Outline for Dynamic Games of Complete Information

I. Examples of *dynamic* games of complete info:
   A. Sequential version of Battle of the Sexes
   B. Sequential version of Matching Pennies

II. Definition of subgame-perfect equilibrium

III. Backward induction in finitely-repeated games:
   A. The finitely-repeated Prisoner’s Dilemma / finitely-repeated Bertrand Oligopoly
   B. The Chain-Store Paradox

IV. Stackelberg Model of Duopoly

V. Bargaining:
   A. Alternating-offer bargaining (Rubinstein, 1982)
   B. Digression on the Nash bargaining solution
   C. Digression on the “Coase Theorem”

VI. Infinitely-repeated games and trigger strategies

VII. The Folk Theorem

VIII. Maximally-collusive equilibria
Subgame-Perfect Equilibrium

**Definition:** In an $n$-player dynamic game of complete information, an $n$-tuple of strategies is said to form a *subgame-perfect equilibrium* (SPE) if the strategies constitute Nash equilibria in every subgame.

(“Game”: An inverted tree.)
(“Subgame”: What is left of the original game after some moves have been played.)

Finitely Repeated Prisoners’ Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>3, 3</td>
<td>0, 4</td>
</tr>
<tr>
<td>Defect</td>
<td>4, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>
Finitely Repeated Bertrand Price Competition

Chain-Store Paradox (Selten, 1978)

Entrant 1

Stay out
(a, 0)

Enter
Incumbent
Fight (0, -c)
Accommodate (a/2, b)

a, b, c > 0
Chain-Store Paradox (Selten, 1978)

![Game Tree Diagram]

\[ (a, 0), (0, -c), \left(\frac{a}{2}, b\right) \]

\[ a, b, c > 0 \]
Stackelberg Model of Duopoly

Firm 1
Selects $q_1$
Firm 2
Selects $q_2$

$\left(q_1[P(q_1 + q_2) - c], q_2[P(q_1 + q_2) - c]\right)$
Solution of Stackelberg Model of Duopoly

First, observe that firm 2 solves:
\[ q_2^* = \arg \max \{ q_2 [P(q_1 + q_2) - c] \}. \]
Consequently, given \( q_1 \), can calculate \( q_2 \) by:
\[ q_2^*(q_1) = \frac{a-q_1-c}{2}. \]
Hence, firm 1 solves:
\[
\begin{align*}
\max_{q_1} \left\{ q_1 \left[ a - q_1 - q_2^*(q_1) - c \right] \right\} &= \\
= \max_{q_1} \left\{ q_1 \left[ a - c - q_1 - \frac{a-q_1-c}{2} \right] \right\} &= \\
= \max_{q_1} \left\{ q_1 \left[ \frac{a-q_1-c}{2} \right] \right\}.
\end{align*}
\]

Giving a first-order condition of:
\[ -\frac{1}{2} q_1 + \frac{a-q_1-c}{2} = 0 \]

Thus, we conclude:
\[ q_1^* = \frac{a-c}{2}; \quad q_2^*(q_1^*) = \frac{a-c}{4}. \]
## Bargaining

<table>
<thead>
<tr>
<th></th>
<th># of Troops</th>
<th># of Troops</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NATO Proposal</strong></td>
<td>9,000</td>
<td>14,000</td>
</tr>
<tr>
<td><strong>Warsaw Pact Proposal</strong></td>
<td>16,000</td>
<td>20,000</td>
</tr>
<tr>
<td><strong>Final Agreement</strong></td>
<td>13,000</td>
<td>17,000</td>
</tr>
</tbody>
</table>
Alternating–Offer Bargaining

\[ (p_1, 1 - p_1) \]

\[ (\delta p_2, \delta [1 - p_2]) \]

\[ (\delta^2 p_3, \delta^2 [1 - p_3]) \] (0, 0)
Alternating–Offer Bargaining

\[ (\delta^2 p_3, \delta^2 [1 - p_3]) \quad (0, 0) \]

Alternating–Offer Bargaining

\[ (\delta p_2, \delta [1 - p_2]) \]

Alternating–Offer Bargaining

\[ (\delta^2 p_3, \delta^2 [1 - p_3]) \quad (0, 0) \]
Alternating–Offer Bargaining

\[
\begin{align*}
\text{Offers } p_2 & \quad \text{Accepts} & \quad \text{Rejects} \\
(\delta p_2, \delta [1 - p_2]) & \quad \text{Offers } p_3 \\
(\delta^2 p_3, \delta^2 [1 - p_3]) & \quad (0, 0)
\end{align*}
\]

Alternating–Offer Bargaining

\[
\begin{align*}
\text{Offers } p_1 & \quad \text{Accepts} & \quad \text{Rejects} \\
(p_1, 1 - p_1) & \quad \text{Offers } p_2 \\
(\delta p_2, \delta [1 - p_2]) & \quad \text{Offers } p_3 \\
(\delta^2 p_3, \delta^2 [1 - p_3]) & \quad (0, 0)
\end{align*}
\]
Alternating–Offer Bargaining

The unique SPE of the infinite-horizon, alternating-offer bargaining game

In any period in which it is the seller’s turn to make an offer:

- The seller offers a price of \( \frac{1}{1+\delta} \).
- The buyer accepts any price \( p \leq \frac{1}{1+\delta} \) and rejects any price \( p > \frac{1}{1+\delta} \).

