Economics 703
Advanced Microeconomics **Prof. Peter Cramton**

Lecture Note 1: Foundations[1]

Outline

A. Introduction and Examples

B. Formal Treatment

1. Existence of Nash Equilibrium
2. Existence without Quasi-concavity
3. Perfect Recall
4. Correlated Equilibria

A. Introduction and Examples

Definition (Roger Myerson [1986]): *Game theory* is the study of mathematical models of conflict and cooperation between *intelligent and rational* decision makers. *Rational* means that each individual's decision-making behavior is consistent with the maximization of subjective expected utility. *Intelligent* means that each individual understands everything about the structure of the situation, including the fact that others are intelligent rational decision makers.

Game 1: Each of three players simultaneously picks a number from [0,1]. A dollar goes to the player whose number is closest to the average of the three numbers. In case of ties, the dollar is split equally.

Description of a Game in Normal Form

player $i \in N = \{1,...,n\}$

strategy $s_i \in S_i$

strategy vector (profile) $s = (s_1,...,s_n) \in S = S_1 \times ... \times S_n$

payoff function $u_i(s): S \rightarrow \Re$, which maps strategies into real numbers

game in normal form $\Gamma = \{S_1,...,S_n;u_1,...,u_n\}$

Game 2 (both pay auction): Ten dollars is auctioned to the highest of two bidders. The players alternate bidding. At each stage, the bidding player must decide either to raise the bid by $1 or to quit. The game ends when one of the two bidders quits in which case the other bidder gets the ten dollars, and *both* bidders pay the auctioneer their bids.

Description of a Game in Extensive Form (See Kreps and Wilson, *Econometrica* 1982)

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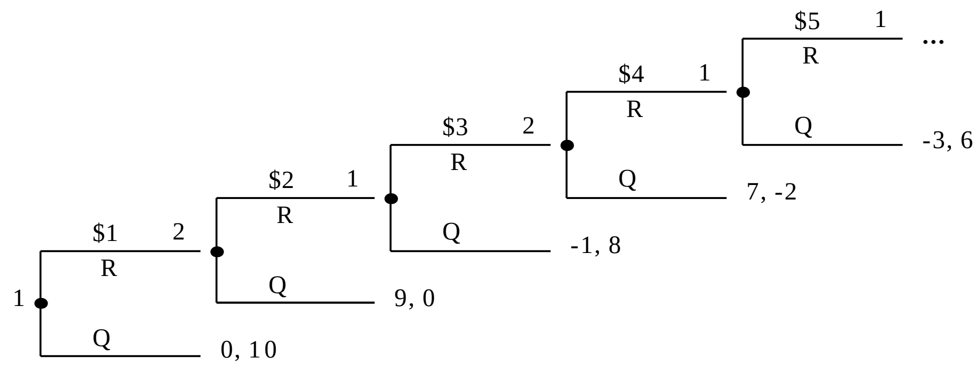
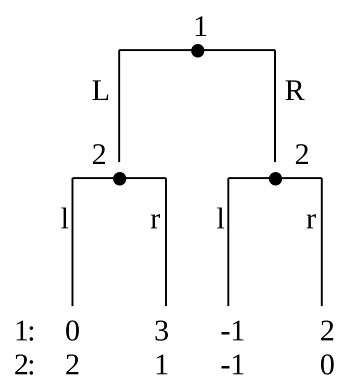
[1]These notes are based without restraint on notes by Robert Gibbons, MIT.

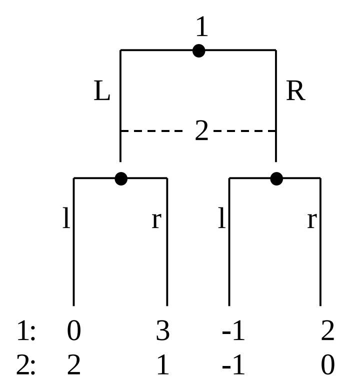
An extensive form answers the questions: Who plays when, what they can do, what they know, what are the payoffs?



Game 3:



Game 4:



Strategy: a complete plan of action (what to do in *every* contingency).

Information Set: for player i is a collection of decision nodes satisfying two conditions: player i has the move at every node in the collection, and i doesn't know which node in the collection has been reached.

Perfect Information: each information set is a single node. (Chess, checkers, go, ...)

