Lecture Note 2:
Dynamics

- Theory
  - Subgame Perfection
  - Repeated Games
  - Finitely Repeated Games
  - Infinitely Repeated Games

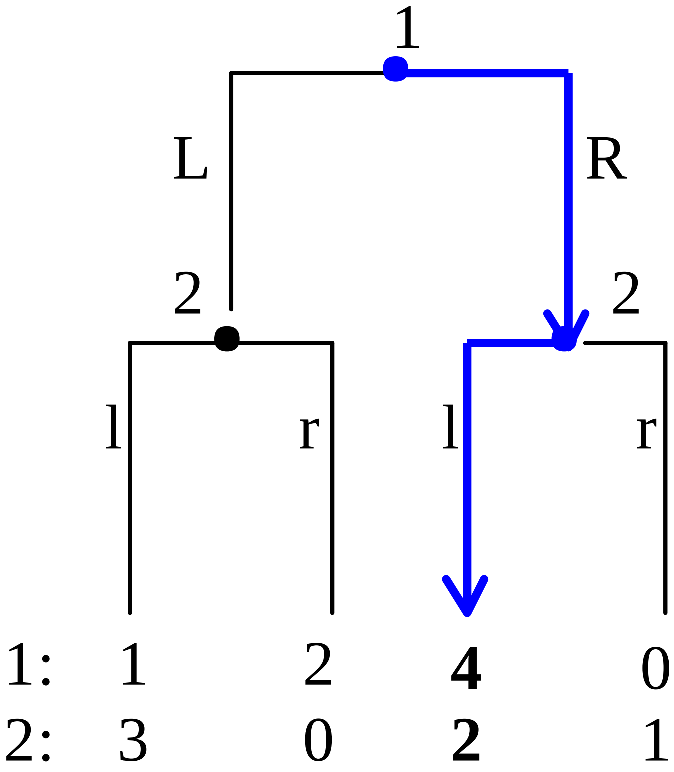
Subgame Perfection

(Selten, 1965)

Nash Equilibrium: each player must act optimally given the other players' strategies, i.e., play a best response to the others' strategies.

Problem: Optimality condition at the beginning of the game. Hence, some Nash equilibria of dynamic games involve incredible threats.

Game in Extensive Form: Backward Induction



Unique equilibrium path

Game in Normal Form

			2		
		ll	lr	rl	rr
1	L	1, 3	**1**, **3**	2, 0	2, 0
	R	**4**, **2**	0, 1	**4**, **2**	0, 1

- Three Nash equilibria in pure strategies: {R,ll}, {L,lr}, and {R,rl}.
- {L,lr}, and {R,rl} involve incredible threats.

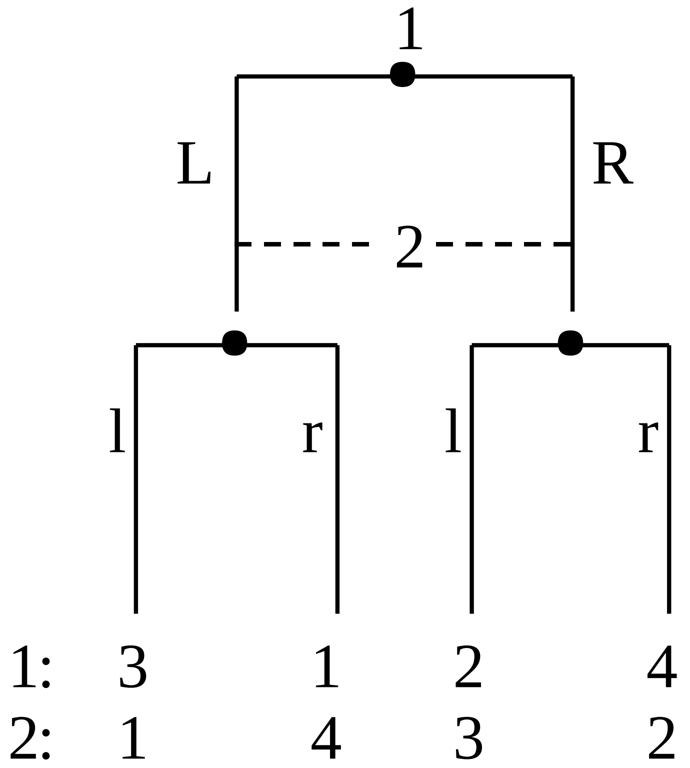
Subgame Perfection with Perfect Information

Consider a game Γ of perfect information consisting of a tree T linking the information sets $i \in I$ (each of which consists of a single node) and payoffs at each terminal node of T. A *subtree* T_i is the tree beginning at information set i, and a *subgame* Γ_i is the subtree T_i and the payoffs at each terminal node of T_i.

Definition

A Nash equilibrium of Γ is *subgame perfect* if it specifies Nash equilibrium strategies in every subgame of Γ. In other words, the players act optimally at every point during the game.

