Economics 703
Advanced Microeconomics

Professor Peter Cramton

Fall 2017

Outline

- Introduction
- Syllabus
- Web demonstration
- Examples

About Me: Peter Cramton

- B.S. Engineering, Cornell University
- Ph.D. Business & Economics, Stanford University
- Associate Professor, Yale University, 1984-93
- National Fellow, Hoover Institution, Stanford University, 1992-93
- Professor of Economics, University of Maryland, since 1993
- Chairman, Market Design Inc., 1995-2016
- Chief Economist, Rivada, since 2014
- Chief Economist, Tremor, since 2017

Course Outline

- Strategic-Form Games
- Extensive-Form Games
- Repeated Games
- Bayesian Games and Bayesian Equilibrium
- Dynamic Games of Incomplete Information
- Refinements of Sequential Equilibrium
- Bargaining Theory
- Market Design
  - Matching
  - Auction Theory and Practice

Logistics

- Meet Tuesday and Thursday, 9:30 to 10:45
  - Sometimes meet 8:05 to 10:45 (double class)
- Problem Sets (about 5) [1/3 of grade]
  - Must be own work; don’t look at past solutions
  - Small discussion groups fine
- Final Exam [2/3 of grade]
  - Thursday, 14 December 2017, 8 am to 11 am
- Office Hours by Appointment
  - Tydings 4101a
  - pcramton@gmail.com

Readings

- Lecture Notes (from course web page)
- Fudenberg and Tirole (FT), *Game Theory*, MIT Press
- Levine (L), *Is Behavioral Economics Doomed?*, Open Press
- Krishna (K), *Auction Theory*, Academic Press
- Milgrom (M), *Putting Auction Theory to Work*, Cambridge Press
- Cramton, Shoham, Steinberg (CSS), *Combinatorial Auctions*, MIT Press
- Web site: www.cramton.umd.edu

Lecture Note 1: Foundations

Introduction and Examples

Definition

Game theory is the study of mathematical models of conflict and cooperation between *intelligent and rational* decision makers.

- *Rational:* each individual maximizes her expected utility
- *Intelligent:* individual understands situation, including fact that others are intelligent rational decision makers

Game 1

- Each of three players simultaneously picks a number from [0,1]
- A dollar goes to the player whose number is closest to the average of the three numbers
- In case of ties, the dollar is split equally

Game 1 in Normal Form (Strategic Form)

- player i $\in$ N = {1,...,n}
- strategy $s_i \in S_i$
- strategy vector (profile)

  $s = (s_1,...,s_n) \in S = S_1 \times ... \times S_n$
- payoff function $u_i(s):S \rightarrow \Re$, which maps strategies into real numbers
- game in normal form $\Gamma = \{S_1,...,S_n;u_1,...,u_n\}$

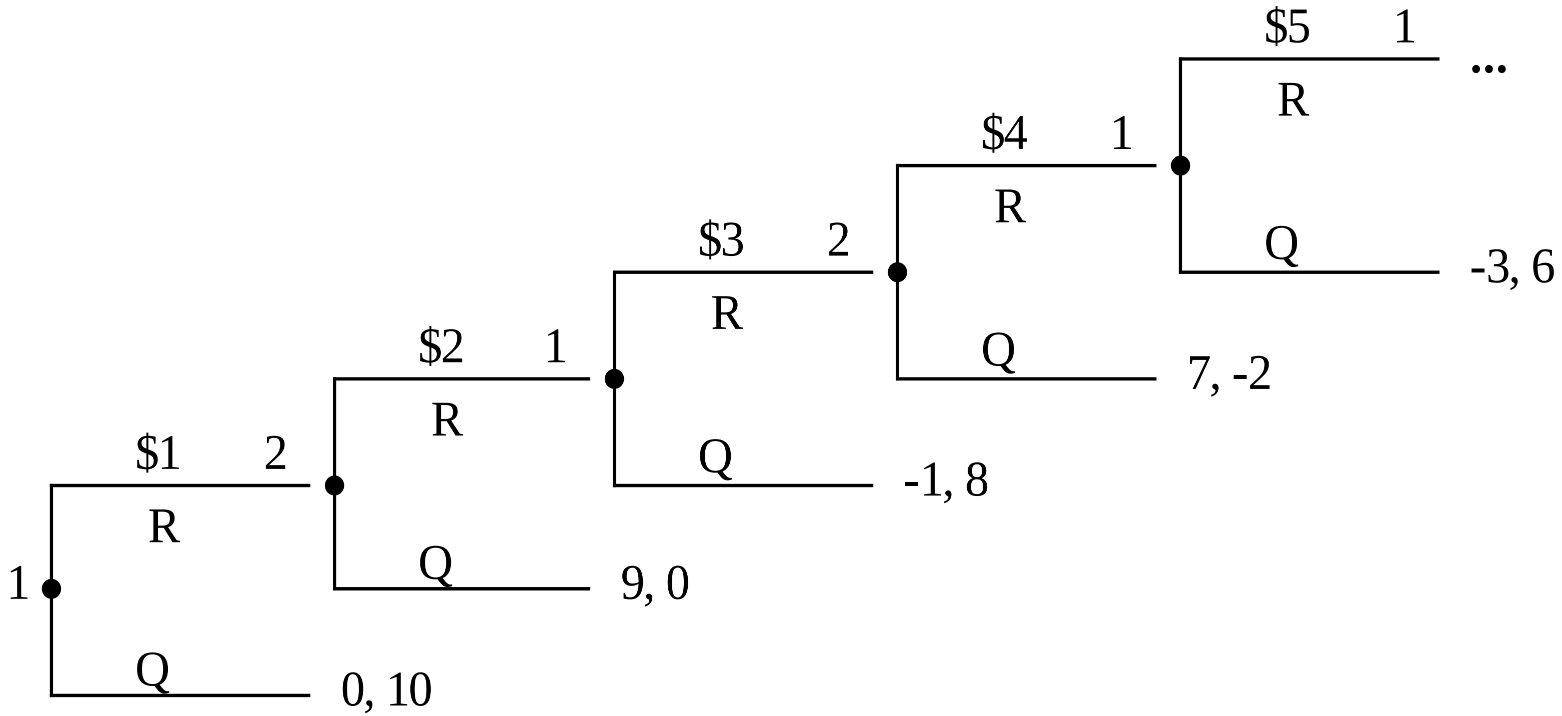
Game 2: Both Pay Auction

- $10 is auctioned to highest of two bidders
- Players alternate bidding
- At each stage, bidding player must decide either to raise bid by $1 or to quit
- Game ends when one of the two bidders quits in which case the other bidder gets the $10, and *both* bidders pay the auctioneer their bids