Finite games of perfect information can be "solved" by backward induction in the extensive form or elimination of weakly dominated strategies in the normal form.

Imperfect Information: at some point in the tree some player is not sure of the complete history of the game so far.

Any extensive-form game can be represented in normal form. Two-person games with finite strategy sets are represented as *bimatrix* games:

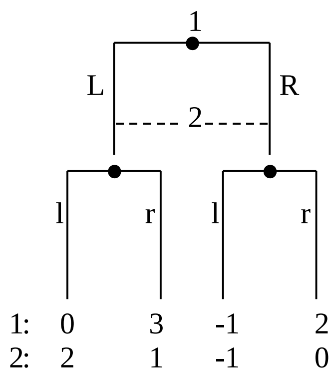
Game 3

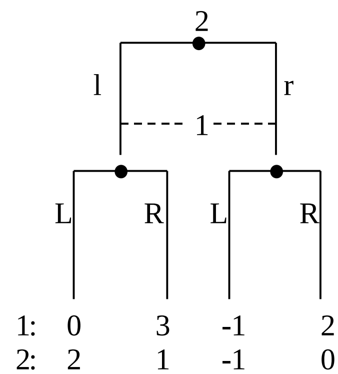
		ll	lr	rl	rr
1	L	0, 2	0, 2	3, 1	3, 1
	R	-1,-1	2, 0	-1, -1	2, 0

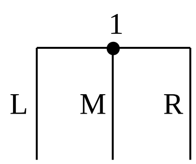
Game 4

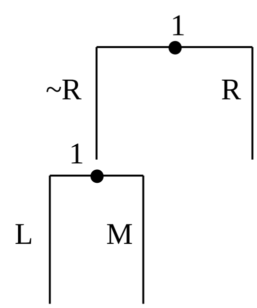
		l	r
1	L	3, 3	-1, $\underline{1}$
	R	$\underline{4}$,-1	$\underline{0}$, $\underline{0}$

Extensive forms contain *more* information than normal forms (e.g., two different extensive forms can have the same normal form):









Game 5 (Prisoners' Dilemma): Two suspects are arrested and charged with a crime. They are held in separate cells. The District Attorney separately offers each the chance to turn state's evidence (i.e., to fink on the other prisoner). A jail sentence of x years has utility -x. The payoffs to the prisoners as a function of their decisions are given by the bimatrix:

		2: Mum	2: Fink
1	Mum	–1, –1	–5, 0
	Fink	0, –5	–4, –4

What is rational?

Dominated Strategy: x *strictly dominates* y if the player gets a higher payoff from playing x than playing y, *regardless* of what the other players do.

x *weakly dominates* y if the player's payoff is at least as great by playing x than y, regardless of what the other players do.

What is rational in game 4?

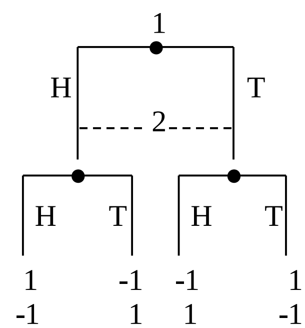
Nash equilibrium: For an n-person game in Normal form, a strategy profile $s^* \in S$ is a *Nash equilibrium* in pure strategies if for all i

$$u_i(s^*) \geq u_i(s^*_{-i}, s_i) \quad \text{for all } s_i \in S_i$$
$$\text{where } s^*_{-i} = (s^*_1, \ldots, s^*_{i-1}, s^*_{i+1}, \ldots, s^*_n).$$

(Each player's strategy is a best response to the others' strategies.)

Game 6 (Matching Pennies): Each of two players simultaneously show a penny. If the pennies match (both heads or both tails), player 1 gets 2's penny. Otherwise, player 2 gets 1's penny.

		2: H	2: T
1	H	1, -1	-1, 1
	T	-1, 1	1,-1



What is rational?

Zero-Sum Game: Sum of the players' payoffs is zero, regardless of outcome.

Mixed strategy: a randomization over pure strategies.

B. Formal Treatment

1. Existence of Nash Equilibrium (Finite Games)

Normal form game: $\Gamma = (S^1,\ldots,S^n; u^1,\ldots,u^n)$

Pure Strategy Profile: $s = \{s^1,\ldots,s^n\} \in S = S^1 \times \ldots \times S^n$

Mixed Strategy Profile: $\sigma = \{\sigma^1,\ldots,\sigma^n\} \in \Delta(S^1) \times \ldots \times \Delta(S^n)$ where

$\sigma^i: S^i \to [0,1]$ and $\sigma^i(s^i) = \Pr(i \text{ plays pure strategy } s^i)$.