Subgame Perfection with Imperfect Information



With imperfect information, each information set consisting of a single node determines a subgame. Hence, there are no (proper) subgames in this example.

Repeated Games

- Stage game: $G = \{A_1,..A_n; u_1,...,u_n\}$.
- Outcome of G: $a = (a_1,...,a_n) \in A = A_1\times...\times A_n$.
- Repeated game G(T): G is repeated T times.
- a^t outcome of t^{th} repetition of G.
- History prior to t^{th} repetition:$h^{t-1}=(a^1,...,a^{t-1}) \in A^{t-1}$.
- Strategy for player i in G(T) is $\sigma_i=(\sigma_i^1,...,\sigma_i^t,...,\sigma_i^T)$, where $\sigma_i^t: A^{t-1} \rightarrow A_i$ maps the history into an action.
- Stage-game payoffs: $\{u_i(a^1),...,u_i(a^T)\}$.

Repeated Games

- Payoffs for G(T):
  - Average

$$U_i(h^T) = \frac{1}{T}\sum_{t=1}^{T} u_i(a^t)$$

  - Discounted sum

$$U_i(h^T) = \sum_{t=1}^{T} \delta^{t-1} u_i(a^t), \quad \text{for } \delta \in [0,1]$$

Repeated Games

- Players condition their behavior in period t on the past history h^{t-1}.

- This may allow the creation of reputations and cooperation that is not possible in the stage game.

Definition

A strategy profile is a *subgame-perfect equilibrium* of G(T) if:

(i) σ is a Nash equilibrium of G(T), and

(ii) for every t < T and all $h^t \in A^t$, $\sigma[h^t]$ is a Nash equilibrium of G(T-t), where $\sigma[h^t]$ is the strategy profile for the game G(T-t) specified by σ following the history h^t.

Finitely Repeated Games (Benoit and Krishna, *Econometrica* 1985)

- *Case 1*: Stage game with a unique NE

		2	
		Mum	Fink
1	Mum	-1, -1	-5, $\underline{0}$
	Fink	$\underline{0}$, -5	$\mathbf{-\underline{4},-\underline{4}}$

(Fink,Fink) unique NE of this stage game G.

Finitely Repeated Games

What if we repeat G many times?

Is cooperation possible?

- No. Second period play cannot reward cooperative first period behavior, since the players cannot commit to playing Mum.
- This is true for any stage game with a unique equilibrium

(Fink,Fink) unique NE of this stage game G.

Finitely Repeated Games

- *Case 2*: Stage game with multiple NE

			2	
		x_2	y_2	z_2
	x_1	$\underline{\mathbf{1}}, \underline{\mathbf{1}}$	$\underline{5}, 0$	$0, 0$
1	y_1	$0, \underline{5}$	$4, 4$	$0, 0$
	z_1	$0, 0$	$0, 0$	$\underline{\mathbf{3}}, \underline{\mathbf{3}}$

Two pure strategy NE: (x_1,x_2) and (z_1,z_2)

Finitely Repeated Games

Can we support actions that are not Nash equilibria of G to be played in a subgame-perfect equilibrium of G(T)?

- Repeat G twice.
- Can achieve (4,4) in the first period by punishing any deviation from (y_1,y_2) by reverting to (x_1,x_2):

$$\sigma_i^1 = y_i, \text{ and } \sigma_i^2(h^1) = \begin{cases} z_i & \text{if } h^1 = (y_1, y_2), \\ x_i & \text{otherwise.} \end{cases}$$

Finitely Repeated Games

Payoffs

Equilibrium		Best Deviation
$(4 + 3)/2$	$\geq$	$(5 + 1)/2$

σ_i is a SPE of G(2)

Infinitely Repeated Games

(Fudenberg and Maskin, *Econometrica* 1986)

- Stage game $G = \{A_1,\ldots,A_n; u_1,\ldots,u_n\}$.
- Infinitely repeated game $G(\infty)$.
- Strategy $\sigma_i = (\sigma_i^1,\ldots,\sigma_i^t,\ldots)$, where $\sigma_i^t : A^{t-1} \rightarrow A_i$.
- Action set A_i = set of all mixed strategies and strategies based on public randomization devices

Infinitely Repeated Games

- *minimax payoff* $v_j^* = \min_{a_{-j}} \max_{a_j} u_j(a_j, a_{-j})$

  Player j can guarantee a payoff of at least v_j^* given any strategy of others.

- *minimax strategy* $M_{-j}^{j} \in \arg\min_{a_{-j}} \max_{a_j} u_j(a_j, a_{-j})$

  If the others play M^j_{-j} they can hold player j to no more than v_j^*.