In any period in which it is the buyer’s turn to make an offer:

- The buyer offers a price of \( \frac{\delta}{1+\delta} \).
- The seller accepts any price \( p \geq \frac{\delta}{1+\delta} \) and rejects any price \( p < \frac{\delta}{1+\delta} \).
Suppose there exists at least one SPE in: (a) the alternating-offer game where the seller moves first; (b) the alternating-offer game where the buyer moves first.

Let $M$ be the supremum of seller payoffs over all SPEs in the game where the seller goes first. Let $m$ be the infimum of buyer payoffs over all SPEs in the game where the buyer goes first.

Observe, for every $n \geq 1$, there exists an SPE $\sigma_n$ such that:

$$\Pi_s(\sigma_n) \geq M - 1/n$$

The following is easily seen to be an SPE of the game where the buyer goes first:

- The buyer offers a division $(\delta \Pi_s(\sigma_n), 1 - \delta \Pi_s(\sigma_n))$. The seller accepts (in period 0)
- Given any deviation, play $\sigma_n$ beginning in period 1.

Observe that:

$$\Pi_B(\sigma_n') = 1 - \delta \Pi_s(\sigma_n) \leq 1 - \delta M + \delta(1/n), \text{ for all } n$$

Hence $m \leq 1 - \delta M$.

But if the buyer offers the seller a division $(\delta M + \epsilon, 1 - \delta M - \epsilon)$, the seller must accept for any $\epsilon > 0$ since $\delta M + \epsilon$ today is better than $M$ tomorrow.

Hence $m \geq 1 - \delta M - \epsilon$, for all $\epsilon > 0$.

Conclude $m = 1 - \delta M$.

Now observe that the following is an equilibrium of the game where the seller goes first.

- The seller offers $[1 - \delta + \delta^2 \Pi_s(\sigma_n), \delta - \delta^2 \Pi_s(\sigma_n)]$. The buyer accepts in period 0.

- In case of any deviation, the buyer offers $[\delta \Pi_s(\sigma_n), 1 - \delta \Pi_s(\sigma_n)]$. The seller accepts in period 1.

- In case of further deviation, the parties play $\sigma_n'$ in all subsequent periods.

Observe that:

$$\Pi_s(\sigma_n'') = 1 - \delta + \delta^2 \Pi_s(\sigma_n) \geq 1 - \delta + \delta^2 M - \delta^2(1/n), \text{ for all } n \geq 0$$

Hence $M \geq 1 - \delta + \delta^2 M$. 


But suppose the seller ever offered the buyer a share smaller than $\delta m$. Surely the buyer would reject. [Similarly, the buyer never makes an offer (which is accepted) in which his own share is less than $m$.] Consequently:

$$M \leq 1 - \delta m = 1 - \delta(1 - \delta M) = 1 - \delta + \delta^2 M.$$  

This demonstrates $M = 1 - \delta + \delta^2 M$, and hence $M = (1 - \delta)/(1 - \delta^2) = 1/(1 + \delta)$.

Then $m = 1 - \delta/(1 + \delta) = 1/(1 + \delta)$.

Reversing the roles, this show that the seller’s payoff in any SPE is uniquely $1/(1 + \delta)$.

Furthermore this can only be realized if the offer is $1/(1 + \delta)$ in the first period.

**Axiom 1**: The solution should not depend on linear transformations of players’ utility functions.

**Axiom 2**: The solution should be individually-rational and Pareto-optimal.

**Axiom 3**: “Independence of Irrelevant Alternatives.”

**Axiom 4**: Symmetry.

**Theorem**: If the feasible set is convex, closed and bounded above, there is a unique solution satisfying Axioms 1-4, and it is given by:

$$\max_{x \geq d} (x_1 - d_1)(x_2 - d_2)$$

where $d$ denotes the disagreement point and $x$ is an element of the feasible set.

<table>
<thead>
<tr>
<th># of steers in herd</th>
<th>Total crop loss</th>
<th>Marginal crop loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>4</td>
</tr>
</tbody>
</table>

Value of crop = $1 per unit
Cost of fence = $9

**Definition:** Let $G$ be a static game. Then the $T$-period *repeated game*, denoted $G(T, \delta)$, consists of game $G$ repeated $T$ times. At each period $t$, the moves from periods $1, \ldots, t - 1$ are known to every player. Payoffs are computed by:

$$u_i = \sum_{t=1}^{T} \delta^{t-1} u_{it}$$

($u_{it}$ denotes the payoff to player $i$ in period $t$)

If $T = \infty$, then $G(T, \delta)$ is referred to as the *infinitely-repeated game*. The **average payoff** to player $i$ is then given by:

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_{it}$$
Finitely Repeated Games