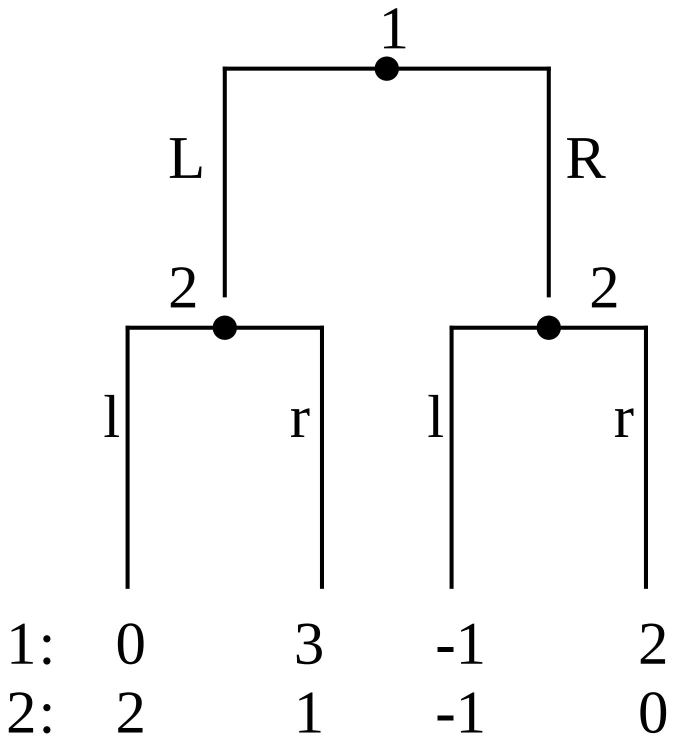
Game in Extensive Form

- Who plays when?
- What can they do?
- What do they know?
- What are the payoffs?

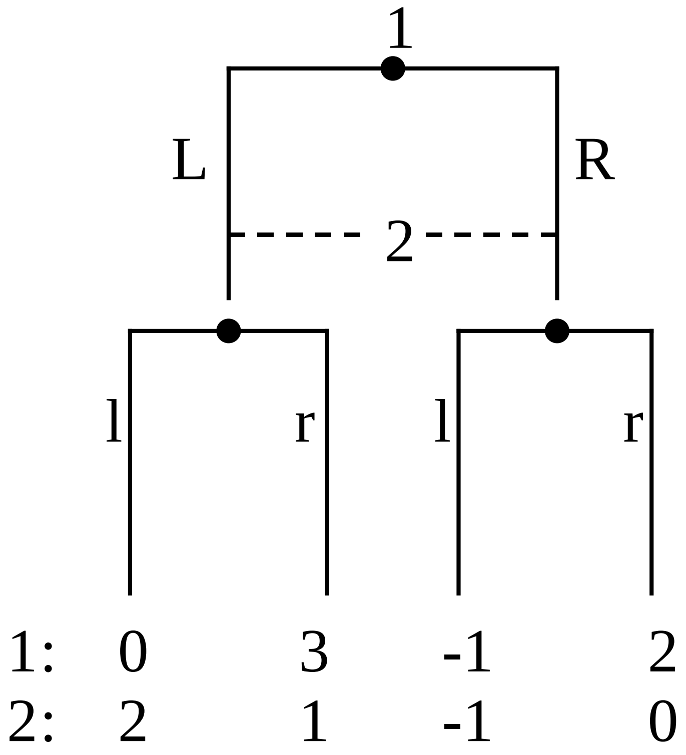
Game 2 in Extensive Form (Game Tree)



