Lecture Note 3: Mechanism Design

- Games with Incomplete Information
  - Imperfect Information vs. Incomplete Information
  - Bayesian Games
  - The Revelation Principle

Imperfect Information vs. Incomplete Information

Definitions

- Game of *imperfect information*: one or more players do not know the full history of the game. (Porter’s model on cartel maintenance).
- Game of *incomplete information*: the players have different private information about their preferences and abilities.

Imperfect Information vs. Incomplete Information

The key to analyzing games of incomplete information is to transform them into games of imperfect information by letting nature move first, randomly selecting each player's payoff function.

Example 1: Symmetric oligopoly model with unknown costs

- Firm i's marginal cost c_i may be either low or high {l,h}.
- Firm i observes its cost (nature's choice of l or h) but not the costs of the other firms.
- In the game of imperfect information, j does not observe natures choice of c_i, but j holds probabilistic beliefs about the likelihoods of nature's choice, summarized by a probability p that nature chose l.

Example 2: First-Price, Sealed-Bid Auction

- Suppose a seller decides to use a first-price, sealed-bid auction to allocate a good to one of two buyers.
- Let nature's choice of the buyers' valuations for the good, v_1 and v_2, be independently and uniformly distributed on [0,1].
- Based on v_i, player i submits a bid $b_i(v_i)$. The player with the highest bid gets the good and pays her bid.

Bayesian Games (Harsanyi, *Management Science* 1967-8)

- normal form game $G = \{A_1,\ldots,A_n; u_1,\ldots,u_n\}$
- Bayesian game $\Gamma = \{A_1,\ldots,A_n; T_1,\ldots,T_n; p_1,\ldots,p_n; u_1,\ldots,u_n\}$
- A_i = strategy set for i, actions: $a = (a_1,\ldots,a_n) \in A = A_1\times\ldots\times A_n$.
- T_i = type space for i, types: $t = (t_1,\ldots,t_n) \in T = T_1\times\ldots\times T_n$
- p_i = beliefs for i, $p_i(t_{-i} \mid t_i)$ = i's belief about types t_{-i} given type t_i.
- u_i = utility function for i, $u_i(a,t)$ depends on both actions a and types t.

Bayesian Games (Harsanyi, *Management Science* 1967-8)

- Beliefs $\{p_1,\ldots,p_n\}$ are ***consistent*** if they can be derived from Bayes' rule from a common joint distribution p(t) on T; i.e., there exists p(t) such that

$$p_i(t_{-i}|t_i) = \frac{p(t)}{p(t_i)} \quad \text{where} \quad p(t_i) = \sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)$$

for all i and t_i.

- Beliefs are consistent if nature moves first and types are determined according to p(t) and each i is informed only of t_i.

Example: First-Price Auction

- Types t_i are the valuations v_i ; $T_i = [0,1]$.
- Actions a_i are the bids b_i ; $A_i = [0,\infty)$.
- $p_i(t_j|t_i) = 1$ for all t_i and t_j.
- $$u_i(a,t) = \begin{cases} t_i - a_i & \text{if } a_i > a_j \\ (t_i - a_i)/2 & \text{if } a_i = a_j \\ 0 & \text{if } a_i < a_j. \end{cases}$$

$b(v_i) = v_i/2$ is the unique symmetric equilibrium bidding strategy

Definitions

- A *strategy* for i is a plan of action for each of i's possible types σ_i:$T_i \rightarrow A_i$. That is, what to do in every possible contingency (each of the possible types).
- A strategy profile $\sigma = (\sigma_1,\ldots,\sigma_n)$ is a *Bayesian equilibrium* of Γ if

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i)u_i[\sigma(t),t] \geq$$

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i)u_i[(\sigma_{-i}(t_{-i}),a_i),t] \quad \forall\ i,\ a_i \in A_i.$$

Bayesian Equilibrium

- Existence of a Bayesian equilibrium when the type sets and pure-strategy spaces are finite follows from the standard existence theorem for finite games.
- Given consistent beliefs, a Bayesian equilibrium of Γ is simply a Nash equilibrium of the game with imperfect information in which nature moves first.
- Any game of incomplete information *with consistent beliefs* can be transformed into a standard normal form game.

Revelation Principle (Myerson, *Econometrica* 1979 and others)

An equilibrium of a Bayesian game Γ can be represented by a simple equilibrium of a modified Bayesian game Γ' as follows:

- $\Gamma = \{A_1,\ldots,A_n; T_1,\ldots,T_n; p_1,\ldots,p_n; u_1,\ldots,u_n\}$
- $\Gamma' = \{A_1',\ldots,A_n'; T_1,\ldots,T_n; p_1,\ldots,p_n; u_1',\ldots,u_n'\}$
- $A_i' = T_i$ (each player reports her private information (possibly dishonestly))
- $u_i'(a',t) = u_i[\sigma(a'),t]$ (by reporting type t_i you get the payoff that t_i gets by playing the equilibrium strategy $\sigma_i(t_i)$ in Γ)

Revelation Principle

- For any Bayesian equilibrium σ of Γ, reporting your true type is a Bayesian equilibrium of Γ'.
- In the game Γ, types are mapped into actions via strategies and then these actions are mapped into outcomes via the utility functions.
- In the game Γ', types are mapped directly into outcomes, by the composition of the utility and strategy functions.

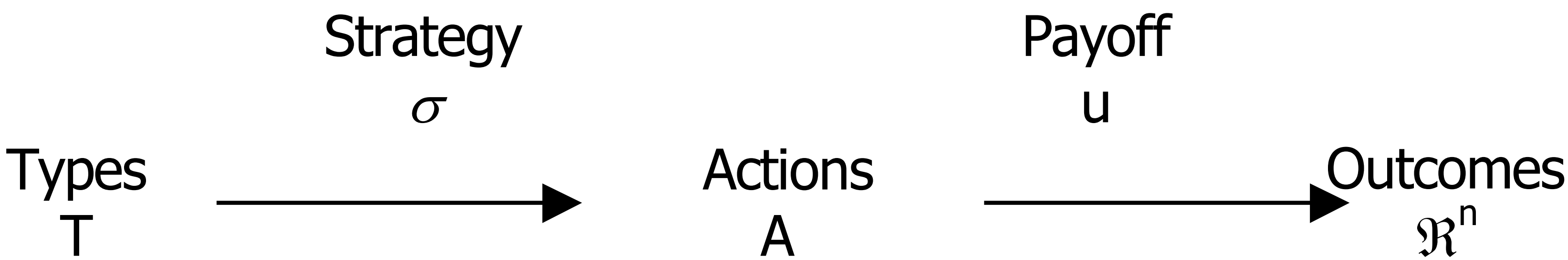
Direct Revelation Game

Original Game (Γ)

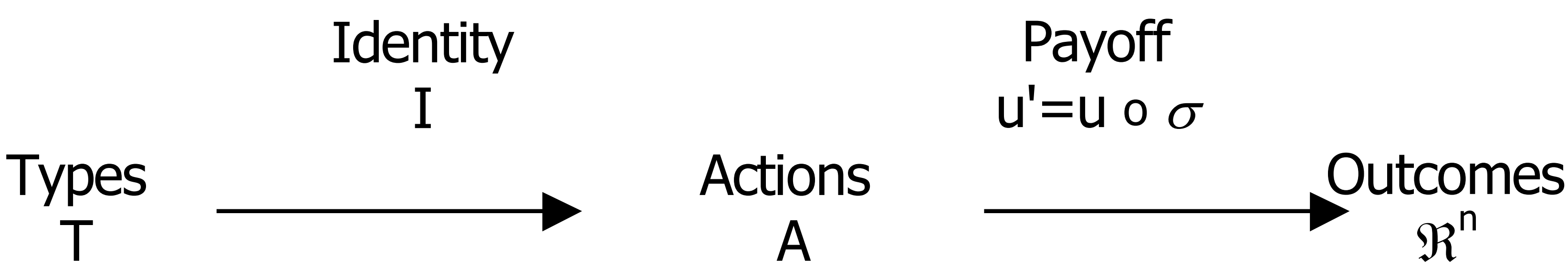
Types T $\xrightarrow{\text{Strategy } \sigma}$ Actions A $\xrightarrow{\text{Payoff } u}$ Outcomes $\Re^n$

Direct Revelation Game

Original Game (Γ)



Direct Revelation Game (Γ')



Direct Revelation Game

- In the direct revelation game, the players' equilibrium strategy profile is simply the identity map I(t) = t.
- A direct mechanism Γ' in which truthful reporting is a Bayesian equilibrium is call *incentive compatible*.
- The *revelation principle* states that without loss of generality, the analysis of Bayesian equilibria can be restricted to incentive compatible direct mechanisms.