Infinitely Repeated Games

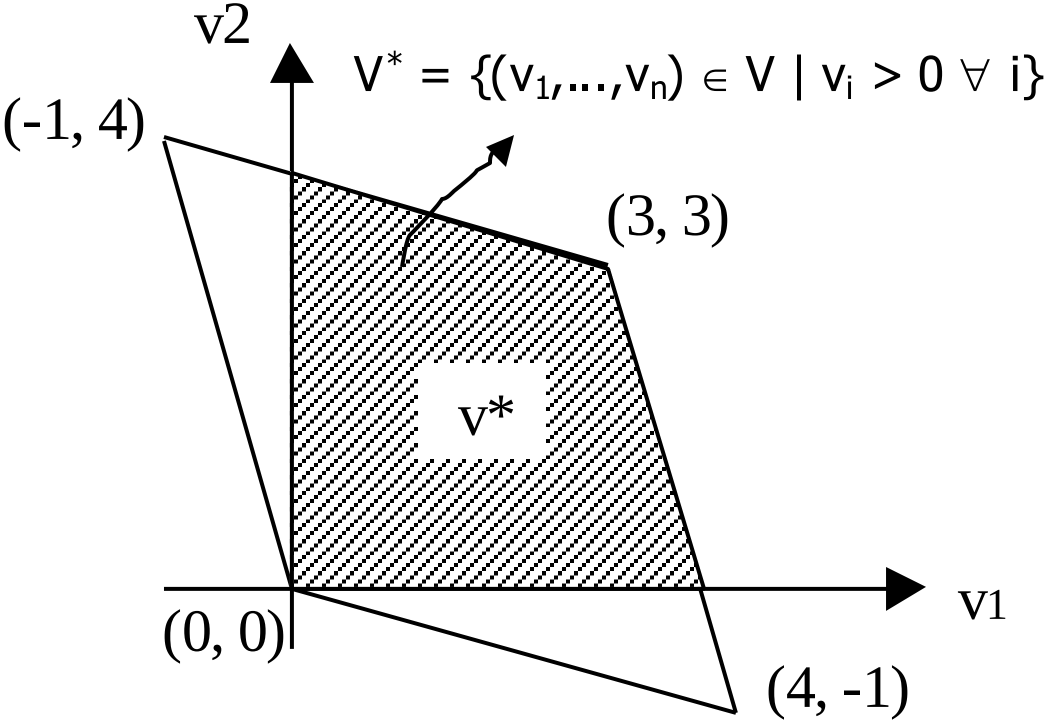
- Payoffs $v_j \geq v_j^*$ are said to be *individually rational* for player j. In any equilibrium of G(T), j's expected payoff must be at least v_j^*.
- Normalize payoff functions $\{u_1,...,u_n\}$ so that
  $v_i^* = 0$ for all i.
- Attainable payoffs
  $U = \{(v_1,...,v_n) \mid \exists\ a \in A$ s.t. $u_i(a) = v_i$ for each $i\}$.
- V = convex hull of U. V is set of payoffs achievable via correlated strategies.

Infinitely Repeated Games

Feasible individually rational payoffs:

$$V^* = \{(v_1,\ldots,v_n) \in V \mid v_i > 0 \ \forall\ i\}$$

		2	
		Mum	Fink
1	Mum	3, 3	-1, <u>4</u>
	Fink	<u>4</u>,-1	**<u>0</u>, <u>0</u>**



Infinitely Repeated Games

- Payoffs in G(∞):

$$U_i(h^\infty) = \sum_{t=1}^{\infty} \delta^{t\text{-}1} u_i(a^t).$$

- Then (1 - δ)$U_i(h^\infty)$ is the average payoff in the sense that if it were received every period then the same total payoff would result.

Folk Theorem

For any $\{v_1,\ldots,v_n\} \in V^*$ if players discount the future sufficiently little ($\exists\ \underline{\delta} \in (0,1)$ s.t. $\forall\ \delta \in (\underline{\delta}, 1)$), there exists a Nash equilibrium of $G(\infty)$ where for all i, i's average payoff is v_i.

Folk Theorem

- Proof:
  - Find a vector of actions $a = (a_1,...,a_n) \in A$ such that $v_i(a) = v_i$ for each i.
  - Let i's strategy be "play a_i until some j deviates from a_j; thereafter play M_i^j."

- Problem: This is not subgame perfect.

Perfect Folk Theorem

(Aumann-Shapley; Rubinstein)

- For any $(v_1,\ldots,v_n) \in V^*$, there exists a perfect equilibrium of $G(\infty)$ with no discounting where, for all i, player i's expected payoff is v_i.
- Payoff without discounting:

$$U_i(h^\infty) = \liminf_{T\to\infty} \tfrac{1}{T}\sum_{t=1}^{T} U_i(a^t).$$

Perfect Folk Theorem

(Proof Sketch)

- Begin by playing a = $(a_1,...,a_n)$ where $u_i(a) = v_i$.
- If j deviates then the others play M^j_{-j}, but only long enough to wipe out j's gain from deviating.
- Afterwards go back to a.
- To induce the punishers to administer the punishment, the punishers are themselves threatened with punishment if they fail to punish the original deviator.

