Outline for Static Games of Incomplete Information

I. Example 1: An auction game with discrete bids
II. Example 2: Cournot duopoly with one-sided asymmetric information
III. Definition of Bayesian-Nash equilibrium
IV. Analysis of first-price auctions
V. Analysis of second-price auctions
VI. Auctions of similar items: "An efficient ascending-bid auction for multiple objects"
VII. Bargaining with two-sided incomplete information

Example 1: Auction with discrete bids

Consider a sealed-bid first-price auction for a single object, where there are only two allowable bids. The two risk-neutral bidders have valuations which are private information and which are drawn from i.i.d. random variables that are uniformly distributed on the interval [0,1]. After observing her own valuation, each of the two bidders simultaneously and independently submits a bid selected from the two-element set $\{0, \frac{1}{3}\}$ (i.e., the only allowable bids are 0 and $\frac{1}{3}$).

The high bidder wins the object and pays the amount of her bid; in the event of a tie, the winner is determined by the toss of a fair coin.

Example 2: Cournot duopoly with one-sided asymmetric information

Demand: Inverse demand function of $P(Q) = a - Q$

Cost: Constant marginal cost

Firm 1: $c_1 = c$
(commonly known)

Firm 2: $c_2 = c_H$, with prob. θ ; c_L with prob. $1 - \theta$
(privately known)

A strategy for firm 2 is a **function** from possible costs to quantities: $q_2(c_H)$ and $q_2(c_L)$

Definition: Let T_1 be the set of possible ***types*** for player 1 and let T_2 be the set of possible ***types*** for player 2. We define (s_1^*, s_2^*) to be a ***Bayesian-Nash equilibrium*** if for every type:

$s_1^*(t_1)$ solves:

$$\max_{a_1 \in A_1} \sum_{t_2 \in T_2} u_1(a_1, s_2^*(t_2); t_1) \bullet p_1(t_2 \mid t_1)$$

and

$s_2^*(t_2)$ solves:

$$\max_{a_2 \in A_2} \sum_{t_1 \in T_1} u_2(s_1^*(t_1), a_2; t_2) \bullet p_2(t_1 \mid t_2)$$

(The above is Gibbons' definition from pg. 151, restricted to two players.)

First-Price Auction

Every player i (simultaneously) submits a sealed bid, b_i.
The player i who submitted the highest bid obtains the good for a price of b_i .

Second-Price Auction

Every player i (simultaneously) submits a sealed bid, b_i.
If player i submitted the highest bid and player j submitted the second highest bid, then player i obtains the good for a price of b_j .

Symmetric Independent Private Values

Values v_i drawn i.i.d. from distribution F on [0,1]
Bidder i knows value v_i; Others only know distribution.

First Price (≡ Dutch)

Symmetric Equilibrium Bidding Strategy

- Bidder's expected profit:

$$\pi(v,b(v)) = (v - b(v))\Pr(Win|b(v)).$$

- By the envelope theorem,

$$\frac{d\pi}{dv} = \frac{\partial\pi}{\partial b}\frac{\partial b}{\partial v} + \frac{\partial\pi}{\partial v} = \frac{\partial\pi}{\partial v}$$

- But then $d\pi/dv = \Pr(Win|b(v)) = \Pr(\text{highest bid})$
  $= \Pr(\text{highest value}) = F(v)^{n-1}$ a.e.

First Price (≡ Dutch)

- By the Fundamental Theorem of Calculus,

$$\pi(v) = \pi(0) + \int_0^v F(u)^{n-1} du = \int_0^v F(u)^{n-1} du,$$

- Substituting into $\pi(v,b(v)) = (v - b(v))\Pr(Win|b(v))$ yields

$$b(v) = v - \frac{\pi(v)}{\Pr(Win)} = v - F(v)^{-(n-1)} \int_0^v F(u)^{n-1} du.$$

Example

- $v \sim U$ on $[0,1]$
- Then $F(v) = v$, so

$$b(v) = v - v/n = v(n-1)/n.$$

- The optimal bid converges to the value as $n \to \infty$, so in the limit the seller is able to extract the full surplus.
- In equilibrium, the bidder bids the expected value of the second highest value given that the bidder has the highest value.