Game 3



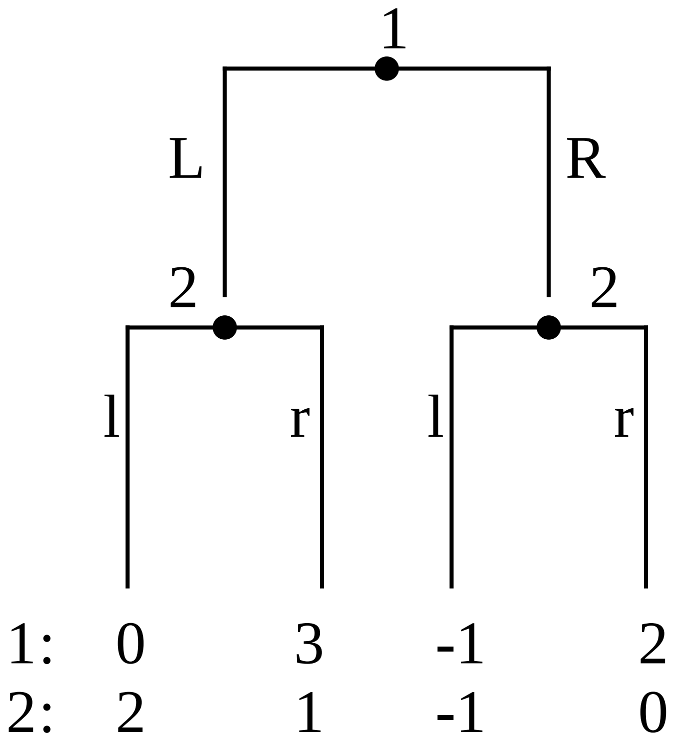
Game 4



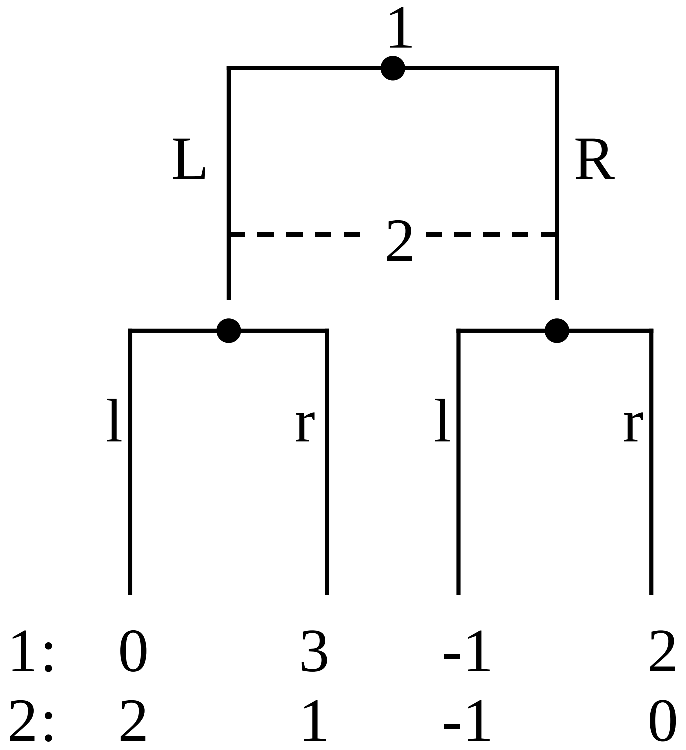
Definitions

- *Strategy*: a complete plan of action (what to do in *every* contingency)
- *Information Set*: for player i is a collection of decision nodes satisfying two conditions: player i has the move at every node in the collection, and i doesn't know which node in the collection has been reached

Game 3: How many info sets?



Game 4: How many info sets?

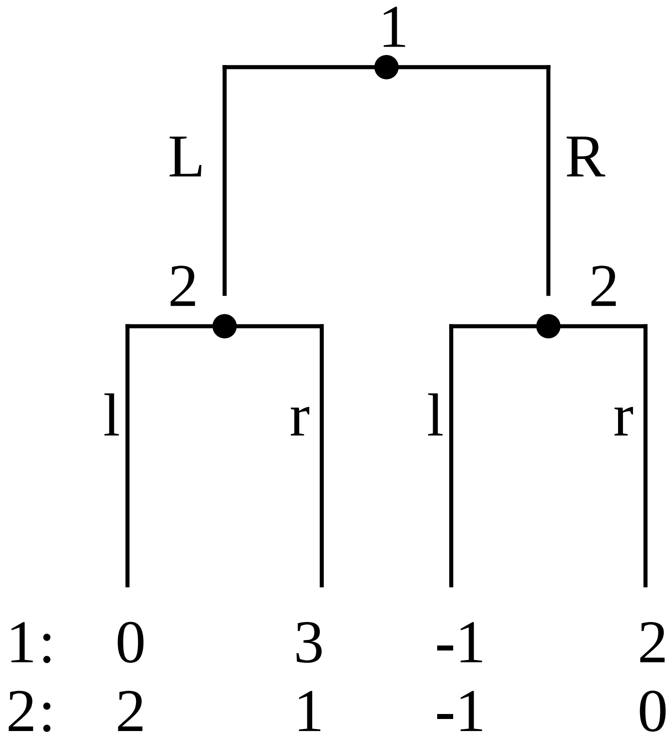


More Definitions

- *Perfect Information*: each information set is a single node (Chess, checkers, go, ...)
- Finite games of perfect information can be "solved" by backward induction in the extensive form or elimination of weakly dominated strategies in the normal form
- *Imperfect Information*: at some point in the tree some player is not sure of the complete history of the game so far

Game 3: Backward Induction

Looking ahead and reasoning back



Lecture Note 1: Foundations

- Introduction and Examples (Continued)
- Formal Treatment
  - Existence of Nash Equilibrium
  - Existence without Quasi-concavity
  - Perfect Recall
  - Correlated Equilibria

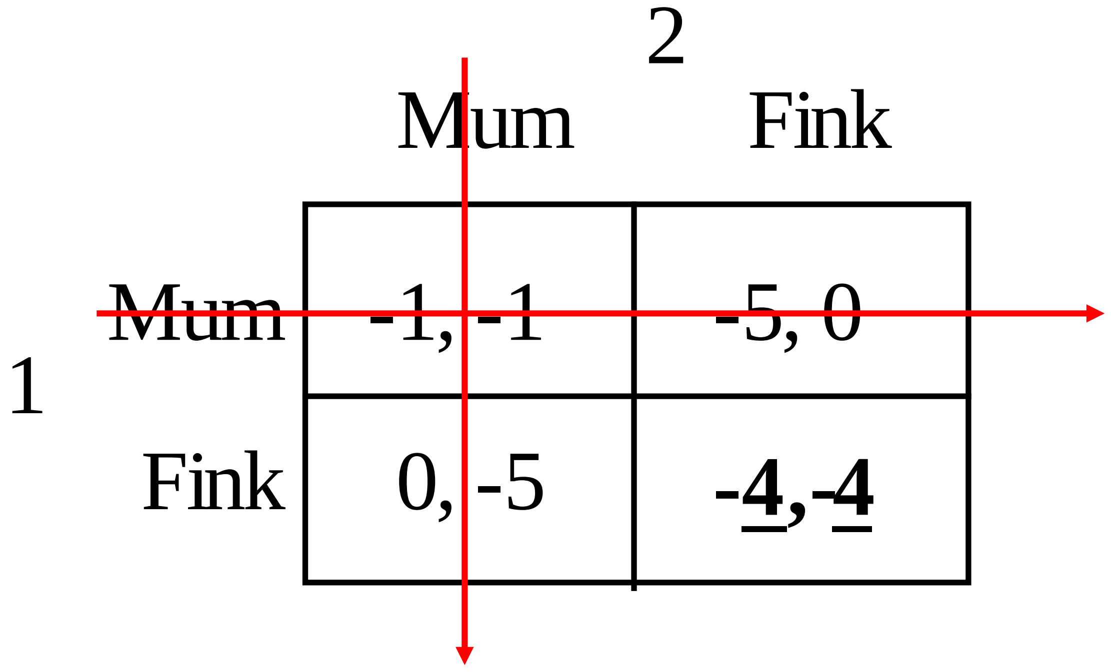
Game 5: Elimination of Dominated Strategy (Prisoner’s Dilemma)

- Two suspects are arrested and charged with a crime.
- They are held in separate cells.
- The DA separately offers each the chance to turn state’s evidence.
- A jail sentence of x years has utility –x.