Provision of a Public Good (Groves, *Econometrica* 1973)

Example: Paving a road

- Three households
- Cost of paving the road: c
- Independent private valuation: $v_i \sim F_i$
- If the road is built, the cost c must be paid from some combination of funds from the three households; no subsidies from outside sources are available.

Procedure 1

- The three households simultaneously announce their values: b_1, b_2, and b_3.
- Decision rule: Pave the road if $b_1+b_2+b_3 \geq c$,

and i pays $$\frac{b_i}{b_1 + b_2 + b_3} \cdot c$$

- Problem: $b_i = v_i$ is not an equilibrium.
- Best response: $b_i < v_i$, the more honest the others are the more you should lie. This misrepresentation leads to inefficiency.

Procedure 2: Groves Mechanism

- Each household simultaneously reports its valuation by making the bid $b_i(v_i)$.
- If $b = b_1 + b_2 + b_3 > c$, then the road is paved, and each contributes the amount the other players' bids fall short of c. That is,

$$u_i(b_1, b_2, b_3, v_i) =$$

$$\begin{cases} 0 & \text{if } b < c \\ v_i - (c - b_j - b_k) & \text{if } b \geq c \text{ and } b_j + b_k < c, \\ v_i & \text{if } b \geq c \text{ and } b_j + b_k \geq c. \end{cases}$$

Groves Mechanism

Claim : Truth-telling is a dominant strategy.

- Your bid *does not influence how much you pay*, only whether the road is paved.
- If $b_{-i} \geq c$, then $u_i = v_i$ and $b_i = v_i$ is a BR.
- If $b_{-i} < c$, then

$$u_i = \begin{cases} v_i - (c - b_{-i}) & \text{if } b_i \geq c - b_{-i} \\ 0 & \text{otherwise} \end{cases}$$

- Your payoff is maximized by bidding so that the road is paved whenever $v_i - (c - b_{-i}) \geq 0$, which is accomplished by bidding $b_i = v_i$.

Groves Mechanism

Problems:

- *Does not satisfy budget balance*

The sum of the payments $3c - 2b \leq c$, since $b \geq c$, so having a benefactor to make up the deficit is essential.

- *Incentive to collude*

The *more* the households value the road, the *more* the benefactor must contribute. Hence, the players have a strong incentive to collude and *overstate* their valuations, so that the benefactor pays a larger share of the road.

Second-Price Auction

- n bidders
- Each has a valuation v_i for the good being auctioned, where each bidders' valuation is private information.
- Suppose the seller uses a second-price auction to allocate the good: each bidder simultaneously submits a bid and the good goes to the highest bidder, who pays the seller a price equal to the *second highest* bid.

Second-Price Auction

Claim: Truth telling is a dominant strategy.

- As in Grooves mechanism, player i's bid does not influence the price paid if i wins, but does affect whether the player wins.
- The optimal bid is such that player i wins whenever p < v_i. This is accomplished by bidding v_i: by bidding b_i < v_i, i stands to lose some profitable opportunities, and by bidding more than v_i, i may lose by winning.

Lecture Note 3: Mechanism Design

- Bilateral Trading Mechanisms
  - War of Attrition
  - Simultaneous Offers
  - The Public Choice Problem

Games of Timing: War of Attrition

- Two animals are fighting for a prize
- Each knows the value of the prize to himself, but not to the other
- The valuations v_i (i=1, 2), are i.i.d. with distribution F and density f on [0,1]
- Each incurs a cost of c for each minute the fight continues

How long should an animal i with valuation v_i wait before conceding?

War of Attrition: Symmetric Bayesian equilibrium

- Let $t_i(v_i)$ be the stopping time for animal i with valuation v_i
- Suppose $t_i' > 0$
- Let $x_i(t_i) = t_i^{-1}(t_i)$ be the valuation of animal i if it concedes at time t_i
- Animal i's payoff is

$$u_i(v_1, v_2, t_1, t_2) = \begin{cases} v_i - ct_j & \text{if } t_j \le t_i \\ -ct_i & \text{if } t_j > t_i. \end{cases}$$

War of Attrition: Symmetric Bayesian equilibrium

- i seeks to maximize her expected utility given animal j's strategy $t_j(\cdot)$; that is for each v_i, t_i is chosen to

$$\max_{t_i} \int_0^{x_{j(t_i)}} [v_i - ct_j] f(v_j) dv_j \quad - ct_i [1 - F(x_j(t_i))].$$

- F.O.C.

$$x_j'(t_i) v_i f(x_j(t_i)) - c[1 - F(x_j(t_i))] = 0.$$

War of Attrition:
Symmetric Bayesian equilibrium

- Imposing symmetry:

$$x'(t) = \frac{c[1 - F(x(t))]}{vf(x(t))}$$

- In equilibrium, x(t) = v, and x'(t) = 1/t'(v), so

$$t'(v) = \frac{vf(v)}{c[1 - F(v)]}$$

War of Attrition:
Symmetric Bayesian equilibrium

- Equilibrium Strategy

$$t(v) = \int_0^v \frac{zf(z)}{c[1 - F(z)]} dz$$

Simultaneous Offers

(Chatterjee & Samuelson, *Operations Research* 1983)

- A seller and a buyer are engaged in the trade of a single object worth s to the seller and b to the buyer.
- Valuations are known privately, as summarized below

Traders Value	Distributed	Payoff	Private Info	Common Knowledge	Strategy (Offer)
Seller s	s~F on $[\underline{s}, \overline{s}]$	u =P – s	s	F, G	p(s)
Buyer b	b~G on $[\underline{b}, \overline{b}]$	v =b – P	b	F, G	q(b)

Simultaneous Offers

- *Independent private value model*: s and b are independent random variables.

- *Ex post efficiency*: trade if and only if $s < b$.

- *Game*: Each player simultaneously names a price; if $p \leq q$ then trade occurs at the price $P = (p + q)/2$; if $p > q$ then no trade (each player gets zero).

Simultaneous Offers

- Payoffs:
  - Seller

$$u(p,q,s,b) = \begin{cases} P - s & \text{if } p \leq q \\ 0 & \text{if } p > q \end{cases}$$

  - Buyer

$$v(p,q,s,b) = \begin{cases} b - P & \text{if } p \leq q \\ 0 & \text{if } p > q \end{cases}$$

where the trading price is $P = (p + q)/2$

Example

- Let F and G be independent uniform distributions on [0,1].