Examples of auctioning similar items

- Treasury bills
- Stock repurchases
- Telecommunications spectrum
- Electric power
- Emissions permits
- Petrochemical products

Nationwide Narrowband Auction July 1994

- Five large licenses
- Can win at most three
- Simultaneous ascending-bid auction
  - Analogous to uniform price auction (all five sold for $80,000,000)

Marginal Values of Bidders

PageNet	AT&T
$V_1 = 123$	$V_1 = 125$
$V_2 = 113$	$V_2 = 125$
$V_3 = 103$	$V_3 = 49$

AirTouch	SkyTel
$V_1 = 75$	$V_1 = 85$
$V_2 = 5$	$V_2 = 65$
$V_3 = 3$	$V_3 = 7$

Efficiency

"Price" = 85, "Revenue" = 425

Profit	PageNet	AT&T
38	$\mathbf{V_1 = 123}$	$\mathbf{V_1 = 125}$
28	$\mathbf{V_2 = 113}$	$\mathbf{V_2 = 125}$
18	$\mathbf{V_3 = 103}$	$V_3 = 49$
84		

AirTouch	SkyTel
$V_1 = 75$	$V_1 = 85$
$V_2 = 5$	$V_2 = 65$
$V_3 = 3$	$V_3 = 7$

Uniform-price auction

Price = 75, Revenue = 375

Profit	PageNet	AT&T
48	**V_1 = 123**	**V_1 = 125**
38	**V_2 = 113**	**V_2 = 125**
	V_3 = 103	V_3 = 49
86		

AirTouch	SkyTel
V_1 = 75	**V_1 = 85**
V_2 = 5	V_2 = 65
V_3 = 3	V_3 = 7

Efficient ascending-bid auction:

Award good when "clinched" at current price

PageNet	Price	AT&T	Price
V_1 = 123	**65**	V_1 = 125	**75**
V_2 = 113	**75**	V_2 = 125	**85**
V_3 = 103	**85**	V_3 = 49	

AirTouch	SkyTel
V_1 = 75	V_1 = 85
V_2 = 5	V_2 = 65
V_3 = 3	V_3 = 7

Efficient ascending-bid auction

Revenue = 385

PageNet	Price	AT&T	Price
V_1 = 123	**65**	**V_1 = 125**	**75**
V_2 = 113	**75**	**V_2 = 125**	**85**
V_3 = 103	**85**	V_3 = 49	

AirTouch	SkyTel
V_1 = 75	V_1 = 85
V_2 = 5	V_2 = 65
V_3 = 3	V_3 = 7

Bargaining with Two-Sided Incomplete Information

Simultaneous Offers

(Chatterjee & Samuelson, *Operations Research* 1983)

- A seller and a buyer are engaged in the trade of a single object worth s to the seller and b to the buyer.
- Valuations are known privately, as summarized below

Traders Value	Distributed	Payoff	Private Info	Common Knowledge	Strategy (Offer)
Seller s	s~F on $[\underline{s}, \overline{s}]$	u = P – s	s	F, G	p(s)
Buyer b	b~G on $[\underline{b}, \overline{b}]$	v = b – P	b	F, G	q(b)

Simultaneous Offers

- *Independent private value model*: s and b are independent random variables.
- *Ex post efficiency*: trade if and only if $s < b$.
- *Game*: Each player simultaneously names a price; if $p \leq q$ then trade occurs at the price $P = (p + q)/2$; if $p > q$ then no trade (each player gets zero).

Simultaneous Offers

- Payoffs:
  - Seller

$$u(p,q,s,b) = \begin{cases} P - s & \text{if } p \leq q \\ 0 & \text{if } p > q \end{cases}$$

  - Buyer

$$v(p,q,s,b) = \begin{cases} b - P & \text{if } p \leq q \\ 0 & \text{if } p > q \end{cases}$$

where the trading price is $P = (p + q)/2$

Example

- Let F and G be independent uniform distributions on [0,1].

