# Outline for Dynamic Games of Incomplete Information

I. General framework of signaling games
II. Example: Job-market signaling model
III. Definition of sequential equilibrium
IV. Analysis of separating equilibrium
V. Analysis of pooling equilibrium
VI. Analysis of hybrid equilibrium



# Signaling

Signaling: Engaging in a costly activity (that is comparatively less costly for the type you are trying to convince your opponent you are) for the purpose of credibly convincing your opponent of your type.

- Why are there labor disputes?
  - Management may be trying to convince union that it is a low-profit type.
- Why do firms pay dividends?
  - Firm may be trying to convince investors that it is a high-returns type.
- Why do firms engage in uninformative advertising?

# Signaling Games

Nature selects a type $t \in T$ for the "Sender", according to a commonly-known probability distribution.

↓

The "Sender" observes $t$ and, on the basis of $t$, chooses a ***message*** $m \in M$ to send to the "Receiver".

↓

The "Receiver" observes $m$ and, on the basis of $m$, selects an ***action*** $a \in A$ that jointly affects the "Sender" and "Receiver".

↓

Payoffs are realized:
$U_S(t, m, a)$ and $U_R(t, m, a)$.

# Job-Market Signaling

Nature randomly selects a worker's productivity, $\eta$.

$\eta$ = H, w/probability $q$

$\eta$ = L, w/probability $1 - q$

↓

The worker learns whether $\eta$ is H or L, and chooses an education level $e \geq 0$.

↓

The two firms observe $e$ and, on the basis of $e$, update their beliefs on whether $\eta$ is H or L. The firms simultaneously make wage offers to the worker.

↓

The worker accepts the higher wage offer. Payoffs are:

$w - c(\eta, e)$ to the worker;

$y(\eta, e) - w$ to the winning firm.

# Basic Requirements of “Sequential Equilibrium” or “Perfect Bayesian Equilibrium”

1) **Beliefs**: After each move by the sender, the Receiver updates his beliefs about the Sender’s type (i.e., maintains a probability distribution over types).
2) **Updating by Bayes’ Rule**: If the Sender sends a message that is sent in equilibrium, the Receiver must update his beliefs by Bayes’ Rule.
3) **Sequential Rationality**: Each player must optimize, according to his beliefs and information.
   - For the Receiver, the choice of action must maximize his expected utility.
   - For the Sender, the choice of message must maximize her utility, given knowledge of her true type and the Receiver’s anticipated response.

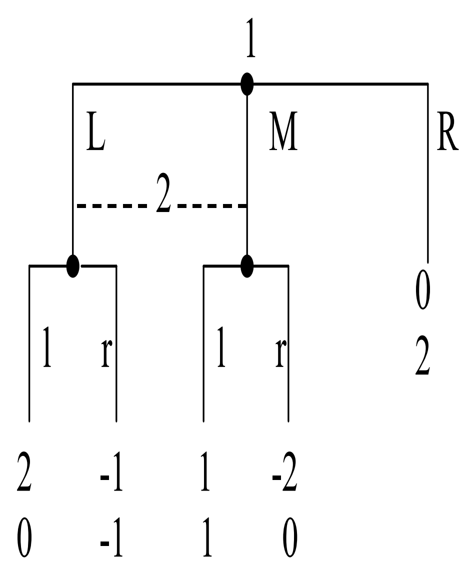
# Subgame Perfection Imperfect Information

*Game 2*:

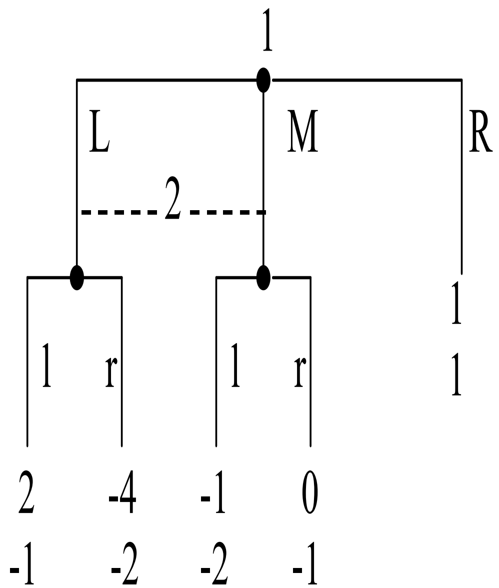
Normal Form

| | | 2: l | 2: r |
|---|---|---|---|
| | L | 2, 0 | -1,-1 |
| 1 | M | 1, 1 | -2, 0 |
| | R | 0, 2 | 0, 2 |

Extensive Form

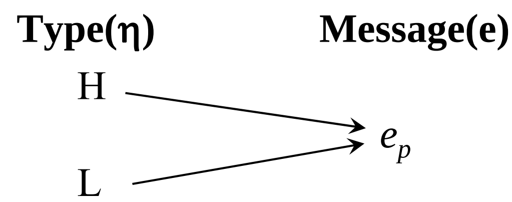


# Game 3




# Three Types of Equilibria

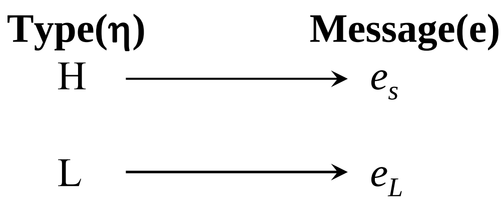
## (1) Pooling Equilibrium




Receiver's beliefs after observing $e_p$ are:

$$\eta = \begin{cases} \text{H}, & \text{w/prob } q \,, \\ \text{L}\,, & \text{w/prob } 1-q \,. \end{cases}$$