- Equilibrium conditions:

$$(1)\ \forall\ s \in [\underline{s}, \overline{s}], p(s) \in \underset{p}{\operatorname{argmax}} E_b\{u(p,q,s,b) | s, q(\cdot)\}$$

$$(2)\ \forall\ b \in [\underline{b}, \overline{b}], q(b) \in \underset{q}{\operatorname{argmax}} E_s\{v(p,q,s,b) | b, p(\cdot)\}$$

Seller's Problem

- Assume p and q are strictly increasing.
- Let $x(\cdot) = p^{-1}(\cdot)$ and $y(\cdot) = q^{-1}(\cdot)$.
- Optimization in (1) can be stated as

$$\max_{p} \int_{y(p)}^{1} [(p + q(b)) / 2 - s] db$$

- First-order condition

$$-y'(p)[p - s] + [1 - y(p)]/2 = 0,$$

since $q(y(p)) = p$

Buyer's Problem

- Optimization in (2) can be stated as

$$\max_{q} \int_{0}^{x(q)} [b - (p(s) + q) / 2] ds$$

- First-order condition

$$x'(q)[b - q] - x(q)/2 = 0,$$

since p(x(q)) = q.

Equilibrium

- Equilibrium condition:

$$s=x(p) \text{ and } b=y(q)$$

- Equilibrium first-order conditions:

(1') $$-2y'(p)[p - x(p)] + [1 - y(p)] = 0,$$

(2') $$2x'(q)[y(q) - q] - x(q) = 0.$$

Solution

- Solving (2') for y(q) and replacing q with p yields

$$(2'') \quad y(p) = p + \frac{1}{2}\frac{x(p)}{x'(p)}, \quad \text{so} \quad y'(p) = \frac{3}{2} - \frac{1}{2}\frac{x(p)x''(p)}{[x'(p)]^2}$$

- Substituting into (1') then yields

$$(1') \quad [x(p) - p]\left[3 - \frac{x(p)x''(p)}{[x'(p)]^2}\right] + \left[1 - p - \frac{1}{2}\frac{x(p)}{x'(p)}\right] = 0$$

Analytical Solution

- Linear Solution:

$$x(p) = \alpha p + \beta.$$

with $\alpha = 3/2$ and $\beta = -3/8$.

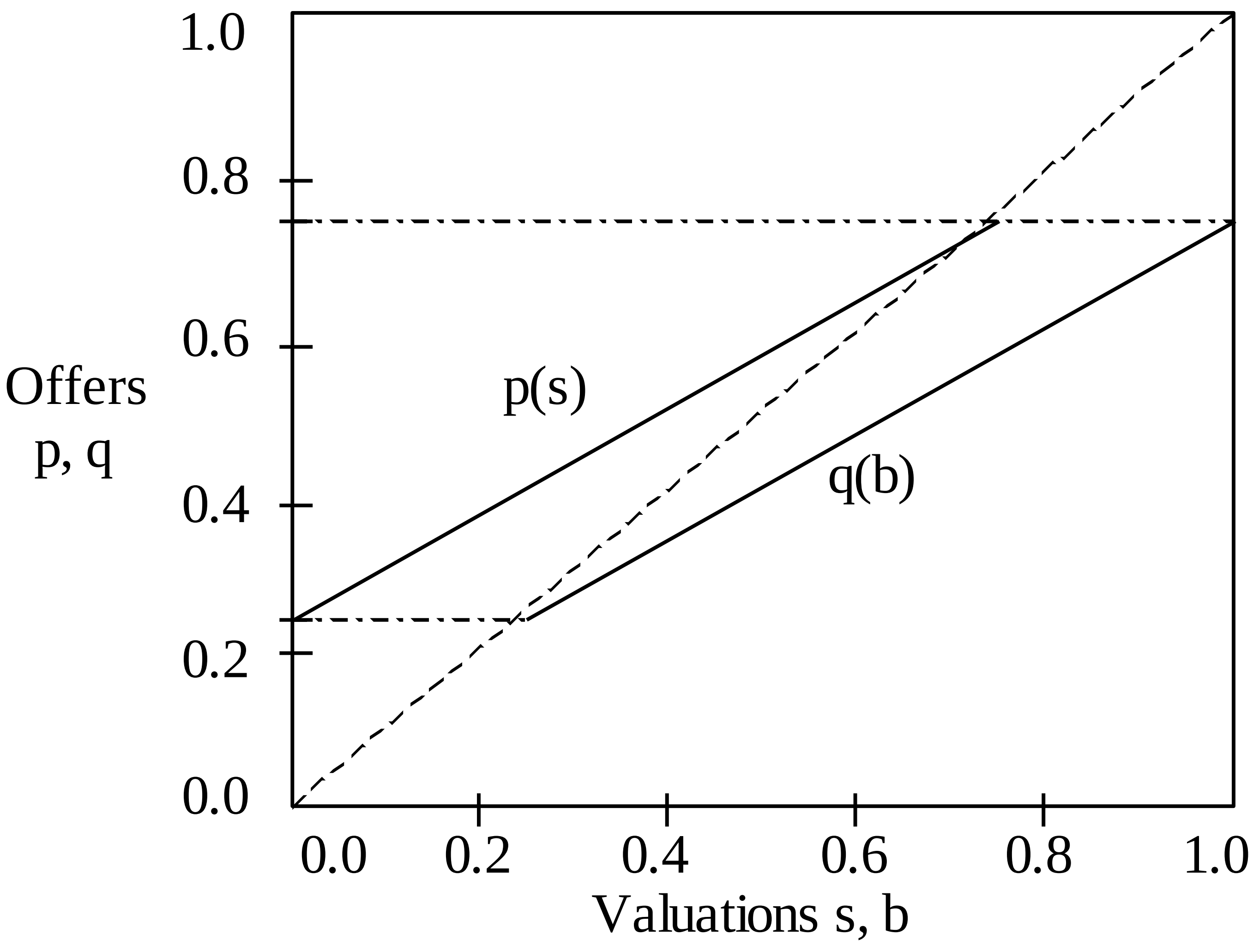
- Using (2") yields

$$y(q) = 3/2\ q - 1/8.$$

- Inverting these functions results in

$$p(s) = 2/3\ s + 1/4 \quad \text{and} \quad q(b) = 2/3\ b + 1/12$$

Figure 1



Outcome

- Trade occurs if and only if

$$p(s) \leq q(b), \text{ or } b - s \geq 1/4$$

- The gains from trade must be at least 1/4 or no trade takes place.

The outcome is inefficient

The Public Choice Problem

- There are two members of society $i \in \{1,2\}$
- Public project: do it/not do it, $d \in \{0,1\}$
- Individual benefits: $v_i \in (-\infty,\infty)$, where v_i is known privately to i
- Ex post efficiency requires:

$$d^*(v_1, v_2) = \begin{cases} 1 & \text{if } v_1 + v_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find mechanism to implement efficient choice rule

Mechanism Design

By the Revelation Principle, restrict attention to incentive-compatible direct mechanisms {d(v),t(v)}

- $v = \{v_1, v_2\}$, $d:\Re^2 \rightarrow \{0,1\}$ determines the decision as a function of the reports

- $t:\Re^2 \rightarrow \Re^2$ determines the transfers between the players where $t(v) = \{t_1(v), t_2(v)\}$ and $t_i(v)$ is the transfer that player i receives.

Mechanism Design

- Find a mechanism that satisfies:

(1) efficient social choice: $d(v) \equiv d^*(v)$, and

(2) dominant-strategy incentive compatibility:

for all, $\hat{v}_j$

$$v_i \in \underset{\hat{v}_i}{\arg\max}\, v_i d(\hat{v}_i, \hat{v}_j) + t_i(\hat{v}_i, \hat{v}_j).$$

Procedure 1: Groves Mechanism

If the reported types are $\hat{v} = (\hat{v}_1, \hat{v}_2)$, then

(D) $$d(\hat{v}) = \begin{cases} 1 & \text{if} \quad \hat{v}_1 + \hat{v}_2 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $$t_i(\hat{v}) = d(\hat{v})\hat{v}_j + h_i(\hat{v}_j)$$

for some function $h_i : \Re \to \Re$.