Folk Theorem w/ Nash Threats

(Friedman)

If $(v_1,...,v_n) \in V^*$ Pareto dominates the payoffs of a Nash equilibrium of the stage game G, then if the players discount the future sufficiently little, there exists a perfect equilibrium of $G(\infty)$ where for all i, i's average payoff is v_i.

Folk Theorem w/ Nash Threats (Proof)

Same as the Folk Theorem, except deviations lead to Pareto-dominated NE forever, instead of the minimax strategies.

Folk Theorem w/ Nash Threats

Prisoners' Dilemma

		2	
		Mum	Fink
1	Mum	3, 3	-1, $\underline{x}$
	Fink	$\underline{x}$,-1	$\underline{0}$, $\underline{0}$

Folk Theorem w/ Nash Threats Prisoners' Dilemma

- Let x > 3, so that (F,F) is an equilibrium in dominant strategies.
- The strategies above support (3,3) as the average payoff in a subgame perfect equilibrium provided

$$\frac{3}{1-\delta} \geq x,$$

so if δ = .9, then x $\leq$ 30 works, and if x = 4, $\delta \geq$ 1/4 works.

Theorem 1

(Fudenberg -Maskin)

Two player case:

For any $(v_1,v_2) \in V^*$, there exists $\underline{\delta} \in (0,1)$ s.t. for all $\delta \in (\underline{\delta},1)$ there exists a subgame perfect equilibrium of $G(\infty)$ in which i's average payoff is v_i when the players discount the future according to δ.

Theorem 1

Proof

- Let M_i be i's minimax strategy against j
- and let $$M_i \in \arg\min_{a_i} \max_{a_j} u_j(a_i, a_j),$$

$$\bar{v}_i = \max_{a_1, a_2} u_i(a_1, a_2).$$

- It suffices to show that $\exists$ N and $\underline{\delta} \in (0,1)$ s.t. for each i,

(1) $v_i > \bar{v}_i(1 - \underline{\delta}) + \underline{\delta}\, p_i$, and

(2) $p_i \equiv (1 - \underline{\delta}^N) u_i(M_1, M_2) + \underline{\delta}^N v_i > 0$

Theorem 1

If (1) and (2) hold then the following strategies are subgame perfect:

Phase 1: play a_i provided (a_1,a_2) was played the previous period. After a deviation, Phase 2.

Phase 2: play M_i for N periods and then start *Phase 1* again. After a deviation from *Phase 2* start *Phase 2* again.

Theorem 1

Does there exist N and $\underline{\delta}$ satisfying (1) *and* (2)?

- $v_i \in (0, \bar{v}_i]$ and $u_i(M_1,M_2) \leq 0$. Hence, there exists $\underline{\delta} \in (0,1)$ such that

$$\underline{\delta} > 1 - \frac{v_i}{\bar{v}_i} \quad \text{and} \quad \underline{\delta}^2 > \frac{-u_i(M_1, M_2)}{v_i - u_i(M_1, M_2)}.$$

- The latter guarantees that (2) holds for N=1, and the former ensures that (1) holds if p_i is sufficiently small because $v_i > (1 - \underline{\delta})$. Q.E.D.

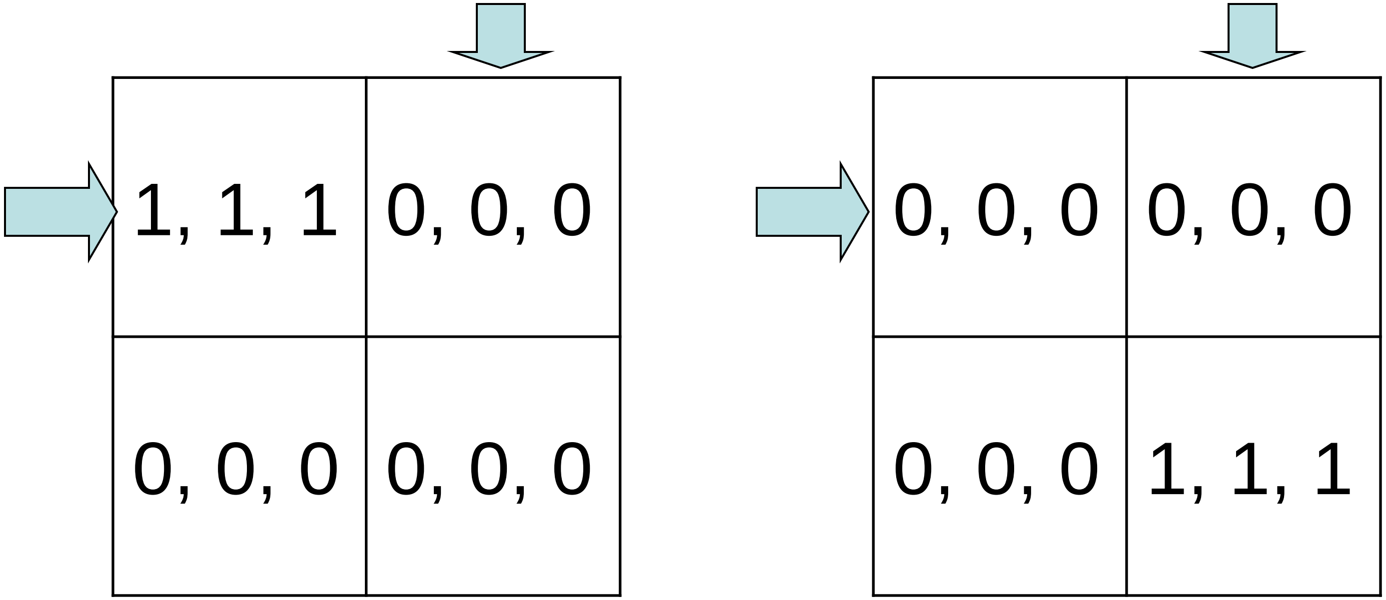
Theorem 2 (Fudenberg -Maskin)