Game 5: Prisoner’s Dilemma

- What is Rational?
- What is Efficient?

Game 5 in Normal Form



		2	
		Mum	Fink
1	Mum	-1, -1	-5, 0
	Fink	0, -5	**-4,-4**

Definitions

- *Dominated Strategy*: x *strictly dominates* y if the player gets a higher payoff from playing x than playing y, *regardless* of what the other players do.
- x *weakly dominates* y if the player's payoff is at least as great by playing x than y, regardless of what the other players do.

Nash Equilibrium

For an n-person game in Normal form, a strategy profile $s^* \in S$ is a *Nash equilibrium* in pure strategies if for all i

$$u_i(s^*) \geq u_i(s^*_{-i}, s_i) \quad \text{for all} \quad s_i \in S_i$$

$$\text{where} \quad s^*_{-i} = (s^*_1, \ldots, s^*_{i-1}, s^*_{i+1}, \ldots, s^*_n).$$

Game 6: Matching Pennies

- Each of two players simultaneously show a penny.
- If the pennies match (both heads or both tails), player 1 gets 2's penny. Otherwise, player 2 gets 1's penny.

Game 6 in Normal Form

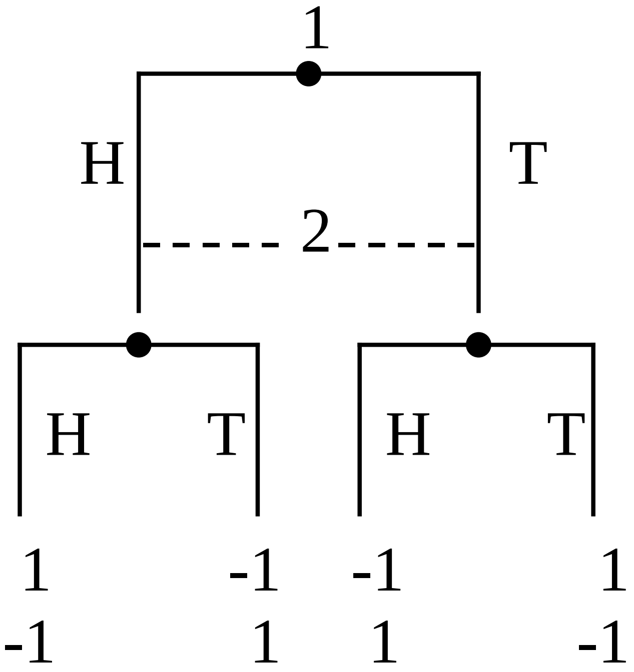
		2		
		y	1-y	
		H	T	EU_1
1	H	$\underline{1}$, -1	-1, $\underline{1}$	y – (1 – y)
	T	-1, $\underline{1}$	$\underline{1}$,-1	-y + (1 – y)

2y – 1 = 1 – 2y

y = ½.

No equilibrium in pure strategies

Game 6 in Extensive Form



Definitions

- *Zero-Sum Game*: Sum of the players' payoffs is zero, regardless of outcome.
- *Mixed strategy*: a randomization over pure strategies.

Existence of Nash Equilibria (Finite Games)

- Normal form game

  $\Gamma = (S^1,\ldots,S^n;\ u^1,\ldots,u^n)$
- Pure Strategy Profile

  $s = \{s^1,\ldots,s^n\} \in S = S^1 \times \ldots \times S^n$
- Mixed Strategy Profile

  $\sigma = \{\sigma^1,\ldots,\sigma^n\} \in \Delta(S^1) \times \ldots \times \Delta(S^n)$ where

  $\sigma^i: S^i \rightarrow [0,1]$ and $\sigma^i(s^i)$ = Pr(i plays pure strategy s^i).

Existence of Nash Equilibria (Finite Games)

- Expected payoff:

$$\mathrm{v}^{\mathrm{i}}(\sigma) = \sum_{\mathrm{s} \in \mathrm{S}} \mathrm{u}^{\mathrm{i}}(\mathrm{s}) \prod_{\mathrm{j}=1}^{\mathrm{n}} \sigma^{\mathrm{j}}(\mathrm{s}^{\mathrm{j}})$$

- Other's strategy:

$$\sigma^{-\mathrm{i}} = \{\sigma^{1}, \ldots, \sigma^{\mathrm{i}-1}, \sigma^{\mathrm{i}+1}, \ldots, \sigma^{\mathrm{n}}\}$$

$$(\sigma^{-\mathrm{i}}, \hat{\sigma}^{\mathrm{i}}) = \{\sigma^{1}, \ldots, \sigma^{\mathrm{i}-1}, \hat{\sigma}^{\mathrm{i}}, \sigma^{\mathrm{i}+1}, \ldots, \sigma^{\mathrm{n}}\}$$

Definition

An n-tuple of mixed strategies $\sigma = (\sigma^1,...,\sigma^n)$ is a Nash Equilibrium if for every i,

$$\mathrm{v}^{\mathrm{i}}(\sigma) \geq \mathrm{v}^{\mathrm{i}}(\sigma^{-i},\hat{\sigma}^{\mathrm{i}})$$

for every

$$\hat{\sigma}^{\mathrm{i}} \in \Delta(\mathrm{S}^{\mathrm{i}}).$$

Theorem 1 (Nash)

Every finite game has a Nash equilibrium in mixed strategies.

Theorem 2

Consider an n-person game

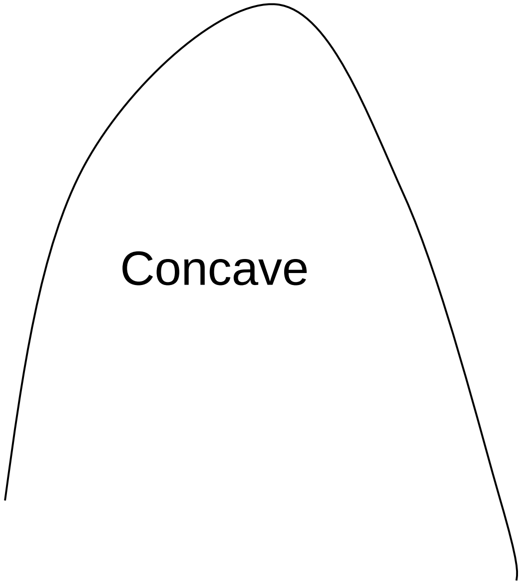
- $\Gamma = \{D^1,\ldots,D^n;v^1,\ldots,v^n\}$
- D^i: set of pure strategies available to i
- $v^i: D^1\times\ldots\times D^n \rightarrow \Re$ is i's payoff function