Procedure 1: Groves Mechanism

Dominant strategy:

v_i solves

$$\max_{\hat{v}_i} v_i d(\hat{v}_i, \hat{v}_j) + t_i(\hat{v}_i, \hat{v}_j)$$

if and only if v_i solves

$$\max_{\hat{v}_i} d(\hat{v}_i, \hat{v}_j)[v_i + \hat{v}_j].$$

But this is the case, since reporting $\hat{v}_i = v_i$ makes $d(\hat{v}_i, \hat{v}_j)$ equal to 1 if and only if $v_i + \hat{v}_j \geq 0$

Truth is a dominant strategy

Procedure 1: Groves Mechanism

Problem: Transfers do not satisfy budget balance.

- Budget balance requires:

$$t_1(v)+t_2(v)=(v_1+v_2)d(v_1,v_2)+h_1(v_2)+h_2(v_1)=0,$$

or

$$h_1(v_2)+h_2(v_1)=\begin{cases}-(v_1+v_2) & \text{if } v_1+v_2\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

- But this cannot happen if h_i is independent of v_i

Procedure 2: Bayesian Game

Solution: weaken incentive compatibility criterion, so that truth is merely a Bayesian equilibrium rather than a dominant strategy.

(2') Bayesian incentive compatibility:

$$v_i \in \underset{\hat{v}_i}{\operatorname{argmax}}\, E_{v_j}[v_i d(\hat{v}_i, v_j) + t_i(\hat{v}_i, v_j) \mid v_i]$$

replacing (2) with (2') allows us to satisfy

(3) Budget balance: $t_1(v) + t_2(v) = 0$ for all v

Procedure 2: Bayesian Game

- $\Gamma = \{A_1, A_2; V_1, V_2; p_1, p_2; u_1, u_2\}$
- $A_i = V_i = \Re$, $u_i(a,v) = v_i d(a) + t_i(a)$
- $p_i(v_j \mid v_i) = f_j(v_j)$, so types are independent

We wish to construct a mechanism satisfying (1), (2'), and (3). Our decision rule must be as in (D) to satisfy (1)

Procedure 2: Bayesian Game

Budget Balance

- Consider the transfers $t_i(v) = g_i(v_i) - g_j(v_j)$, where

$$g_i(v_i) = \int_{-\infty}^{\infty} v_j d(v_i, v_j) f_j(v_j) dv_j \quad .$$

Clearly, the transfers balance.

Procedure 2: Bayesian Game

Incentive Compatibility

- Player i chooses the report to solve

$$\max_{\hat{v}_i} \int_{-\infty}^{\infty} [v_i d(\hat{v}_i, v_j) + t_i(\hat{v}_i, v_j)] f_j(v_j) dv_j \quad .$$

- Substituting the definition of $t_i(\cdot)$ yields

$$\max_{\hat{v}_i} \int_{-\infty}^{\infty} (v_i + v_j) d(\hat{v}_i, v_j) f_j(v_j) dv_j \quad ,$$

Procedure 2:
Bayesian Game

Incentive Compatibility (cont.)

- Previous expression is equivalent to

$$\max_{\hat{v}_i} \int_{-\hat{v}_i}^{\infty} (v_i + v_j) f_j(v_j) dv_j \quad ,$$

when the decision rule in (D) is used.

- First-order condition

$$(v_i - \hat{v}_i) f_j(-\hat{v}_i) = 0 \quad \text{so} \quad \hat{v}_i = v_i$$

Procedure 2: Bayesian Game

Problem: It may not satisfy individual rationality

(4) Interim individual rationality:

$$U_i(v_i) = \int_{-\infty}^{\infty} [v_i d(v_i, v_j) + t_i(v_i, v_j)] f_j(v_j) dv_j$$

$$\geq 0 \quad \text{for all } v_i \in V_i.$$

Procedure 2: Bayesian Game

Substituting for d(·) and t(·) yields:

$$U_i(v_i) = \int_{-v_i}^{\infty} (v_i + v_j) f_j(v_j) dv_j$$

$$- \int_{-\infty}^{\infty} v_i f_i(v_i)[1\text{-}F_j(\text{-}v_i)] dv_i$$

$$= v_i[1\text{-}F_j(\text{-}v_i)] + \int_{-\hat{v}_i}^{\infty} v_j f_j(v_j) dv_j$$

$$- \int_{-\infty}^{\infty} v_i f_i(v_i)[1\text{-}F_j(\text{-}v_i)] dv_i \quad .$$

Procedure 2: Bayesian Game

- Note that $U_i'(v_i) = 1 - F_j(-v_i) \geq 0$, so if interim individual rationality fails, it fails for the lowest values of v_i.
- Assume that the means and variances of v_i and v_j are finite. Then as $v_i \rightarrow -\infty$ the first and second terms approach zero.
- If the integral in the third term is positive then for sufficiently low values of v_i,

$U_i(v_i) < 0$.

Lecture Note 3: Mechanism Design

- Bilateral Trading Mechanisms
  - A General Model
  - Efficiency in Games with Incomplete Information
  - Durability

A General Model (Myerson and Satterthwaite, *JET* 1983)

Direct Revelation Game:

- Bilateral Exchange with independent private value.
- s ~ F with positive pdf f on $[\underline{s}, \overline{s}]$
- b ~ G with positive pdf g on $[\underline{b}, \overline{b}]$
- F and G are common knowledge

In the DRG, the traders report their valuations and then an outcome is selected. Given the reports (s,b), an outcome specifies a probability of trade (p) and the terms of trade (x).

Definition

Direct Mechanism

A direct mechanism is a pair of outcome functions $\langle p,x \rangle$, where:

– p(s,b) is the probability of trade given the reports (s,b), and

– x(s,b) is the expected payment from the buyer to the seller.

Payoffs

Ex post utilities:

- Seller's ex post utility:

$$u(s,b) = x(s,b) - sp(s,b)$$

- Buyer's ex post utility:

$$v(s,b) = bp(s,b) - x(s,b)$$

Both traders are risk neutral and there are no income effects

Payoffs

Define:

$$X(s) = \int_{\underline{b}}^{\overline{b}} x(s,b)g(b)db \qquad Y(b) = \int_{\underline{s}}^{\overline{s}} x(s,b)f(s)ds$$

$$P(s) = \int_{\underline{b}}^{\overline{b}} p(s,b)g(b)db \qquad Q(b) = \int_{\underline{s}}^{\overline{s}} p(s,b)f(s)ds.$$

X(s) is the seller's expected revenue given s

Y(b) is the buyer's expected payment given b

P(s) is the seller's probability of trade

Q(b) is the buyer's probability of trade

Payoffs

- Interim Utilities:

$$U(s) = X(s) - sP(s) \quad V(b) = bQ(b) - Y(b)$$

- The mechanism $\langle p,x \rangle$ is *incentive compatible* if for all s, b, s', and b':

$$\text{(IC)} \quad U(s) \geq X(s') - sP(s') \quad V(b) \geq bQ(b') - Y(b')$$

- The mechanism $\langle p,x \rangle$ is *individually rational* if for all $s \in [\underline{s}, \overline{s}]$ and $b \in [\underline{b}, \overline{b}]$

$$\text{(IR)} \quad U(s) \geq 0 \qquad V(b) \geq 0.$$

Lemma 1
(Mirrlees, Myerson)

The mechanism $\langle p,x \rangle$ is IC if and only if $P(\cdot)$ is decreasing, $Q(\cdot)$ is increasing, and

$$U(s) = U(\bar{s}) + \int_{s}^{\bar{s}} P(t)dt$$

(IC')

$$V(b) = V(\underline{b}) + \int_{\underline{b}}^{b} Q(t)dt$$

Lemma 1: Proof

Only if:

