Exercises (not graded):

A. Three players simultaneously pick a point on the interval \([0,1]\). The player closest to the average of the three points wins $1. If there is a tie, then the dollar is split equally among them. More formally, the players simultaneously choose strategies \(s_i \in S_i = [0,1]\). The average of their choices is \(\bar{s} = (s_1 + s_2 + s_3)/3\).

Player \(i\)'s payoff function is

\[
U_i(s_1, s_2, s_3) = \begin{cases} 
1/t & \text{if } i \in \arg\min_j |s_j - \bar{s}|; \\
0 & \text{otherwise},
\end{cases}
\]

where \(t\) is the number of players who tie (their choices are equally close to the average).

(i) What are the pure-strategy equilibria of this game?

(ii) What are the mixed-strategy equilibria if the possible strategies are limited to playing 0 or 1, rather than \([0,1]\)?

B. Change the payoff functions given above from argmin to argmax (each player tries to be the farthest from the mean). Describe the pure-strategy equilibria of the new game in terms of the following possibilities:

(i) all three players choose the same strategy,

(ii) exactly two of the three choose the same strategy, and

(iii) no two players choose the same strategy.

Note: The existence theorems for pure-strategy equilibria discussed in class do not apply to the games in exercises A and B. Nonetheless, pure-strategy equilibria exist.]
Problems:

1. Consider the following variant of the game described in Exercises A and B: We still have three symmetric players, and each player $i$ must select a strategy $s_i \in S_i = [0,1]$. Here, a player $i$ can receive two kinds of payoffs:

   a. If player $i$ chooses $s_i \notin \min_j |s_j - \bar{s}|$, they receive a payoff of 1.
   
   b. If player $i$ chooses $s_i \notin \max_j |s_j - \bar{s}|$, they receive a payoff of 1.

   A player who satisfies both conditions above receive both payoffs, i.e. total payoff 2.

   Either describe the pure strategy equilibria of this game, or prove there are none.

2. Consider the road intersection/stoplight game described in class in the context of correlated equilibria.

   Two players each approach an intersection, and must choose between (W)aiting and (P)roceeding with the following payoffs:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1,-1</td>
<td>0,1</td>
</tr>
<tr>
<td>P</td>
<td>1,0</td>
<td>-10,-10</td>
</tr>
</tbody>
</table>

   a. Describe the set of correlated equilibria possible in this game that avoid crashes, i.e. give the triples $(a, b, c)$ of probabilities that a valid correlated equilibrium could place on the strategy profiles (W,W), (W,P), (P,W) respectively (with probability 0 placed on the profile (P,P))

   b. What if players 1 and 2 only appear (independently) with probabilities $p_1, p_2 \in [0,1]$, and players must commit to their strategies before discovering whether or not their opponent has shown up? Assume a player who does not appear always receives payoff 0; and facing a player who does not show up is identical to facing a player who always (W)aits. Describe the set of equilibria without crashes – given as triples as described in part a –as a function of $p_1, p_2$. Does this set of equilibria favor one player over the other, and if so when and by how much?

   c. It is possible for a stoplight to adjust its schedule when cars are only present on one of the intersecting roads, to allow those cars to proceed immediately. Similarly, we could consider allowing a correlated equilibrium to condition on whether each of players 1 and 2 show up. What are the efficiency gains compared to part b from doing so? If a city has limited resources, which intersections should they focus on (in terms of $p_1$ and $p_2$)?
3. Consider a two-player game where all payoffs are distinct. E.g. each payoff for each player is an independent draw from \(U[0,1]\) and we ignore the measure 0 event that some number is drawn twice.

a. Assume each player has exactly two available strategies. How many pure-strategy Nash equilibria are there? How many mixed-strategy equilibria are there? How are these related?

b. How does your answer to a change if each player has exactly 3 available strategies?

c. How do your answers to a and b change if we extend the game to three players?