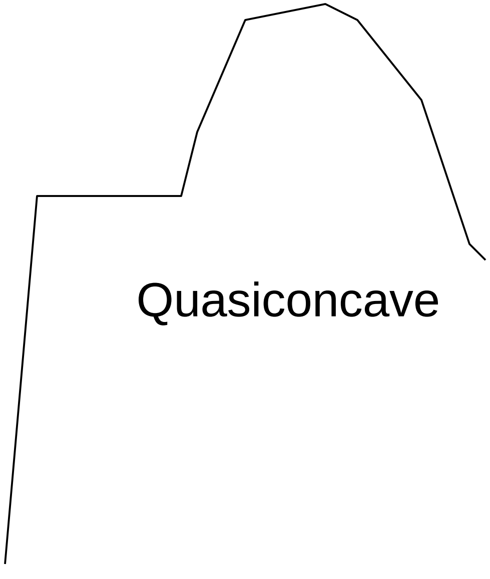
If each D^i is a compact convex subset of a Euclidean space, and each v^i is continuous and quasiconcave in d^i, then Γ has a Nash equilibrium in pure strategies.

Definitions

- *Quasiconcave*

$$v^{i}(d^{-i},\alpha d^{i}+(1-\alpha)\hat{d}^{i})\geq \min\{v^{i}(d^{-i},d^{i}),v^{i}(d^{-i},\hat{d}^{i})\}$$





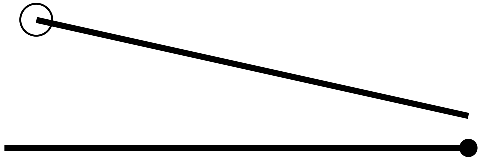
Definitions

- *Upper hemicontinous*

  A correspondence φ: T $\rightarrow$ V is *upper hemicontinuous* at a point $x \in T$ if $x_r \rightarrow x$ and $y_r \rightarrow y$, where $y_r \in \varphi(x_r)$ for every r, implies $y \in \varphi(x)$.

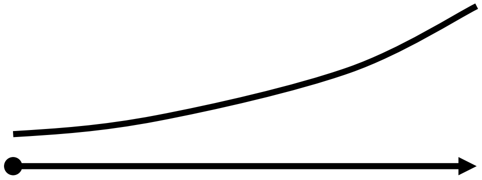
For best response to exist need maximum to exist

