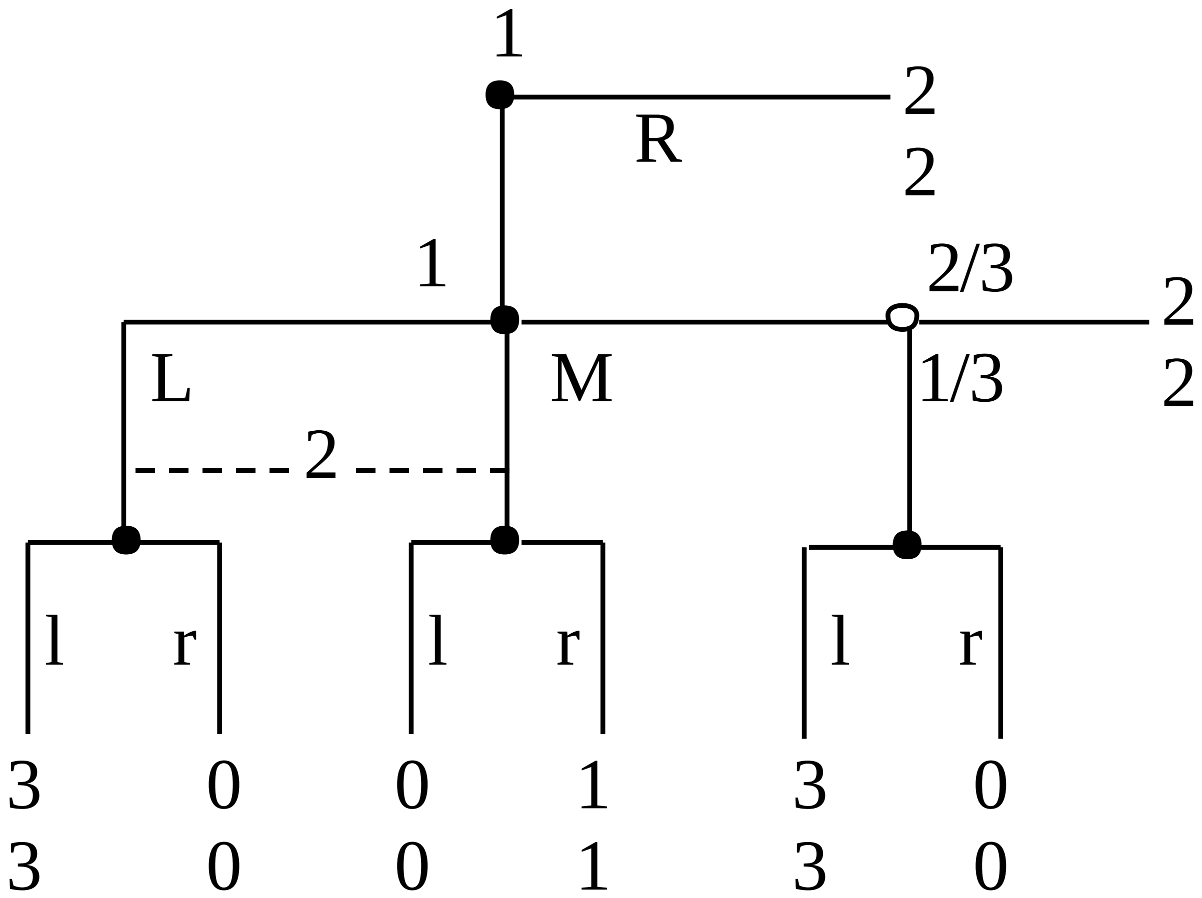
Requirement 2

- A further invariance requirement is that strategically stable equilibria depend only on the *reduced normal form*, determined by eliminating all pure strategies that are convex combinations of other pure strategies.

Game 6



Game 6’



Invariance

- G-6' is the same as G-6 with the explicit inclusion of the mixed strategy in which R is played with probability 2/3 and L is played with probability 1/3.

- Both (L,l) and (R,r) are sequential equilibria in G-6, but in G-6' (L,l) is the only sequential equilibrium: the randomized strategy strictly dominates M for player 1, so that 2 must but zero weight on node M and so 2 must play l in which case 1 plays L.

Admissibility and Iterated Dominance

Strategies that are not *(weakly) dominated* are called *admissible*.

- Both perfect and proper equilibria satisfy admissibility (dominated strategies are never played with positive probability on or off the equilibrium path), but sequential equilibria can involve weakly dominated strategies.

Iterated Dominance

- Problem: requiring *iterated dominance* could be incompatible with existence.

- Example:

		2	
		L	R
	U	$\underline{3}, \underline{2}$	$\underline{2}, \underline{2}$
1	M	$1, \underline{1}$	$0, 0$
	D	$0, 0$	$1, 1$

Iterated Dominance

- The equilibrium (either (3,2) or (2,2)) found by iterated dominance depends on the order in which dominated strategies are eliminated. Both (2,2) and (3,2) are inadmissible.
- In order to get existence, it is necessary to *include* all equilibria that satisfy iterative dominance, rather than *exclude* equilibria that can be eliminated by iterative dominance.