Assume the dimensionality of V^* = n. Then for any $(v_1,\ldots,v_n) \in V^*$, $\exists\ \underline{\delta} \in (0,1)$ s.t. for all $\delta \in (\underline{\delta},1)$ $\exists$ a SPE of $G(\infty)$ in which i's average payoff is v_i when players have discount factor δ.

- The dimensionality condition is needed since discounting may prevent the creation of sufficiently strong punishments to punish those who fail to punish a deviator, so instead those who successfully administer a punishment are rewarded

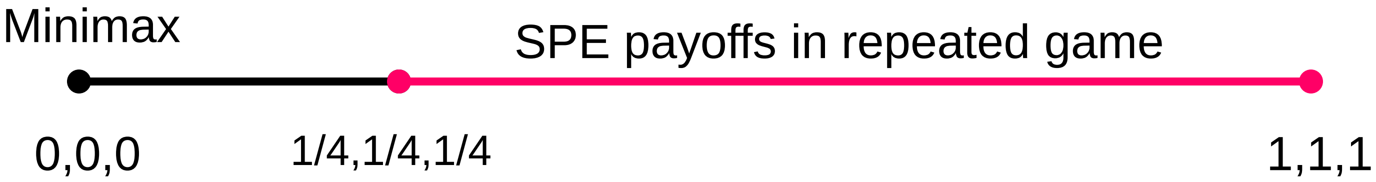
3-player example: any reward, rewards all

Player 3 selects matrix



1, 1, 1	0, 0, 0
0, 0, 0	0, 0, 0

0, 0, 0	0, 0, 0
0, 0, 0	1, 1, 1



Minimax

SPE payoffs in repeated game

0,0,0

1/4,1/4,1/4

1,1,1

Lecture Note 2:
Dynamics (continued)

- **Applications**
  - Cartel Maintenance
  - Sequential Bargaining

Cartel Maintenance

(Porter, *Jet* 1983)

Static oligopoly model:

- firms i $\in$ (1,...,N); simultaneous selection of quantity q_i of homogenous good
- quantity vector: $q=(q_i,q_{-i})$
- total production $Q = q_1 + ... + q_n$
- price $\tilde{p}(Q) = \tilde{\theta} p(Q)$ where $p(Q) = a - bQ$ and $\tilde{\theta}$ has distribution $F(\theta)$ on $[0,\infty)$ and mean μ.
- costs $C(q_i) = c_0 + c_1 q_i$ where $0 < c_1 < \mu a$.
- profit $\pi_i(q) = \mu[a - b(q_i + Q_{-i})]q_i - c_0 - c_1 q_i$.

Cartel Maintenance: Nash Equilibrium

- Γ = {$S_1,\ldots,S_n$; $\pi_1,\ldots,\pi_n$} where S_i = $[0,\infty)$ for all i.
- $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$ is a (Cournot) NE if $\pi_i(\bar{q}_i, \bar{q}_{-i}) \geq \pi_i(q_i, \bar{q}_{-i})$ for all i and $q_i \geq 0$.
- FOC: $$\frac{\partial \pi_i(q_i, \bar{q}_{-i})}{\partial q_i} = 0.$$
- Unique Cournot NE satisfies:

$$\bar{q}_i = \frac{\mu a - c_1}{\mu b(N+1)} = \frac{A}{B(N+1)} \equiv s$$

where A = $\mu a - c_1$ and B = μb.

Cartel Maintenance: Collusion Outcome

- Maximize aggregate profits subject to the constraint that every firm stay in operation:

$$\max_{q} \sum_{i=1}^{N} \pi_i(q).$$

- Solution: $q_i = r \equiv A/2BN$, for $i \in (1,...,N)$.