- Continuous function on compact set has a maximum; hence, require:
  - closed



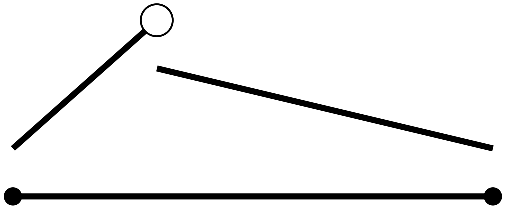
or no max

  - bounded



or no max

  - continuous

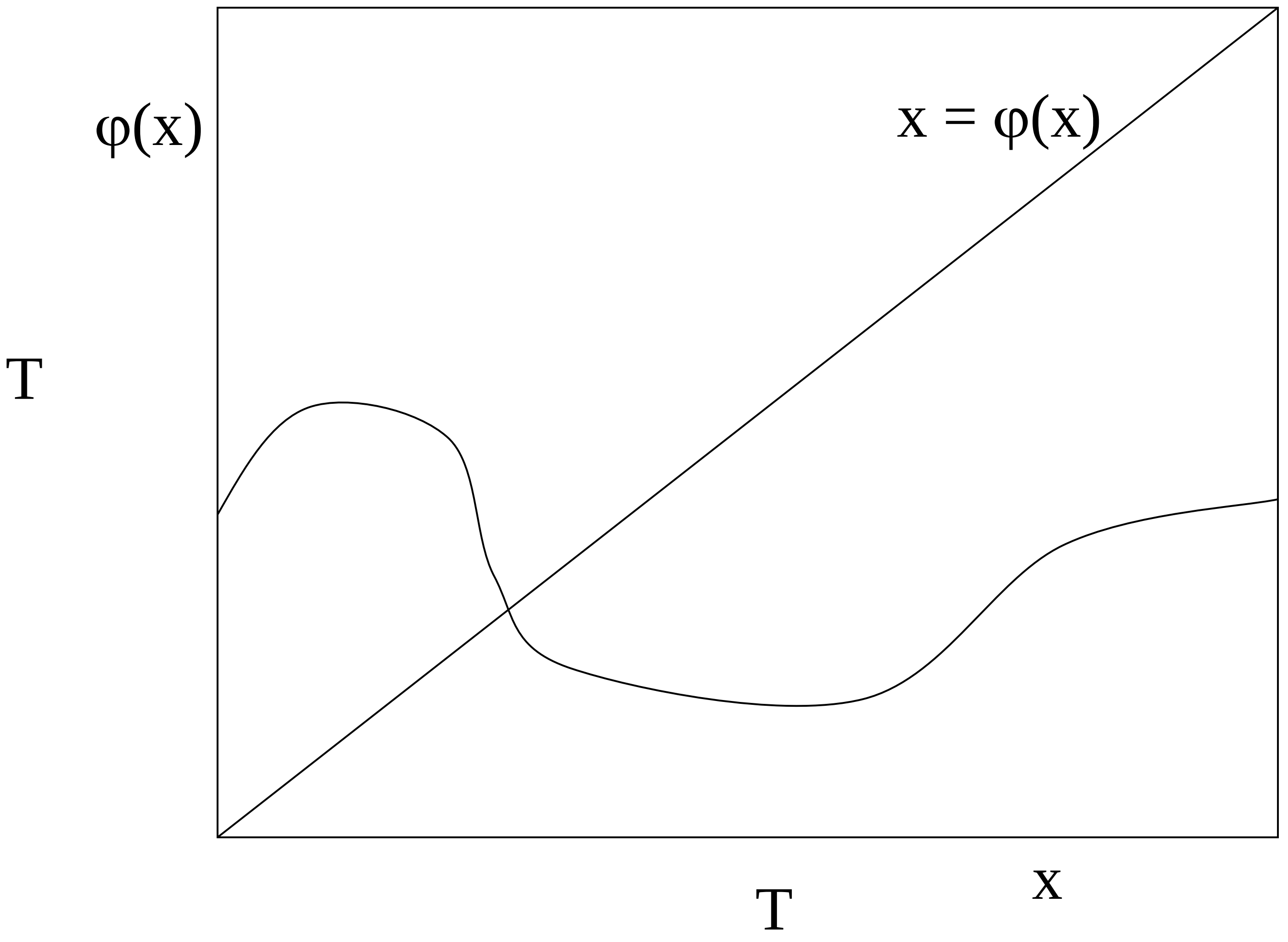


or no max

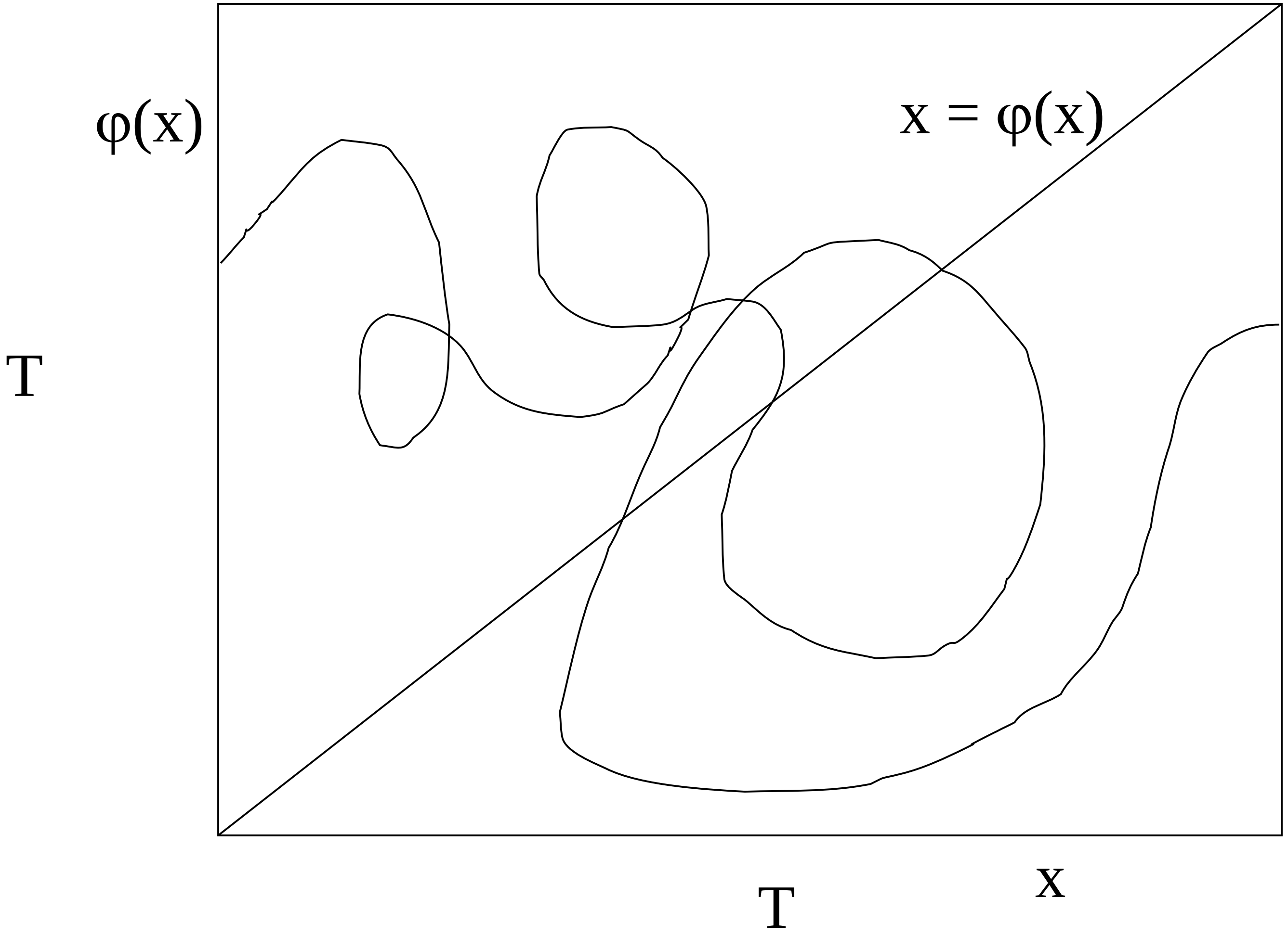
Theorem 3
(Kakutani)

If T is a nonempty compact convex subset of a Euclidean space, and φ is an upper hemicontinuous nonempty convex-valued correspondence from T to T, then φ has a fixed point, i.e., there is an $x \in T$ such that $x \in \varphi(x)$.