- By definition, $U(s) = X(s) - sP(s)$ and $U(s') = X(s') - s'P(s')$. This and (IC) imply

$$U(s) \geq X(s') - sP(s') = U(s') + (s' - s)P(s'), \text{ and}$$

$$U(s') \geq X(s) - s'P(s) = U(s) + (s - s')P(s)$$

- Putting these inequalities together yields

$$(s' - s)P(s) \geq U(s) - U(s') \geq (s' - s)P(s')$$

Lemma 1: Proof

- Taking s' > s implies that P(·) is decreasing
- Dividing by (s' - s) and letting s' $\rightarrow$ s, then yields dU(s)/ds = -P(s)
- Integrating produces (IC')
- The same is true for the buyer

Lemma 1: Proof

If:

- To prove (IC) for the seller, note that it suffices to show that

  s[P(s) - P(s')] + [X(s') - X(s)] $\leq$ 0 for all s, s' $\in [\underline{s}, \bar{s}]$

- Substituting for X(s') and X(s) using (IC') and the definition of U(s) yields

$$X(s) = sP(s) + U(\bar{s}) + \int_{s}^{\bar{s}} P(t)dt \quad .$$

Lemma 1: Proof

If:

- Then it suffices to show for every s,*s'*$\in[\underline{s},\bar{s}]$ that

$$0 \geq s[P(s) - P(s')] + s'P(s') + \int_{s'}^{\bar{s}} P(t)dt \quad - sP(s) - \int_{s}^{\bar{s}} P(t)dt$$

$$= (s' - s)P(s') + \int_{s'}^{s} P(t)dt \quad = \int_{s'}^{s} [P(t) - P(s')]dt \quad ,$$

which holds because P(·) is decreasing.

- The proof for the buyer is similar.

Lemma 2

An incentive compatible mechanism $\langle p,x \rangle$ is individually rational if and only if

(IR') $U(\bar{s}) \geq 0$ and $V(\quad) \geq 0.$

Lemma 2

Proof

- Clearly, (IR') is necessary for $\langle p,x \rangle$ to be IR
- By Lemma 1, $U(\cdot)$ is decreasing; hence, (IR') is sufficient as well

Corollary

- An incentive-compatible, individually rational mechanism $\langle p,x\rangle$ satisfies

$$(*)\ U(\overline{s}) + V(\underline{b}) =$$

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \left[b - \frac{1-G(b)}{g(b)} - s - \frac{F(s)}{f(s)} \right] p(s,b) f(s) g(b)\, ds\, db \geq 0.$$

Lemma 2: Corollary

Proof

- Using (IC') and the definition of U(s) yields

$$X(s) = sP(s) + U(\bar{s}) + \int_{s}^{\bar{s}} P(t)dt.$$

Lemma 2: Corollary

- Taking the expectation with respect to s (and substituting in the definitions of X(s) and P(s)) shows that

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} x(s,b)f(s)g(b)dsdb \;=$$

$$U(\overline{s}) + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} sp(s,b)f(s)g(b)dsdb$$

$$+ \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} p(s,b)F(s)g(b)dsdb.$$

Lemma 2: Corollary

- The third term in the right hand side follows, since

$$\int_{\underline{s}}^{\bar{s}} \int_{s}^{\bar{s}} p(t,b)f(s)\,dt\,ds$$

$$= \int_{\underline{s}}^{\bar{s}} \int_{\underline{s}}^{t} p(t,b)f(s)\,ds\,dt \quad = \int_{\underline{s}}^{\bar{s}} p(s,b)F(s)\,ds\,.$$

Lemma 2: Corollary

- Preceding analogously for the buyer yields

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} x(s,b)f(s)g(b)dsdb$$

$$= -V(\underline{b}) + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} bp(s,b)f(s)g(b)dsdb$$

$$- \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} p(s,b)f(s)[1-G(b)]dsdb.$$

- Equating the right-hand sides of the last two equations and applying (IR') completes the proof

Theorem

If it is not common knowledge that gains exist (the supports of the traders' valuations have non-empty intersection), then no incentive-compatible, individually rational trading mechanism can be ex-post efficient.

Proof

- A mechanism is ex-post efficient if and only if trade occurs whenever $s \leq b$:

$$p(s,b) = \begin{cases} 1 & \text{if } s \leq b \\ 0 & \text{if } s > b. \end{cases}$$

Proof

- To prove that ex-post efficiency cannot be attained, it suffices to show that the inequality (*) in the Corollary fails when evaluated at this p(s,b). Hence,

$$\int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\min\{b,\bar{s}\}} \left[b - \frac{1 - G(b)}{g(b)} - s - \frac{F(s)}{f(s)} \right] f(s)g(b)dsdb$$

Proof

$$= \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\min\{b,\overline{s}\}} [bg(b)+G(b)-1]f(s)dsdb - \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\min\{b,\overline{s}\}} [sf(s)+F(s)]dsg(b)db$$

$$= \int_{\underline{b}}^{\overline{b}} [bg(b)+G(b)-1]F(b)db - \int_{\underline{b}}^{\overline{b}} \min\{bF(b),\overline{s}\}g(b)db \text{ (by parts)}$$

$$= -\int_{\underline{b}}^{\overline{b}} [1-G(b)]F(b)db + \int_{\overline{s}}^{\overline{b}} (b-\overline{s})g(b)db$$

$$= -\int_{\underline{b}}^{\overline{b}} [1-G(b)]F(b)db + \int_{\overline{s}}^{\overline{b}} [1-G(b)]db \text{ (by parts)}$$

$$= -\int_{\underline{b}}^{\overline{s}} [1-G(t)]F(t)dt < 0, \text{ since } \underline{b} < \overline{s}.$$

Proof

- The second term in the second line follows, since by integrating by parts

$$\int_{\underline{s}}^{x} [sf(s) + F(s)]ds = xF(x).$$

Since ex-post efficiency is unattainable, we need a weaker efficiency criterion with which to measure a mechanism's performance

Definition

Pareto optimality:

- An allocation is Pareto optimal if there does not exist an alternative allocation that makes no parties worse off and at least one party strictly better off

Efficiency in Games with Incomplete Information

"A decision rule is efficient if and only if no other *feasible* decision rule can be *found* that may make some individuals *better off* without ever making any other individual worse off."

Efficiency in Games with Incomplete Information

Problems:

- What is meant by a feasible decision rule? Are we to recognize incentive constraints?
- What is meant by better off or worse off? On what information should the expectation be conditioned? Three alternatives are:

1. ***Ex ante information***: a planner's information at the beginning of the game (no knowledge of types).
2. ***Interim information***: a player's private information at the beginning of the game.
3. ***Ex post information***: all the private information.

Efficiency in Games with Incomplete Information

Problems:

- Who is to "find" the potentially better decision rule, and at what information stage? If a player proposes a particular decision rule after learning her private information, the other players may infer something about the player's type from the information.

Bayesian Game

- $\Gamma = \{A_1,\ldots,A_n;T_1,\ldots,T_n;p_1,\ldots,p_n;u_1,\ldots u_n\}$
- Each action set A_i and type set T_i is finite
- Beliefs p_i are consistent.
- Let D be the set of probability distributions over $A = A_1\times\ldots\times A_n$.
- A decision rule (or direct mechanism) $\delta:T\rightarrow D$ maps reports into a randomization over feasible actions.
- The utility function $u_i(d,t):D\times T\rightarrow\mathfrak{R}$ maps the decision and types into payoffs.