- Problem: $(r,...,r)$ is not an equilibrium in Γ.

Cartel Maintenance: Infinitely Repeated Game

How close to the cooperative outcome can we get in $G(\infty)$?

- periods {1,...,t,...}
- firm i selects q_i^t and then observes p_t but not q^t_{-i}.
- random disturbance $\tilde{\theta}_t$ is iid according to $F(\theta)$.
- expected payoff $E\left\{\sum_{t=0}^{\infty} \beta^t \pi_i(q^t)\right\}$, with $\beta \in (0,1)$.
- No proper subgames here: no firm reaches a singleton information set because no firm ever learns the complete history of the game.

Cartel Maintenance: Trigger Strategy

Trigger-price strategy (q,p,T)

- q: cooperative quantity
- p: trigger price
- T: duration of punishment phase

For all i:

(a) i plays q_i in period 0;

(b) if q_i was played in period t and $p_t \geq p$ then i plays q_i in period t+1; and

(c) if q_i was played in period t and $p_t < p$ then i plays s for the T-1 periods t+1, t+2, ... , t+T-1, and then plays q_i in period t+T.

Cartel Maintenance: Trigger Strategy

- Value function:

$$\mathrm{V_i(q)} = \pi_{\mathrm{i}}\mathrm{(q)} + \mathrm{Pr(p_t \geq p)}\beta \mathrm{V_i(q)}$$

$$+ \mathrm{Pr(p_t < p)}\left[\sum_{\mathrm{t=1}}^{\mathrm{T\text{-}1}} \beta^{\mathrm{t}} \pi_{\mathrm{i}}\mathrm{(s)} + \beta^{\mathrm{T}} \mathrm{V_i(q)}\right].$$

- Solving for $V_i(q)$

$$\mathrm{V_i(q)} = \frac{\pi_{\mathrm{i}}\mathrm{(s)}}{1-\beta} + \frac{\pi_{\mathrm{i}}\mathrm{(q)} - \pi_{\mathrm{i}}\mathrm{(s)}}{1-\beta+(\beta-\beta^{\mathrm{T}})\,\mathrm{Pr(p_t < p)}},$$

where $Pr(p_t<p)=Pr(\theta_t p(Q)<p)=Pr(\theta_t<p/p(Q))=$ $F(p/p(Q))$.

Cartel Maintenance: Trigger Strategy

- The trigger-price strategy (q,p,T) is a Nash equilibrium if for all i

$$V_i(q_i, q_{-i}) \geq V_i(\tilde{q}_i, q_{-i}) \quad \text{for all } \tilde{q}_i \geq 0.$$

Propositions

Solution: $q^*(p,T) = \{q_1^*(p,T),\ldots,q_N^*(p,T)\}$

Value function: $V_i^*(p,T) \equiv V_i(q^*(p,T))$.

- *Proposition 1.* Cournot-Nash quantity $q_i^* = s$ for all i is a Nash equilibrium for all p,T.
- *Proposition 2.* Given p and T, $q_i^* = q_j^*$ for all i,j.
- *Proposition 3.* For all p and T, $q_i^* \in (s/N, s]$ where $s/N < r < s$.
- *Proposition 4.* If $F(\theta)$ is convex then $V_i(q)$ is concave in q_i, so the foc is sufficient.

Optimal Punishment

- Find optimal trigger price strategy (p,T). Interior solutions are characterized by

$$\frac{\partial V_i^*}{\partial p} = \frac{\partial V_i^*}{\partial T} = 0.$$

Optimal Punishment

- *Proposition 5.* For interior p^* and T^*,

$$q_i^* = \begin{cases} r\left(\dfrac{N + \eta^* + (N+1)(a/A)}{N + \eta^* + 1}\right) & \text{for } \eta^* > \eta^0 \\ s & \text{otherwise,} \end{cases}$$

where $\eta^* = \dfrac{f(p^* / p(Nq_i^*))}{F(p^* / p(Nq_i^*))} \cdot \dfrac{p^*}{p(Nq_i^*)}$

and $\eta^0 = \dfrac{N+1}{N-1} \cdot [(N+1)(a/A) - N].$

Proposition 5: Corollaries

- $q_i^* \in (r,s]$;
- $dq_i^*/d\eta^*$ is less than (equals) zero when η^* is greater than (is less than or equal to) η°;
- as $\eta^* \to \infty$, $q_i^* \to r$.