- Equilibrium conditions:

$$(1)\ \forall\ s \in [\underline{s}, \bar{s}], p(s) \in \arg\max_{p} E_b \{u(p, q, s, b) | s, q(\cdot)\}$$

$$(2)\ \forall\ b \in [\underline{b}, \bar{b}], q(b) \in \arg\max_{q} E_s \{v(p, q, s, b) | b, p(\cdot)\}$$

Seller's Problem

- Assume p and q are strictly increasing.
- Let $x(\cdot) = p^{-1}(\cdot)$ and $y(\cdot) = q^{-1}(\cdot)$.
- Optimization in (1) can be stated as

$$\max_{p} \int_{y(p)}^{1} [\tfrac{1}{2}(p + q(b)) - s] db$$

- First-order condition

$$-y'(p)[p - s] + [1 - y(p)]/2 = 0,$$

since $q(y(p)) = p$

Buyer's Problem

- Optimization in (2) can be stated as

$$\max_{q} \int_{0}^{x(q)} [b - (p(s) + q) / 2]ds$$

- First-order condition

$$x'(q)[b - q] - x(q)/2 = 0,$$

since $p(x(q)) = q$.

Equilibrium

- Equilibrium condition:

$$s=x(p) \text{ and } b=y(q)$$

- Equilibrium first-order conditions:

(1') $-2y'(p)[p - x(p)] + [1 - y(p)] = 0,$

(2') $2x'(q)[y(q) - q] - x(q) = 0.$

Solution

- Solving (2') for y(q) and replacing q with p yields

$$(2'') \quad y(p) = p + \frac{1}{2}\frac{x(p)}{x'(p)}, \quad \text{so} \quad y'(p) = \frac{3}{2} - \frac{1}{2}\frac{x(p)x''(p)}{[x'(p)]^2}$$

- Substituting into (1') then yields

$$(1') \quad [x(p) - p]\left[3 - \frac{x(p)x''(p)}{[x'(p)]^2}\right] + \left[1 - p - \frac{1}{2}\frac{x(p)}{x'(p)}\right] = 0$$

Analytical Solution

- Linear Solution:

$$x(p) = \alpha p + \beta.$$

with $\alpha = 3/2$ and $\beta = -3/8$.

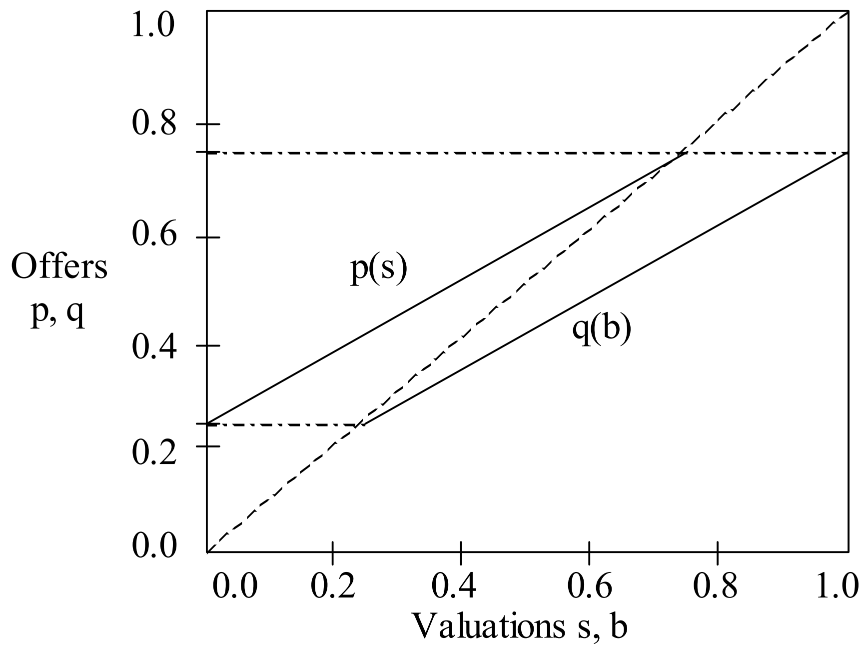
- Using (2'') yields

$$y(q) = 3/2\ q - 1/8.$$

- Inverting these functions results in

$$p(s) = 2/3\ s + 1/4 \quad \text{and} \quad q(b) = 2/3\ b + 1/12$$

Figure 1



In the sealed-bid double auction, trade occurs if and only if:

$$p_b(v_b) \geq p_s(v_s)$$
$$\tfrac{2}{3} v_b + \tfrac{1}{12} \geq \tfrac{2}{3} v_s + \tfrac{1}{4}$$
$$\tfrac{2}{3}(v_b - v_s) \geq \tfrac{1}{6}$$
$$\mathbf{v_b \geq v_s + \tfrac{1}{4}}$$