Expected payoff:

$$v^i(s) = \sum_{s \in S} u^i(s) \prod_{j=1}^{n} s^j(s^j)$$

Other's strategy: $s^i = \{s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n\}$

$$(s^{-i}, \hat{s}^i) = \{s^1, \ldots, s^{i-1}, \hat{s}^i, s^{i+1}, \ldots, s^n\}$$

Def: An n-tuple of mixed strategies $\sigma = (\sigma^1, \ldots, \sigma^n)$ is a Nash Equilibrium if for every i,

$v^i(s) \geq v^i(s^{-1}, \hat{s}^i)$ for every $\hat{s}^i \in \Delta(S^i)$.

Theorem 1 (Nash): Every finite game has a Nash equilibrium in mixed strategies.

Theorem 2: Consider an n-person game $\Gamma = \{D^1, \ldots, D^n; v^1, \ldots, v^n\}$, where D^i is the set of pure strategies available to i and $v^i: D^1 \times \ldots \times D^n \to \Re$ is i's payoff function. If each D^i is a compact convex subset of a Euclidean space, and each v^i is continuous and quasiconcave in d^i, then Γ has a Nash equilibrium in pure strategies.

Quasiconcave: $v^i(d^{-i}, a d^i + (1-a)\hat{d}^i) \geq \min\{v^i(d^{-i}, d^i), v^i(d^{-i}, \hat{d}^i)\}$, i.e., the payoff from a convex combination of two strategies is at least as great as the payoff from the worst of the two strategies.

Def: A correspondence φ from a subset T of Euclidean space to a compact subset V of Euclidean space is *upper hemicontinuous* at a point $x \in T$ if $x_r \to x$ and $y_r \to y$, where $y_r \in \varphi(x_r)$ for every r, implies $y \in \varphi(x)$. φ is upper hemicontinuous if it is upper hemicontinuous at every $x \in T$.

Theorem 3 (Kakutani): If T is a nonempty compact convex subset of a Euclidean space, and φ is an upper hemicontinuous nonempty convex-valued correspondence from T to T, then φ has a fixed point, i.e., there is an $x \in T$ such that $x \in \varphi(x)$.

Proof of Theorem 2: For each i, define a "best response" correspondence φ^i from $D = D^1 \times \ldots \times D^n$ to D^i as follows. For any $d \in D$, let $\varphi^i(d)$ be the set of strategies which maximize i's payoff given the others strategies are d^{-i}, i.e., $\varphi^i(d) = \{d^i \in D^i \mid v^i(d^{-i}, d^i) \geq v^i(d^{-i}, \hat{d}^i)$ for every $\hat{d}^i \in D^i\}$. φ^i is nonempty since D^i is compact and v^i is continuous. φ^i is convex-valued since v^i is quasiconcave in d^i. φ^i is upper hemicontinuous, since v^i is continuous. (Consider a sequence $d_r \to d \in D$ and a sequence $d^i_r \to d^i \in D^i$, where $d^i_r \in \varphi^i(d_r)$ for every r. For any $\hat{d}^i \in D^i$, $v^i(d_r^{-i}, d_r^i) \geq v_i(d_r^{-i}, \hat{d}^i)$, so since v_i is continuous $v^i(d^{-i}, d^i) \geq v^i(d^{-i}, \hat{d}^i)$, i.e., $d^i \in \varphi^i(d)$.) Define $\varphi: D \to D$ by $\varphi(d) = \varphi^1(d) \times \ldots \times \varphi^n(d)$. D is a compact subset of Euclidean space since each D^i is, and φ is an upper hemicontinuous nonempty convex-valued correspondence since each φ^i is. So by Kakutani's theorem, φ has a fixed point. But a fixed point of φ is just a Nash equilibrium of Γ. Q.E.D.

Proof of Theorem 1: Let $D^i = \Delta(S^i)$. Each D^i is then a compact convex subset of Euclidean space, and each v^i is continuous and quasiconcave in d^i (indeed linear). Hence, by Theorem 2, there is a Nash equilibrium in which each player chooses a "pure" strategy from $\Delta(S^i)$. Q.E.D.