Sequential Bargaining (Rubinstein, *Econometrica* 1982)

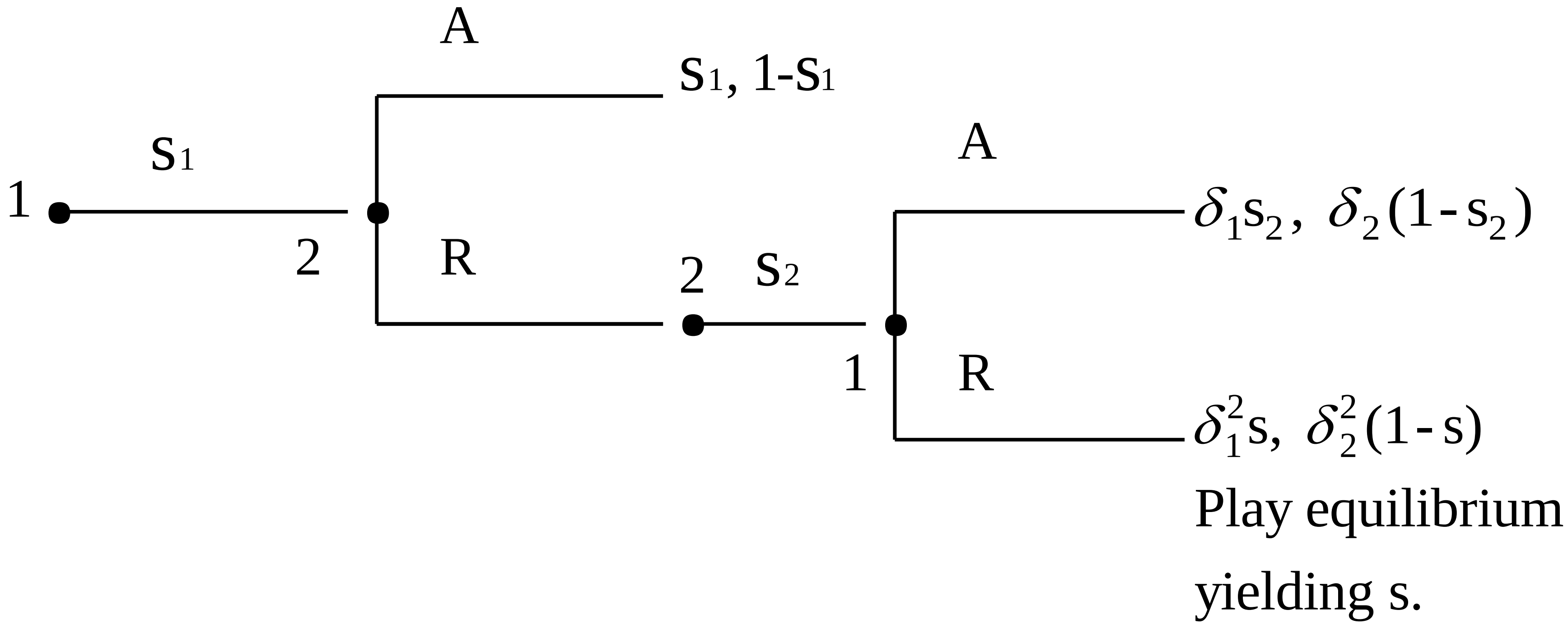
- Two players bargain over the division of a pie.
- They alternate in making offers: player 1 makes an offer which player 2 can accept or reject; if 2 rejects then 2 makes a proposal which 1 can accept or reject; and so on.
- Each offer takes one period and players discount payoffs received in period t by the discount factors δ_1^t and δ_2^t, respectively.
- Let all proposals represent player 1's share of the pie. So if 1 offers s_1 in the first period, 2 rejects it and offers s_2 in the second period, and 1 accepts, then the payoffs are $\delta_1 s_2$ for 1 and $\delta_2(1 - s_2)$ for 2.

Bargaining Solution

- If the players discount future payoffs, then the game has a unique subgame-perfect equilibrium in which trade occurs immediately (efficient bargaining outcome).

- In this unique subgame-perfect equilibrium outcome 1 offers $s_1 = (1 - \delta_2)/(1 - \delta_1\delta_2)$, and 2 accepts.

Bargaining Solution Uniqueness



Proof of Uniqueness

Backward Induction: Suppose there exists an equilibrium in which 1 receives s if the game reaches period three.

- In period two, 1 will only accept offers that are at least as great as δ_1s, so 2's best response is to offer $s_2^* = \delta_1 s$, since $1 - \delta_1 s > \delta_2(1 - s)$.
- In period one, 2 will accept 1's offer of s_1 if and only if $1 - s_1 \geq \delta_2(1 - \delta_1 s)$, so that 1's best response is to offer $s_1^* = 1 - \delta_2 + \delta_1\delta_2 s$.