Bayesian Game

- A decision rule $\delta \in \Delta \equiv \{\delta:T\to D\}$ is incentive compatible if for all i and $t_i \in T_i$

(IC) $$\sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i)u_i(\delta(t),t) \ge \sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i)u_i(\delta(t_{-i},\hat{t}_i),t)$$

for all $\hat{t}_i \in T_i$

- Let $\Delta^* = \{\delta:T\times D\to\Re\}$ be the set of all incentive-compatible decision rules. By the revelation principle, we can restrict attention to $\delta\in\Delta^*$.

Expected Utility

For a decision rule δ(·), the expected utility at each of the three information stages are

- (1) *Ex Ante Utility*: $U_i(\delta) = \sum_{t \in T} p(t) u_i(\delta(t), t)$
- (2) *Interim Utility*: $U_i(\delta | t_i) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} | t_i) u_i(\delta(t), t)$
- (3) *Ex Post Utility*: $U_i(\delta | t) = u_i(\delta(t), t)$

Domination

- γ *ex ante dominates* δ iff $U_i(\gamma) \geq U_i(\delta)$ $\forall$ i with at least one strict inequality;

- γ *interim dominates* δ iff $U_i(\gamma|t_i) \geq U_i(\delta|t_i)$ $\forall$ i and $t_i \in T_i$ with at least one strict inequality;

- γ *ex post dominates* δ iff $U_i(\gamma|t) \geq U_i(\delta|t)$ $\forall$ i and $t \in T$ with at least one strict inequality.

Efficiency

- δ is *ex post (classically) efficient* iff there does not exist $\gamma \in \Delta$ that ex post dominates δ
- δ is *ex ante (incentive) efficient* iff there does not exist $\gamma \in \Delta^*$ that ex ante dominates δ
- δ is *interim (incentive) efficient* iff there does not exist $\gamma \in \Delta^*$ that interim dominates δ

Trading Game (Myerson and Satterthwaite)

- Ex ante efficient mechanism that maximizes the expected gains from trade:

$$\int_{\underline{s}}^{\overline{s}} U(s)f(s)ds + \int_{\underline{b}}^{\overline{b}} V(b)g(b)db$$

Trading Game (Myerson and Satterthwaite)

- Myerson and Satterthwaite show that the ex ante efficient decision rule (probability of trade) is:

$$p^{\alpha}(s,b) = \begin{cases} 1 & \text{if } c(s,\alpha) \le d(b,\alpha) \\ 0 & \text{if } c(s,\alpha) > d(b,\alpha) \end{cases}$$

where

$$c(\alpha,s) = s + \alpha\frac{F(s)}{f(s)} \qquad d(\alpha,b) = b - \alpha\frac{1-G(b)}{g(b)}$$

and α is chosen so that $U(\overline{s}) = V(\underline{b}) = 0$.

Remarks

- If $\alpha = 0$, then p^{α} is ex post efficient (all the weight on the objective function).
- If $\alpha = 1$, p^{α} maximizes the expression in (*); a constrained maximization.
- The ex ante efficient trading rule has the property that, given the reports, trade either occurs with probability one or not at all.

Example

Valuations are uniformly distributed on [0,1]

- Ex ante efficient mechanism: linear equilibrium in which trade occurs if and only if the gains from trade are at least 1/4 (Chatterjee & Samuelson)
- If the traders cannot commit to walking away from gains from trade, then they would be unable to implement this mechanism
- So long as it is not common knowledge that gains exist, the traders will, with positive probability, make incompatible demands in situations where gains from trade exist

Accomplishments

- Characterization of the set of all BE of all bargaining games in which the players' strategies map their private valuations into a probability of trade and a payment from buyer to seller
- Proof that ex post efficiency is unattainable if it is uncertain that gains from trade exist
- Determination of the set of ex ante efficient mechanisms
- Proof that ex ante efficiency is incompatible with sequential rationality

Durability

Are there further restrictions on the feasible set of decision rules δ *that should be made*?

- If the players have limited abilities to make binding commitments, then this limitation poses a further restriction on the set of feasible decision rules
- A second restriction on the set of feasible decision rules can come from the process of deciding on which decision rule to implement

Durability

Is it ever the case that a player, knowing her type, could suggest an alternative decision rule γ that the others would surely prefer?

Example

- Each of two players, 1 and 2, is of one of two types, a or b
- The players' utilities as a function of their types and which of three possible decisions {A,B,C} are:

	1a	1b	2a	2b
d=A	2	0	2	2
d=B	1	4	1	1
d=C	0	9	0	-8

Decision Rule

- The ex ante efficient decision rule that maximizes the sum of the players' payoffs is:

$$\delta(1a,2a) = A \qquad \delta(1a,2b) = B$$
$$\delta(1b,2a) = C \qquad \delta(1b,2b) = B.$$

- No outsider could suggest an alternative decision rule that would make some type better without making another type worse off

Decision Rule

Problem:

- If player 1's type is 1a, then player 1 can suggest to 2 that decision A be adopted, and 2 would surely accept such a proposal
- In the words of Holmstrom and Myerson, the decision rule δ is not durable

A decision rule is *durable* iff the players would never unanimously approve a change to any other decision rule

Lecture Note 3: Mechanism Design

- Multilateral Trading Mechanisms
  - Dissolving a Partnership
  - Optimal Auctions

Dissolving a Partnership
(Cramton, Gibbons, and Klemperer, 1987)

- n traders. Each trader $i \in \{1,...,n\}$ owns a share $r_i \geq 0$ of the asset, where $r_1 + ... + r_n = 1$
- As in MS, player i's valuation for the entire good is v_i
- The utility from owning a share r_i is $r_i v_i$
- Private values, v_i's are iid $\sim F(\cdot)$ on $[\underline{v}, \overline{v}]$
- A *partnership* (r,F) is fully described by the vector of ownership rights $r = \{r_1,...,r_n\}$ and the traders' beliefs F about valuations

Dissolving a Partnership

MS Case:

- n = 2 and r = {1,0}
- There does not exist a BE σ of the trading game such that:

  (1) σ is (interim) individually rational and

  (2) σ is ex post efficient

CGK Case:

- If the ownership shares are not too unequally distributed, then it is possible to satisfy both (1) and (2), (satisfying IC, IR, EE and BB)

Dissolving a Partnership

A partnership (r,F) can be *dissolved efficiently* if there exists a Bayesian Equilibrium σ of a Bayesian trading game such that σ is interim individually rational and ex post efficient

Theorem

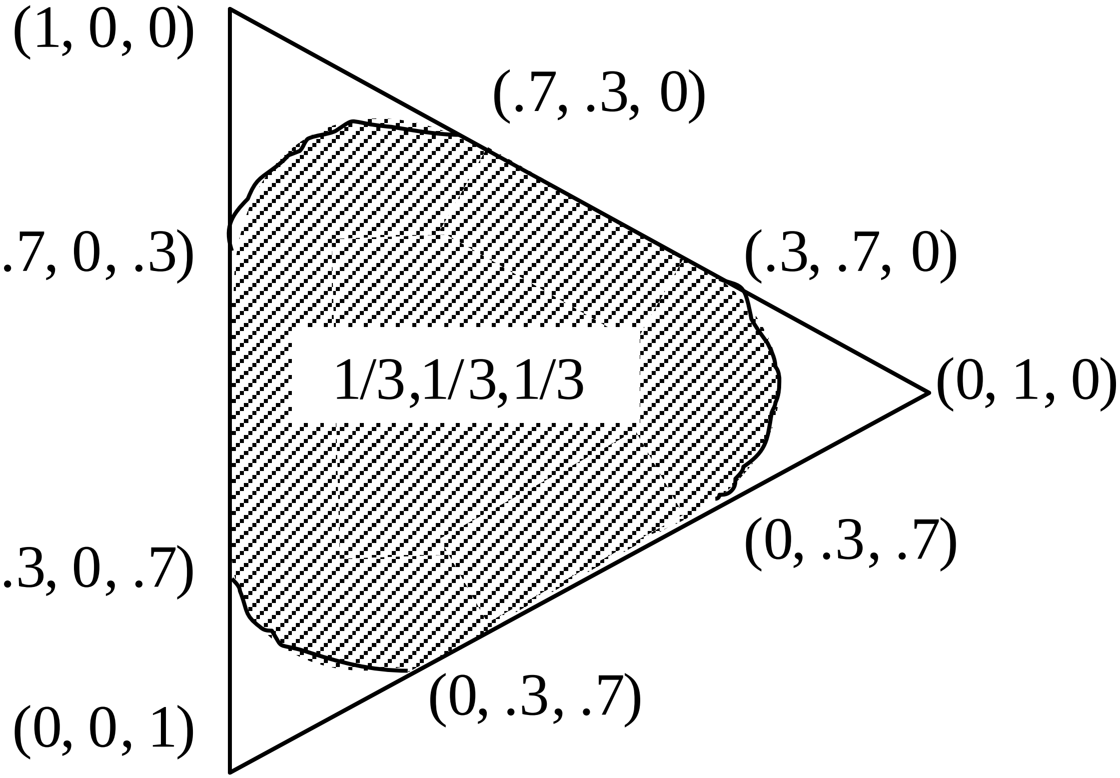
- The partnership (r,F) can be dissolved efficiently if and only if