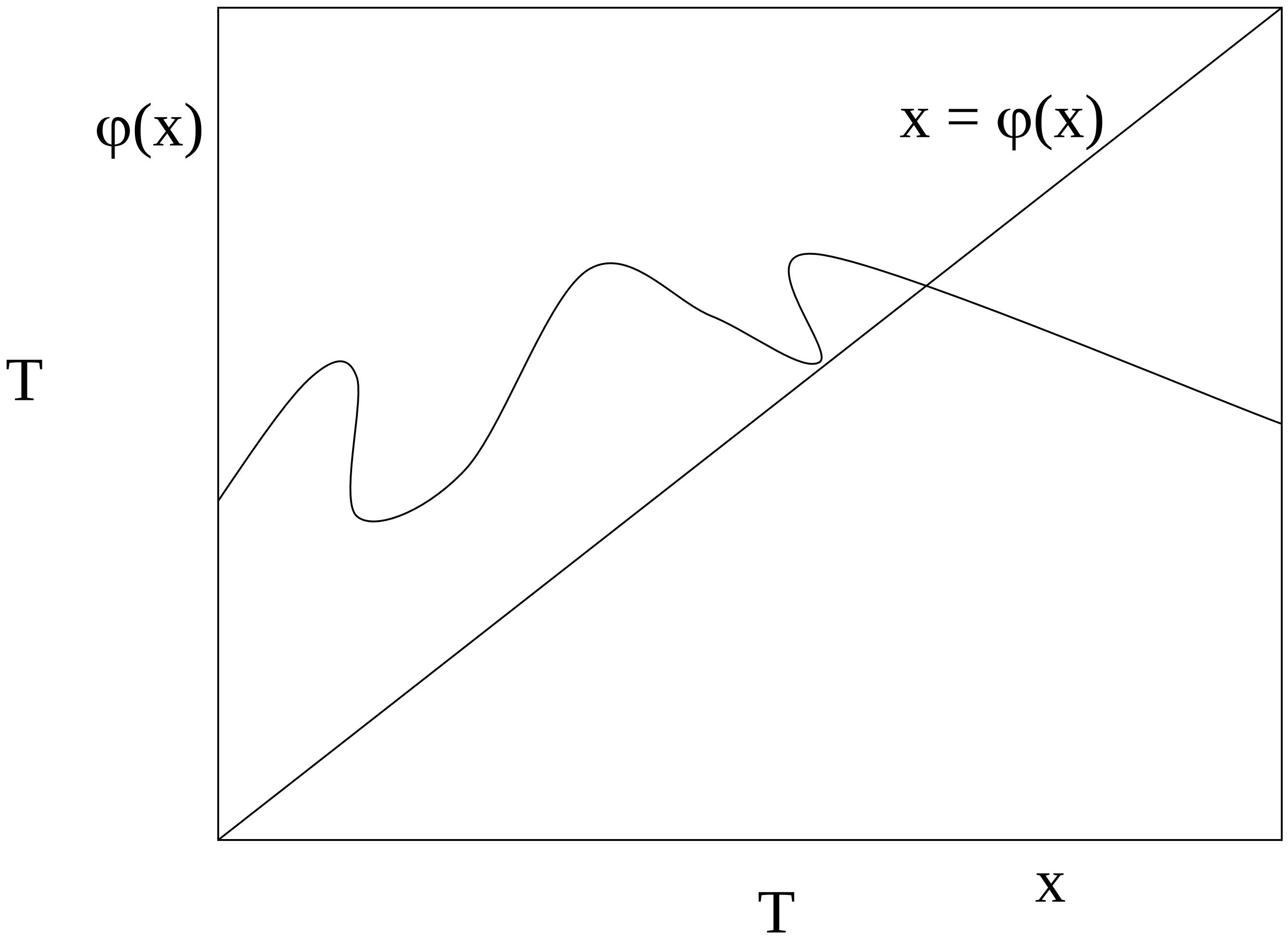
“Proof” of Kakutani Fixed Point Theorem



“Proof” of Kakutani Fixed Point Theorem



Usually an odd number of equilibria



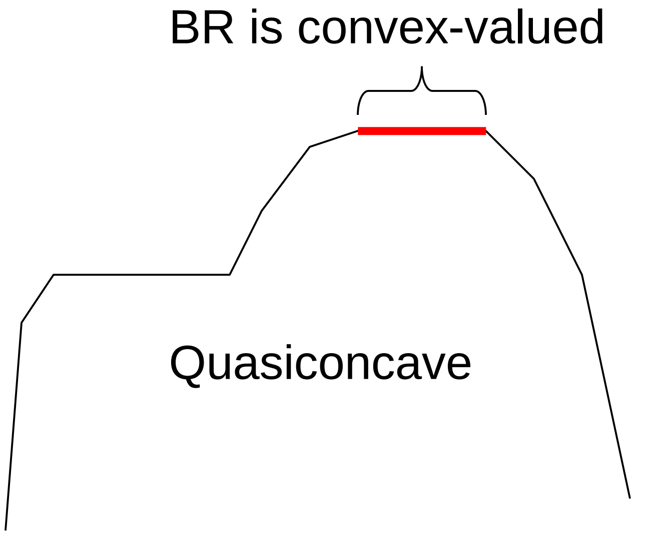
Proof of Theorem 2

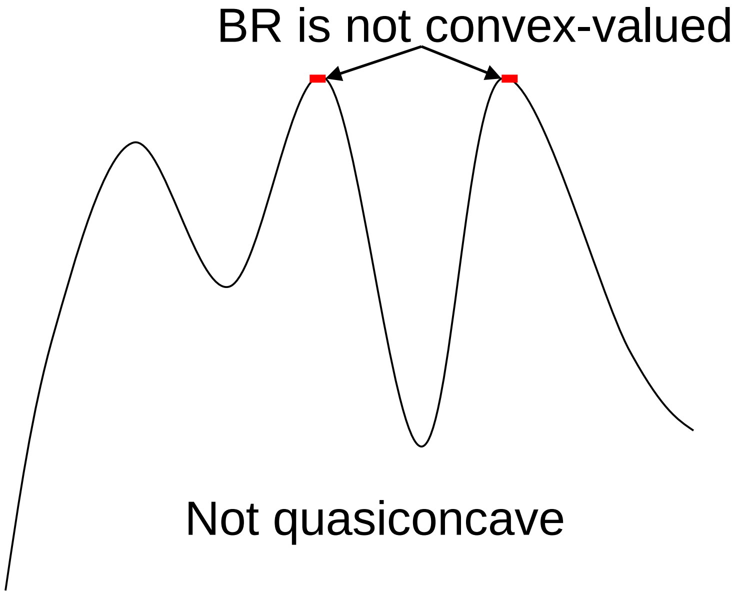
For each i, define "best response" correspondence $\varphi^i : D = D^1 \times \ldots \times D^n \rightarrow D^i$ as:

$$\varphi^i(d) = \{d^i \in D^i \mid v^i(d^{-i}, d^i) \geq v^i(d^{-i}, d^{i\prime}) \ \forall \ d^{i\prime} \in D^i\}.$$

- φ^i is nonempty since D^i is compact and v^i is continuous
- φ^i is convex-valued since v^i is quasiconcave in d^i
- φ^i is upper hemicontinuous, since v^i is continuous

φ^i is convex-valued since v^i is quasiconcave in d^i





Proof of Theorem 2 (Cont.)