Proof of Uniqueness

- So given any SPE payoff s, there exists another subgame-perfect equilibrium payoff

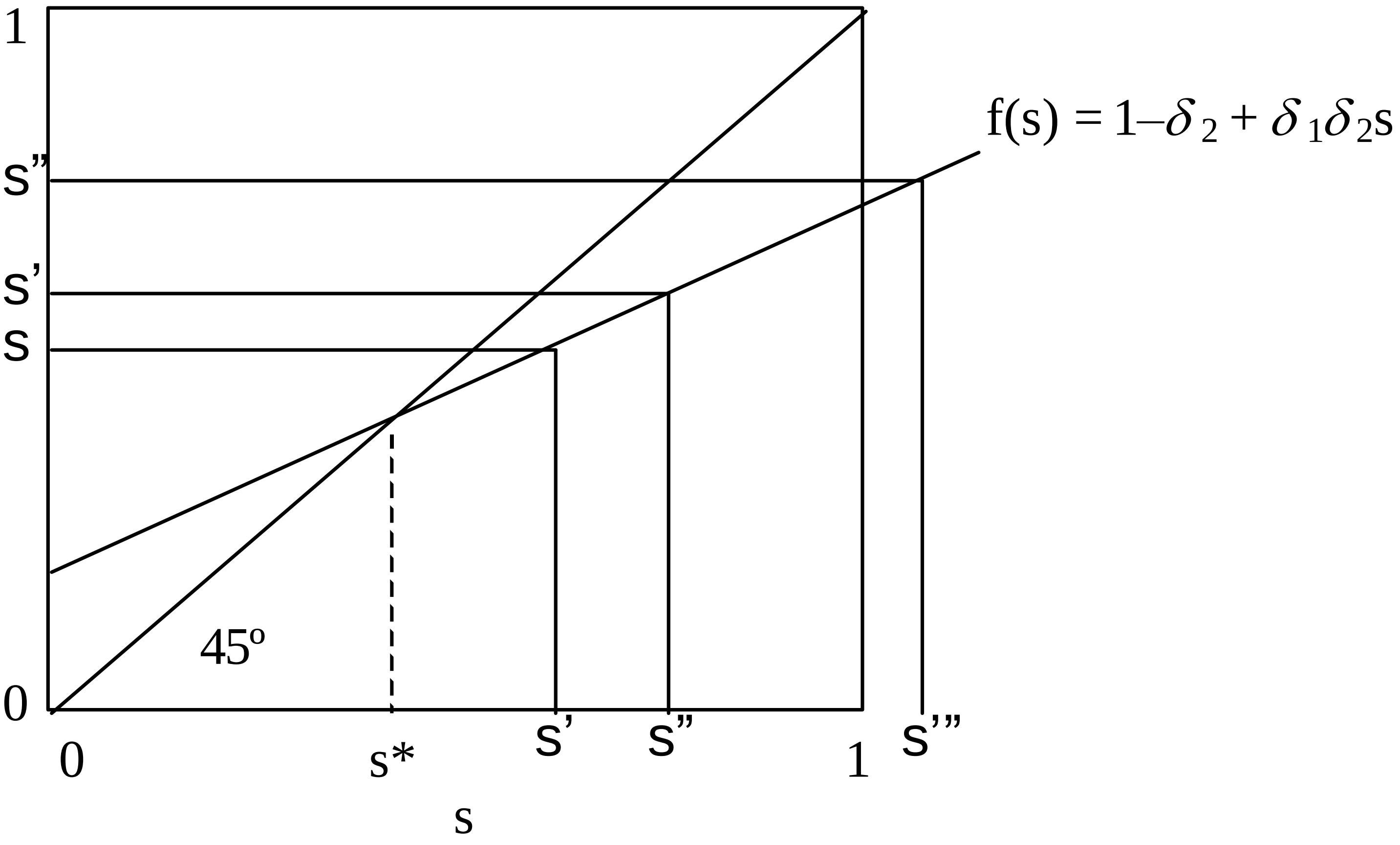
$$s' = f(s) = 1 - \delta_2 + \delta_1\delta_2 s.$$

- Given s there also exists

$$s'' = f^{-1}(s) = \frac{s + \delta_2 - 1}{\delta_1\delta_2}$$

where s'' supports s just as s supports s'.

Proof of Uniqueness



Unique Solution: fixed point s^* that satisfies $s^* = f(s^*)$.

Bargaining Solution Existence

Consider the following strategies:

- 1 always offers $x = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$
- 2 always offers $y = 1 - \frac{1 - \delta_1}{1 - \delta_1 \delta_2}$

1 only accepts offers of y or more

2 only accepts offers of x or less.

Proof of Existence

Since strategies are the same in every subgame, just prove that the strategies are Nash (best response).

- Given player 2's strategy, player 1 can get x in any odd period or y in any even period.
- Since $x > \delta_1 y$, player 1's best response is to get x immediately. And given player 1's strategy, player 2 can get 1 - x in any odd period or 1 - y in any even period.
- Since $(1 - x) = \delta_2(1 - y)$, player 2 is just as well off taking 1 - x immediately as waiting a period for 1 - y.

Proof of Existence

Necessary conditions for an equilibrium: Make offer so other is indifferent to accept

- 2 is indifferent between

  1 - x today and 1 - y tomorrow, and

- 1 is indifferent between

  y today and x tomorrow.