$$(*) \quad \sum_{i=1}^{n}\left[\int_{v_i^*}^{\overline{v}}[1-F(u)]u\,dG(u)-\int_{\underline{v}}^{v_i^*}F(u)u\,dG(u)\right]\geq 0$$

where v_i^* solves $F(v_i)^{n-1} = r_i$ and $G(u) = F(u)^{n-1}$

Example

- n=3, $F(v_i) = v_i$.
- Then (*) becomes $\sum_{i=1}^{3} r_i^{3/2} \leq 3/4$



Proposition

For any distribution F, the one-owner partnership $r = \{1,0,0,...,0\}$ cannot be dissolved efficiently.

- The one-owner partnership can be interpreted as an auction
- Ex post efficiency is unattainable because the seller's value v_1 is private information: the seller finds it in her best interest to set a reserve price above her value v_1
- An optimal auction maximizes the seller's expected revenue over the set of feasible (ex post inefficient) mechanisms

Theorem

- If a partnership (r,F) can be dissolved efficiently, then the unique symmetric equilibrium of the following bidding game is interim individually rational and achieves ex-post efficiency: given an arbitrary minimum bid $\underline{b}$,
  - the players choose bids $b_i \in [\underline{b},\infty)$
  - the good goes to the highest bidder
  - *each* bidder i pays

$$p_i(b_1,\ldots,b_n) = b_i - \frac{1}{n-1}\sum_{j\neq i}^{n} b_j$$

Theorem

– each player receives a side-payment, independent of the bidding,

$$c_i(r_1,\ldots,r_n) = \int_{\underline{v}}^{v_i^*} u\,dG(u) - \frac{1}{n}\sum_{j=1}^{n}\int_{\underline{v}}^{v_j^*} u\,dG(u).$$

Optimal Auctions (Myerson, 1981)

- Let the n buyers be indexed by $i \in \{1,...,n\}$
- Let each buyer i's willingness to pay for the object be
- t_i independently drawn from $f_i(\cdot)$ $t_i \in [\underline{a}_i, \overline{a}_i]$
- The seller's type, t_0, is *common knowledge*

Bayesian Auction

- A *Bayesian auction* consists of bids spaces $\{B_1,\ldots,B_n\}$ and outcome functions

$$\tilde{p}_i : B \to [0,1] \quad \text{and} \quad \tilde{x}_i : B \to \Re$$

where $B = B_1 \times \ldots, \times B_n$, $\tilde{p}_i$ is the probability that player i gets the object when the bids are $b = \{b_1,\ldots,b_n\} \in B$, and $\tilde{x}_i$ is the payment from i to the seller when the bids are $b \in B$. For each b,

$$\sum_{i=1}^{n} \tilde{p}_i \leq 1$$

which allows for the possibility that the seller may keep the object.

Utility Function

- The utility functions $\tilde{u}_i(b,t)$ are

$$\tilde{u}_i(b,t) \equiv t_i \tilde{p}_i(b) - \tilde{x}_i(b)$$

where t = $\{t_1,\ldots,t_n\}$.

Bayesian Game

- $\Gamma = \{B_1,\ldots,B_n; T_1,\ldots,T_n; f_1,\ldots,f_n; \tilde{u}_1,\ldots, \tilde{u}_n\}$

- A *strategy* for bidder i in this game is $b_i: T_i \rightarrow B_i$
- A strategy profile $b=\{b_1,\ldots,b_n\}$ is a *Bayesian equilibrium* if for each $t_i \in T_i$, the prescribed bid $b_i(t_i)$ is a best response to the n - 1 other strategies b_{-i}.

Maximization Problem

- Choose $\{B_i, \tilde{p}_i, \tilde{x}_i\}$ to maximize the expected revenue

$$E_b\left\{\left[1-\sum_{i=1}^{n}\tilde{p}_i(b)\right]t_0+\sum_{i=1}^{n}\tilde{x}_i(b)\right\}$$

- subject to

$$E_{b_{-i}}\left\{t_i\tilde{p}_i(b_i,b_{-i})-\tilde{x}_i(b_i,b_{-i})|b_i=b_i(t_i)\right\}\geq 0.$$

Seller's Problem

Choose outcome functions $p_i(t)$ and $x_i(t)$ to

(ER) $$\max \int_T \left\{ t_0 \left(1 - \sum_{i=1}^{n} p_i(t) \right) + \sum_{i=1}^{n} x_i(t) \right\} f(t)dt$$

subject to

- $\forall$ i, $\forall$ t, $p_i(t) \geq 0$ and $p_1(t) + \ldots + p_n(t) \leq 1$
- (IC) $V_i(t_i) \equiv v_i(t_i,t_i) \geq v_i(\tau_i,t_i)\ \forall\ i,\ \forall\ \tau_i,\ t_i \in T_i$
- (IR) $V_i(t_i) \geq 0$ for all i.

Definition

Probability of Trade

$$P_i(t_i) \equiv \int_{T_{-i}} p_i(t) f_{-i}(t_{-i}) dt_{-i}$$

is the conditional probability that player i gets the object when i's type is t_i.

Lemma 1

$\{p_i(\cdot),x_i(\cdot)\}$ satisfies (IC) and (IR) iff $\forall$ i

(i) $P_i(t_i)$ is weakly increasing

(ii) $V_i(t_i) = V_i(\underline{a}_i) + \int_{\underline{a}_i}^{t_i} P_i(\tau_i)d\tau_i$ for all $t_i \in T_i$

(iii) $V_i(\underline{a}_i) \geq 0.$

Lemma 2

If $\{p_i(\cdot),x_i(\cdot)\}$ satisfies (IC) and (IR), then (ER) becomes

(ER')

$$t_0 - \sum_{i=1}^{n} V_i(\underline{a}_i) + \int_T \left[\sum_{i=1}^{n} \left\{ t_i - \frac{1-F_i(t_i)}{f_i(t_i)} - t_0 \right\} p_i(t) \right] f(t)dt.$$

Lemma 2

Proof: By definition,

$$V_i(t_i) = \int_{T_{-i}} [t_i p(t_i, t_{-i}) - x_i(t_i, t_{-i})] f_{-i}(t_{-i}) dt_{-i},$$

and by (ii) and the definition of $P_i(t_i)$,

$$V_i(t_i) = V_i(\underline{a}_i) + \int_{\underline{a}_i}^{t_i} P_i(\tau_i) d\tau_i$$

$$= V_i(\underline{a}_i) + \int_{\underline{a}_i}^{t_i} \int_{T_{-i}} p_i(\tau_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i} d\tau_i.$$

Lemma 2

Proof (cont.): Rearranging terms yields

$$\int_{T_{-i}} x_i(t) f_{-i}(t_{-i}) dt_{-i} =$$

$$- V_i(\underline{a}_i) + \int_{T_{-i}} \left[t_i p_i(t) - \int_{\underline{a}_i}^{t_i} p_i(\tau_i, t_{-i}) d\tau_i \right] f_{-i}(t_{-i}) dt_{-i}$$

Lemma 2

Proof (cont.): Integrating with respect to $f_i(t_i)$ produces

$$\int_T x_i(t)f(t)dt = -V_i(\underline{a}_i) + \int_T \left[t_i p_i(t) - \int_{\underline{a}_i}^{t_i} p_i(\tau_i, t_{-i})d\tau_i \right] f(t)dt$$

$$= -V_i(\underline{a}_i) + \int_T \left[t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right] p_i(t)f(t)dt$$

after changing the order of integration.