Define φ: D $\rightarrow$ D by $\varphi(d) = \varphi^1(d)\times\ldots\times\varphi^n(d)$. D is a compact subset of Euclidean space since each D^i is, and φ is an upper hemicontinuous nonempty convex-valued correspondence since each φ^i is. So by Kakutani's theorem, φ has a fixed point. But a fixed point of φ is just a Nash equilibrium of Γ.

Q.E.D.

Proof of Theorem 1

Let $D^i = \Delta(S^i)$. Each D^i is then a compact convex subset of Euclidean space, and each v^i is continuous and quasiconcave in d^i (indeed linear). Hence, by Theorem 2, there is a Nash equilibrium in which each player chooses a "pure" strategy from $\Delta(S^i)$.

Q.E.D.

Existence w/o Quasiconcavity

(Dasgupta and Maskin, *RES* 1986)

- Strategy set is Continuous
- Payoffs functions are not quasiconcave

$\rightarrow$ Can't use Kakutani's FPT.

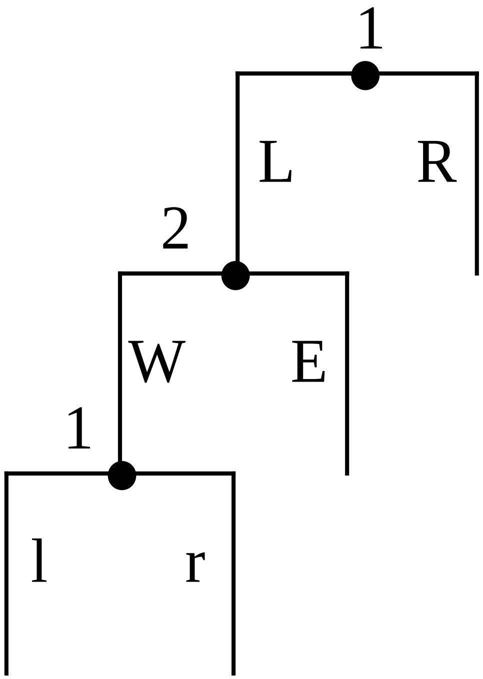
Theorem (Glicksberg). Let each D^i be a non-empty and compact subset of Euclidean space. Let each v^i be continuous. Then Γ has a Nash equilibrium in mixed strategies.

Perfect Recall

A game has *perfect recall* if each player knows whatever he knew previously, including his previous actions.

- *Example*: bridge (2 player or 4 player game).
- A *behavioral strategy* specifies a probability distribution over feasible actions at each information set.

Games with Perfect Recall



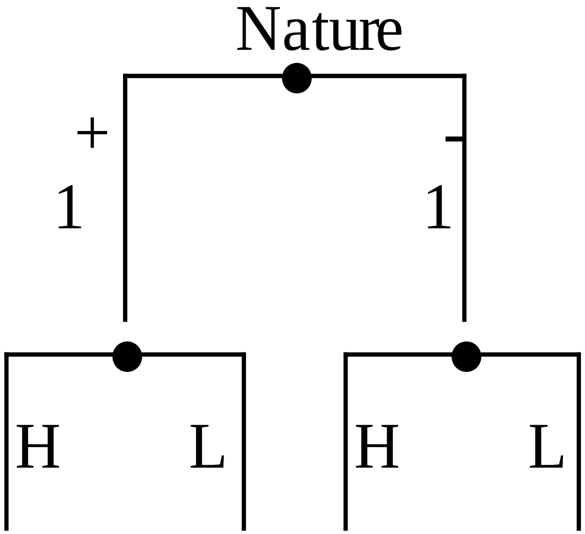
$S_1 = \{Ll, Lr, R\}$

Mixed: $(p_1, p_2, 1 - p_1 - p_2)$

Behavior: $p = \Pr(L)$

$q = \Pr(l|L, W)$

Games with Perfect Recall



S_1 = {HH, HL, LH, LL}

Mixed: (p_1, p_2, p_3, p_4)

Behavior: p = Pr(H|+) q = Pr(H|-)

Correlated Equilibria

(Aumann, *JME* 1974)

Example 1: Battle of the Sexes

		2	
		L	R
1	U	**2**, **1**	0, 0
	D	0, 0	**1**, **2**

Nash equilibria:	Payoffs:
(U,L)	(2,1)
(D,R)	(1,2)
(2/3U,1/3L)	(2/3,2/3)

→By correlating strategies players can get (3/2, 3/2).

Correlated Equilibria

Example 2: Coordination Game

		2	
		L	R
1	U	6, 6	$\underline{\mathbf{2}}$, $\underline{\mathbf{7}}$
	D	$\underline{\mathbf{7}}$, $\underline{\mathbf{2}}$	0, 0

Correlated Equilibria

Example 2: Coordination Game

Nash equilibria:	Payoffs:
(U,R)	(2,7)
(D,L)	(7,2)
(2/3 U, 2/3 L)	(14/3,14/3)

Correlated Equilibria

Can they do better?

- Let a random device pick A, B, or C with probability 1/3 each.
- 1 is told whether A is chosen.
- 2 is told whether C is chosen.
- 1 plays D if A and U otherwise; 2 plays R if C and L otherwise.

This yields payoff (5, 5)

Definition

For a normal form game, a *correlated strategy* is a probability distribution p(s) over the set of pure-strategy n-tuples S.

Definition

The correlated strategy p(s) is a *correlated equilibrium* of the mediated game if for every i and for all s^*_i such that $p(s^*_i) > 0$,

$$\sum_{s^*_{-i} \in S_{-i}} u_i(s^*_{-i}, s^*_i) p(s^*_{-i} | s^*_i) \geq$$

$$\sum_{s^*_{-i} \in S_{-i}} u_i(s^*_{-i}, s_i) p(s^*_{-i} | s^*_i) \quad \text{for all } s_i \in S_i$$

Correlated Equilibria

- *Theorem:* Every point in the convex hull of the Nash-equilibrium payoffs is a correlated-equilibrium payoff.

- *Proof*: Use a mutually observable randomizing device.

Correlated Equilibria

- *Theorem*: The CE payoffs are a convex polyhedron defined by linear inequalities, unlike the n-1st degree equations that determine Nash equilibria.
- *Proof*: The linear inequalities determine a convex polyhedron in the space of correlated strategies, which determines a convex polyhedron of correlated-equilibrium payoffs.