Stable Equilibria

Desireable properties for a set-valued definition of stability

- *Existence*: Every game has at least one solution.
- *Connectedness*: Every solution is connected; i.e., the solution is a connected set in the simplex of mixed strategies.
- *Backwards Induction*: A solution of a tree contains a backwards induction (either sequential or perfect) equilibrium of the tree.
- *Invariance*: A solution of a game is also a solution of any equivalent game.

Stable Equilibria

- *Admissibility*: The players' strategies are undominated at any point in a solution.
- *Iterated Dominance*: A solution of a game G contains a solution of any game G' obtained from G by deletion of a dominated strategy.
- *Forward Induction*: A solution must remain so after deletion of a strategy which is an inferior response for all strategies contained in the solution.

Proposition 1

The set of Nash equilibria of any game has finitely many connected components. At least one is such that for any equivalent game (i.e., same reduced normal form) and for any perturbation of the normal form there is a Nash equilibrium close to this component.

Hyperstable Equilibrium

S is a *hyperstable set of equilibria* in a game G if it is minimal with respect to the following property:

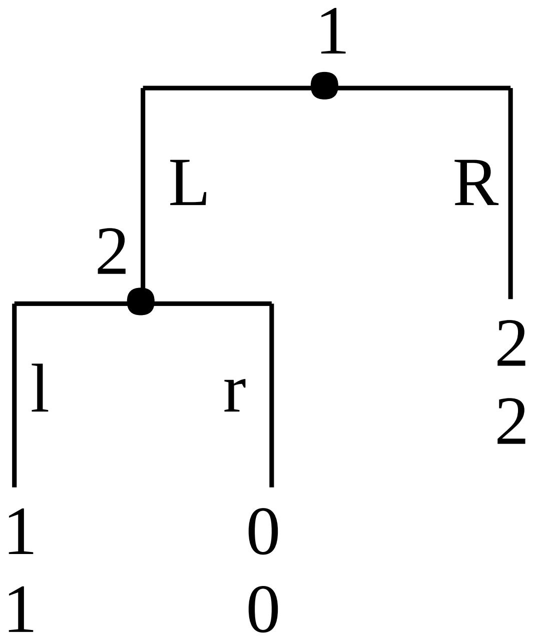
- *Property (H):* S is a closed set of Nash equilibria of G such that, for any equivalent game, and for any perturbation of the normal form of that game, there is a Nash equilibrium close to S.

Hyperstable Equilibrium

- Hyperstable Equilibria allow any perturbation of the game (both payoffs and strategies).

- A hyperstable equilibrium satisfies existence, a version of connectedness, backwards induction, invariance, and iterated dominance, but does not satisfy admissibility.

Example



	l	r
R	$\underline{2}$, $\underline{2}$	$\underline{2}$, $\underline{2}$
L	1, $\underline{1}$	0, 0

- The unique hyperstable set is the full interval from (R,l) to (R,r), but only (R,l) is admissible.

Fully Stable Equilibrium

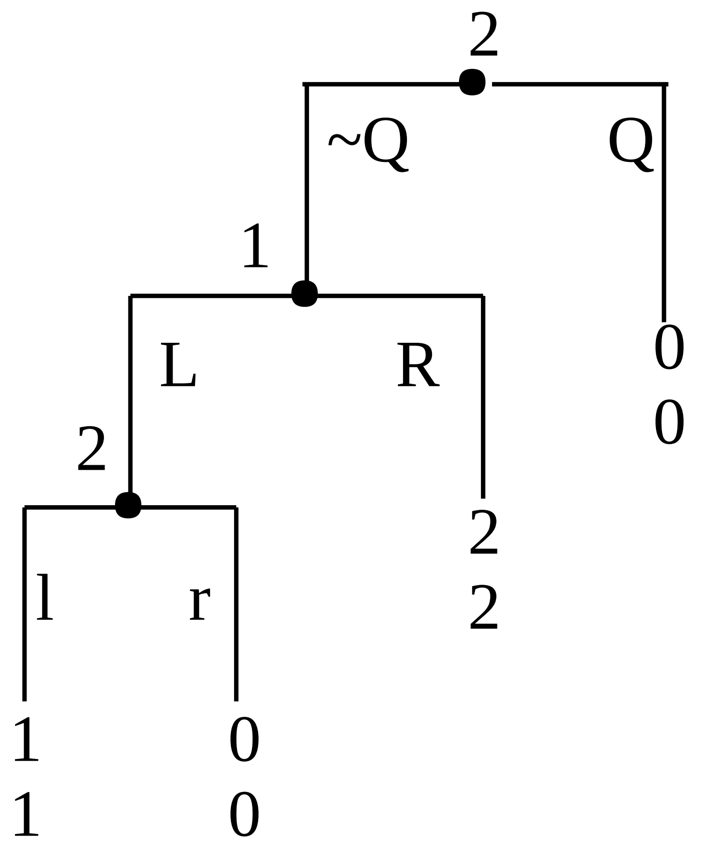
S is a *fully stable* set of equilibria of a game G if it is minimal with respect to the following property:

- *Property (F):* S is a closed set of Nash equilibria of G satisfying: for any $\varepsilon>0$ there exists a $\delta>0$ such that, whenever each player's strategy set is restricted to some compact convex polyhedron contained in the interior of the simplex and at (Hausdorff) distance less than δ from the simplex, then the resulting game has an equilibrium point ε-close to S.