Substituting into (ER) then completes the proof.

Lemma 2

Solving the problem of optimal auction design

- For fixed $\{p_i(\cdot)\}$ choose

(EP) $$x_i(t) = t_i p_i(t) - \int_{\underline{a}_i}^{t_i} p_i(t_{-i}, \tau_i) d\tau_i,$$

- Suppose the seller sets $V_i(\underline{a}_i) = 0$ for each i. Then the problem has simplified to choosing $\{p_i(\cdot)\}$ to maximize (ER') subject to (i) and the feasibility constraint.

Lemma 2
Optimal Auction Design

Solving the problem of optimal auction design

- Consider choosing $\{p_i(\cdot)\}$ to maximize (ER') subject only to the feasibility constraint.

- Define: $$c_i(t_i) \equiv t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

and for fixed t let j maximize $c_i(t_i)$ over $i \in \{1,...,n\}$ such that $c_j(t_j) \geq c_i(t_i)$ for all i.

Optimal Auction Design

Point-wise Optimization

For each fixed t,

- if $c_j(t_j) - t_0 > 0$ then set $p_j(t) = 1$ and $p_i(t) = 0$ for all $i \neq j$, and

- if $c_j(t_j) - t_0 \leq 0$ then set $p_i(t) = 0$ for all i.

This defines an optimal auction if (i) holds.

Optimal Auction Design

Regular Case:

For each i, $c_i(t_i)$ is weakly increasing in t_i

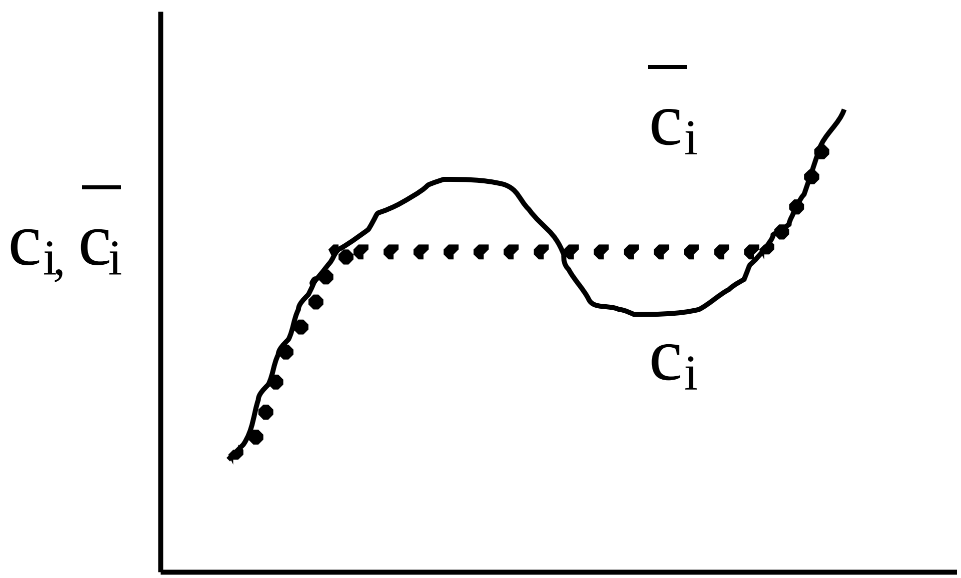
Consider $\tau_i < t_i$

- Then $c_i(\tau_i) \leq c_i(t_i)$, and so $p_i(\tau_i,t_{-i}) \leq p_i(t_i,t_{-i})$ for any t_{-i}
- Therefore $P_i(t_i)$ is weakly increasing, completing the optimization

Optimal Auction Design

Irregular Case:

Define new functions $\bar{c}_i(t_i)$ that are constructed from the functions $c_i(t_i)$ and are guaranteed to be increasing.



Efficiency

Is the optimal auction ex post efficient ?

To maximize expected revenue when the buyers know their types but the seller does not, the seller may need to design an auction that sometimes *fails* to award the object to the player with the highest willingness to pay.

Inefficiency 1: Seller withholds

Example 1:

- Let $t_0 = 0$, and $F_i(t_i) = t_i$ for t_i on [0,1]
- Then $c(t_i) = t_i - (1 - t_i) = 2t_i - 1$ and $c_i(t_i) - t_0 > 0$ if and only if $t_i > 1/2$

The seller sets a *reserve price* that deters half the types from collecting the object, even though this would be efficient since $t_0 = 0$ because this reservation price increases the bids of the other half of the types.

Inefficiency 2: Seller misassigns

Example 2: Asymmetric Bidders

- Let n = 2
- $f_1(t_1) = 1/(\bar{a}_1 - \underline{a}_1)$ on $[\underline{a}_1, \bar{a}_1]$
- $f_2(t_2) = 1/(\bar{a}_2 - \underline{a}_2)$ on $[\underline{a}_2, \bar{a}_2]$
- Let $t_0 = 0$
- Then $c_i(t_i) = 2t_i - \bar{a}_i$, and it could happen that $2t_1 - \bar{a}_1 > 2t_2 - \bar{a}_2 > 0$ and $t_2 > t_1$,

so that 1 gets the object even though 2 values it more

Practical Results: Second-Price Auction

- For each i, $T_i = T_1$, $f_i(t_i) = f_1(t_1)$, and

$$c_1(t_1) = t_1 - \frac{1 - F_1(t_1)}{f_1(t_1)}$$

- Suppose that $c_1(t_1)$ is strictly increasing, so $c^{-1}(\cdot)$ exists.
- For fixed t, let j denote the bidder with highest type: $t_j > t_i$ for all $i \neq j$.

Practical Results: Second-Price Auction

- In the optimal auction, bidder j gets the object if $c(t_j)$ - $t_0 > 0$, or $t_j > c^{-1}(t_0)$, and pays

(EP) $$x_j(t) = t_j p_j(t) - \int_{\underline{a}_j}^{t_j} p_j(t_{-j}, \tau_j) d\tau_j,$$

which is simply

$$\max\{c^{-1}(t_0), \max_{i \neq j} t_i\},$$

the second-highest type.

Practical Results:

Revenue Equivalence Theorem

One form of the *Revenue Equivalence Theorem* states that if

(1) $V(\underline{a}_i) = 0$ for all i, and

(2) for each t, $p_j(t) = 1 \text{ if } t_j > \max_{j \neq i} t_i,$

then the seller's expected revenue is the expected value of the second-highest type.

Remarks

- In the absence of reservation prices, symmetric equilibria of the English, Dutch, first-price (sealed bids), and second-price (sealed bids) auctions satisfy these conditions
- Predominance of the English auction in practice:
  - violation of the *private values* assumption
  - noisy signals about a common value