2. Existence without Quasiconcavity (Dasgupta and Maskin, *RES* 1986)

What if the strategy set is continuous, but the payoff functions are not quasiconcave? Then we can't use Kakutani's fixed point theorem, so look at equilibria in mixed strategies.

Theorem (Glicksberg). Let each D^i be a non-empty and compact subset of Euclidean space. And let each v^i be continuous. Then Γ has a Nash equilibrium in mixed strategies.

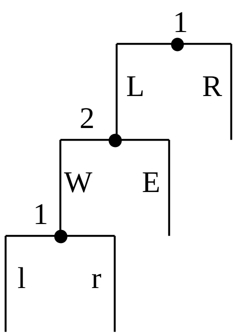
[Since any function on a finite set is continuous, Nash's theorem is an immediate corollary.]

Dasgupta and Maskin extend this result by relaxing the requirement that the payoffs be continuous.

3. Perfect Recall

A game has *perfect recall* if each player knows whatever he knew previously, including his previous actions. Example: bridge (2 player or 4 player game).

A *behavioral strategy* specifies a probability distribution over feasible actions at each information set.

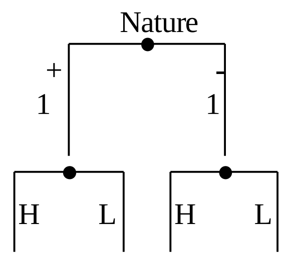


$S_1 = \{Ll, Lr, R\}$

Mixed: $(p_1, p_2, 1 - p_1 - p_2)$

Behavior: $p = Pr(L)$

$q = Pr(l|L, W)$



$S_1 = \{HH, HL, LH, LL\}$

Mixed: (p_1, p_2, p_3, p_4)

Behavior: $p = Pr(H|+)$ $q = Pr(H|-)$

Kuhn (1953) showed that in games of perfect recall, mixed strategies can be described by behavior strategies: they induce identical distributions over terminal nodes.

4. Correlated Equilibria (Aumann, *Journal of Mathematical Economics* 1974)

Example (Battle of the Sexes):

		2: L	2: R
1	U	2, 1	0, 0
	D	0, 0	1, 2

Nash equilibria:	Payoffs:
(U,L)	(2,1)
(D,R)	(1,2)
(2/3U,1/3L)	(2/3,2/3)

But by correlating strategies on a mutually verifiable coin toss then players can get (3/2,3/2). Need communication!

Example (Coordination Game):

		2	
		L	R
1	U	6, 6	2, 7
	D	7, 2	0, 0

Nash equilibria:	Payoffs:
(U,R)	(2,7)
(D,L)	(7,2)
(2/3U,1/3L)	(14/3,14/3)

Can they do better? Let a random device pick A, B, or C with probability 1/3 each. 1 is told whether A is chosen; 2 is told whether C is chosen. 1 plays D if A and U otherwise; 2 plays R if C and L otherwise. Yields payoff (5,5). For a normal form game, a *correlated strategy* is a probability distribution p(s) over the set of pure-strategy n-tuples S.

A mediator recommends strategy according to randomization device $p(\cdot)$ that is common knowledge among the players. Given a recommended strategy s^*_i, player i holds beliefs about other's strategies $p(s^*_{-i}|s^*_i)$ derived from the correlated strategy by Bayes' rule.

Def: The correlated strategy p(s) is a *correlated equilibrium* of the mediated game if for every i and for all s^*_i such that $p(s^*_i) > 0$,

$$\sum_{s^*_{-i} \in S_{-i}} u_i(s^*_{-i}, s^*_i) p(s^*_{-i}|s^*_i) \geq \sum_{s^*_{-i} \in S_{-i}} u_i(s^*_{-i}, s_i) p(s^*_{-i}|s^*_i) \text{ for all } s_i \in S_i$$

Theorem: Every point in the convex hull of the Nash-equilibrium payoffs is a correlated-equilibrium payoff.

Proof: Use a mutually observable randomizing device.

Theorem: The correlated-equilibrium payoffs are a convex polyhedron defined by linear inequalities, unlike the n-1st degree equations that determine Nash equilibria.

Proof: The linear inequalities in the definition of a correlated equilibrium determine a convex polyhedron in the space of correlated strategies, which determines a convex polyhedron of correlated-equilibrium payoffs.