Fully Stable Equilibrium

- A fully stable equilibrium allows any perturbation of the players strategies, but the payoffs cannot be perturbed.
- A fully stable equilibrium satisfies existence, connectedness, backwards induction, invariance, and iterated dominance, but fails to satisfy admissibility.

Example



	l	r	Q
R	$\underline{2}$, $\underline{2}$	$\underline{2}$, $\underline{2}$	$\underline{0}$, 0
L	1, $\underline{1}$	0, 0	$\underline{0}$, 0

- Here only (R,l) is admissible, but the unique fully stable set is the interval from (R,l) to (R,r).

Stable Equilibria

A set S of equilibria is *stable* in a game G if it minimal with respect to the following property:

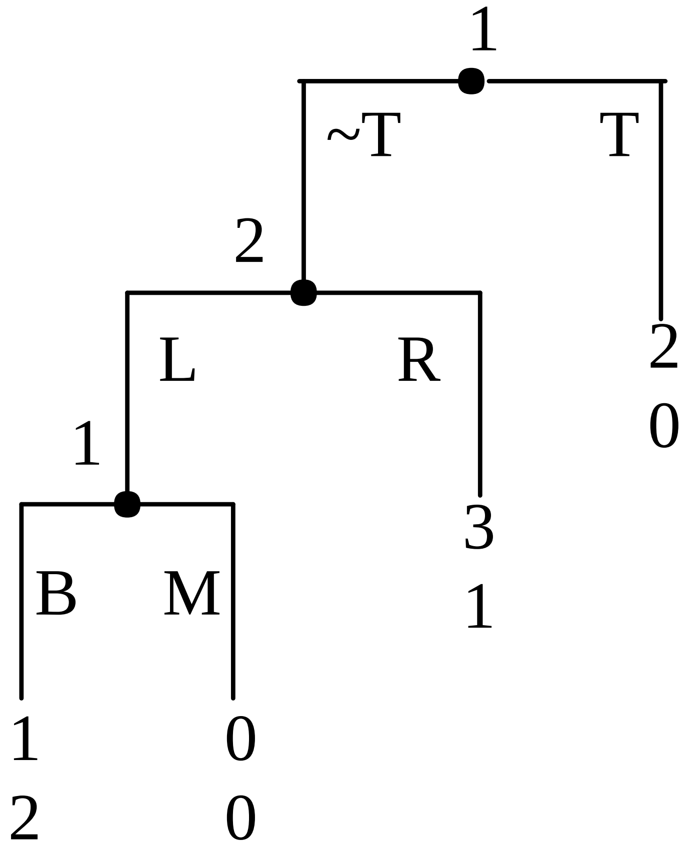
- *Property (S):* S is a closed set of Nash equilibria of G satisfying: for any ε>0 there exists δ_0>0 such that for any completely mixed strategy vector $\sigma_1,\ldots,\sigma_n$ and for any $\delta_1,\ldots,\delta_n$ with $0 < \delta_i < \delta_0$, the perturbed game where every strategy s of player i is replaced by $(1-\delta_i)s + \delta_i\sigma_i$ has an equilibrium ε-close to S.

Stable Equilibria

- Same definition as full stability, but perturbed strategies are restricted to simplices with faces parallel to the faces of the original simplex.
- If we say "some" instead of "any" $\sigma_1,\ldots,\sigma_n$ and $\delta_1,\ldots,\delta_n$, then we get perfect equilibria. Indeed, *truly perfect* is the corresponding point-based equilibrium concept (but need set-based concept to get existence).
- A stable equilibrium satisfies existence, a version of connectedness, invariance, admissibility, iterated dominance, and forward induction. Unfortunately, a stable equilibrium need not satisfy connectedness or backward induction.

Forward Induction vs Backward Induction

- Example:



	L	R
T	$\underline{2}$, $\underline{0}$	2, $\underline{0}$
M	0, 0	$\underline{3}$, $\underline{1}$
B	1, $\underline{2}$	$\underline{3}$, 1

Forward Induction vs Backward Induction

- (T,L) is the unique backward induction equilibrium, but if we delete row B since it is an inferior response at this equilibrium, then (M,R) becomes the forward induction equilibrium.
- The unique stable set is (T,L) to (T, 1/2 L).
- The equilibrium outcome is always (2,0), but row B is no longer inferior to all points in the stable set, so forward induction is satisfied.