## (2) Separating Equilibrium



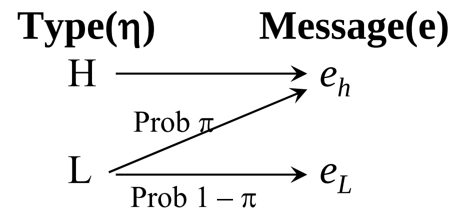

Receiver's beliefs after observing $e_s$ are:

$$\eta = \begin{cases} \text{H}, & \text{w/prob } 1, \\ \text{L}, & \text{w/prob } 0. \end{cases}$$

Receiver's beliefs after observing $e_L$ are:

$$\eta = \begin{cases} \text{H}, & \text{w/prob } 0, \\ \text{L}, & \text{w/prob } 1. \end{cases}$$

## (3) Partially-Pooling or Hybrid Equilibrium




Receiver's beliefs after observing $e_h$ are:

$$\eta = \begin{cases} \text{H}, & \text{w/prob } \dfrac{q}{q+(1-q)\pi}, \\ \text{L}, & \text{w/prob } \dfrac{(1-q)\pi}{q+(1-q)\pi}. \end{cases}$$

This is calculated as follows:

Prob [H and $e_h$] = $q \bullet 1 = q$

Prob [L and $e_h$] = $(1 - q)\,\pi$.

Hence:

$$\text{Prob}[\text{H} \mid e_h] = \frac{\text{Prob}[\text{H} \,\&\, e_h]}{\text{Prob}[\text{H} \,\&\, e_h] + \text{Prob}[\text{L} \,\&\, e_h]}.$$

Receiver's beliefs after observing $e_L$ are:

$$\eta = \begin{cases} \text{H}, & \text{w/prob } 0, \\ \text{L}, & \text{w/prob } 1. \end{cases}$$

## Completion of Description of Separating Equilibria

Receiver's (firm's) beliefs:

$$\mu(\mathrm{H}|e) = \begin{cases} 1, \text{ for } e \geq e_s \\ 0, \text{ for } e < e_s \end{cases}$$

Firm's offer is the worker's expected productivity:

$$w(e) = \begin{cases} y(\mathrm{H}, e), \text{ for } e \geq e_s \\ y(\mathrm{L}, e), \text{ for } e < e_s \end{cases}$$

Self-selection constraints for worker:

$$w(e_L) - c(\mathrm{L}, e_L) \geq w(e_s) - c(\mathrm{L}, e_s)$$

↑ ↑

(for low type)

$$w(e_s) - c(\mathrm{H}, e_s) \geq w(e_L) - c(\mathrm{H}, e_L)$$

↑ ↑

(for high type)

## Notation

$e^*(\mathrm{L})$ denotes the education level that the low type would select under complete information.

$e^*(\mathrm{L})$ maximizes:

$$\max_{e \geq 0} \{y(\mathrm{L}, e) - c(\mathrm{L}, e)\}$$

Analogously, $e^*(\mathrm{H})$ maximizes:

$$\max_{e \geq 0} \{y(\mathrm{H}, e) - c(\mathrm{H}, e)\}$$

In the separating equilibrium:

$$e_{\mathrm{L}} = e^*(\mathrm{L})$$

**but**:

$$e_s > e^*(\mathrm{H})$$

## Completion of Description of Pooling Equilibrium

Receiver's (firm's) beliefs:

$$\mu(\mathrm{H}|e) = \begin{cases} q, \text{ for } e = e_p \\ 0, \text{ for } e \neq e_p \end{cases}$$

Firm's offer is the worker's expected productivity. Define:

$$w_p = q \bullet y(\mathrm{H}, e_p) + (1 - q) \bullet y(\mathrm{L}, e_p)$$

Then:

$$w(e) = \begin{cases} w_p, \text{ for } e = e_p \\ y(\mathrm{L}, e), \text{ for } e \neq e_p \end{cases}$$

Self-selection constraints for worker:

$w_p - c(\mathrm{L}, e_p) \geq y(\mathrm{L}, e) - c(\mathrm{L}, e)$,

↑ (for low type) ↑ for all $e \geq 0$

$w_p - c(\mathrm{H}, e_p) \geq y(\mathrm{L}, e) - c(\mathrm{H}, e)$,

↑ (for high type) ↑ for all $e \geq 0$

## Completion of Description of Hybrid Equilibrium

Define $q' = \dfrac{q}{q + (1-q)\pi}$ .

Receiver's (firm's) beliefs are:

$$\mu(\mathrm{H}|e) = \begin{cases} q', \text{ for } e \geq e_h \\ 0, \text{ for } e < e_h \end{cases}$$

Firm's offer is the worker's expected productivity:

$$w(e) = \begin{cases} q'\, y(\mathrm{H}, e) + (1 - q')\, y(\mathrm{L}, e), & \text{for } e \geq e_h \\ y(\mathrm{L}, e), & \text{for } e < e_h \end{cases}$$

Constraints for worker:

$w(e_L) - c(\mathrm{L}, e_L) = w(e_h) - c(\mathrm{L}, e_h)$

(low type randomizes, so must be indifferent)

$w(e_h) - c(\mathrm{H}, e_h) \geq w(e_L) - c(\mathrm{H}, e_L)$

(self-selection for high type)

And note that $e_L = e^*(\mathrm{L})$.