- Stage game with multiple NE

\[
\begin{array}{ccc}
  & x_2 & y_2 & z_2 \\
 x_1 & 1, 1 & 5, 0 & 0, 0 \\
 y_1 & 0, 5 & 4, 4 & 0, 0 \\
 z_1 & 0, 0 & 0, 0 & 3, 3 \\
\end{array}
\]

Two pure strategy NE: \((x_1, x_2)\) and \((z_1, z_2)\)

Trigger Strategy Equilibria

First, define a **main equilibrium path** to be an action suggested for every player \(i\) \((i = 1, \ldots, n)\) and for every period:

\[
\tilde{s} = (s_{i1}, \ldots, s_{in_1}), (s_{i2}, \ldots, s_{in_2}), (s_{i3}, \ldots, s_{in_3}), ...
\]

Second, define \((s_1^*, \ldots, s_n^*)\) to be a Nash equilibrium of the static game \(G\).

A **trigger strategy** for player \(i\) in the repeated game \(G(T, \delta)\) is given by:

\[
\sigma_{it} = \begin{cases} 
  s_{it}, & \text{if } every \text{ player has played according to } \tilde{s} \text{ in all previous periods } 1, \ldots, t-1 \text{ (or if } t=1), \\
  s_i^*, & \text{if there has been any prior deviation by } any \text{ player.}
\end{cases}
\]
Example: Infinitely-Repeated Cournot Game

Description of trigger strategies:
\[
q_{it} = \begin{cases} \frac{a-c}{4}, & \text{if } q_{is} = \frac{a-c}{4} = q_{2s}, \text{ for all } s = 1, \ldots, t-1, \\ \frac{a-c}{3}, & \text{otherwise}. \end{cases}
\]

Payoff along equilibrium path:
\[
\sum_{t=0}^{\infty} \delta^t \left( \frac{a-c}{4} \right) \left( a - \frac{a-c}{4} - \frac{a-c}{4} - c \right) = \left( \frac{1}{1-\delta} \right) \frac{1}{8} (a-c)^2.
\]

Payoff from optimally deviating:
\[
\frac{3}{8} (a-c) \left( a - \frac{5}{8} (a-c) - c \right) + \sum_{t=1}^{\infty} \delta^t \left( \frac{a-c}{3} \right) \left( a - \frac{2}{3} (a-c) - c \right)
\]
\[
= \frac{9}{64} (a-c)^2 + \delta \left( \frac{1}{1-\delta} \right) \frac{1}{9} (a-c)^2.
\]

Example: Infinitely-Repeated Cournot Game

This is a trigger-strategy equilibrium if and only if:
\[
\text{Payoff}_i(\text{equilibrium path}) \geq \text{Payoff}_i(\text{optimally deviating})
\]
\[
\left( \frac{1}{1-\delta} \right) \frac{1}{8} \left( a-c \right)^2 \geq \frac{9}{64} \left( a-c \right)^2 + \delta \left( \frac{1}{1-\delta} \right) \frac{1}{9} \left( a-c \right)^2.
\]
\[
\frac{1}{8} \geq \frac{9}{64} (1-\delta) + \frac{1}{9} \delta.
\]
\[
\left[ \frac{\frac{9}{64} - \frac{1}{8}}{\frac{9}{64}} \right] \delta \geq \frac{9}{64} - \frac{1}{8}.
\]
\[
\left[ \frac{9 \times 9 - \frac{64}{64 \times 9}}{64 \times 9} \right] \delta \geq \frac{1}{64}.
\]
\[
\left[ \frac{17}{64 \times 9} \right] \delta \geq \frac{1}{64}.
\]
\[
\delta \geq \frac{9}{17}.
\]
The Folk Theorem

**Definition:** The \( n \)-tuple \((x_1, \ldots, x_n)\) of payoffs to the \( n \) players is called *feasible* if it arises from the play of pure strategies or if it is a convex combination of payoffs from pure strategies.

**Theorem:** Let \((e_1, \ldots, e_n)\) be the payoffs from a Nash equilibrium of \(G\) and let \((x_1, \ldots, x_n)\) be any feasible payoffs from \(G\). If \(x_i > e_i\) for every player \(i\), then there exists a subgame-perfect equilibrium of \(G(\infty, \delta)\) that attains \((x_1, \ldots, x_n)\) as the average payoff, *provided that \(\delta\) is sufficiently close to 1.*

Example: Prisoners’ Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Remain Silent</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I</strong></td>
<td>((-1, -1))</td>
<td>((-5, 0))</td>
</tr>
<tr>
<td><strong>II</strong></td>
<td>((-5, 0))</td>
<td>((-4, -4))</td>
</tr>
</tbody>
</table>
Example: Battle of the Sexes

\[
\begin{array}{c|cc}
 & \text{Boxing} & \text{Ballet} \\
\hline
\text{Boxing} & (2, 1) & (0, 0) \\
\text{Ballet} & (0, 0) & (1, 2) \\
\end{array}
\]

Maximally-Collusive Equilibria

Example: Two-Phase Strategies in Repeated Cournot.

If \(q_{1,t-1} = (a-c)/4\) and \(q_{2,t-1} = (a-c)/4\)

Start

\begin{align*}
\text{“Collusive Phase”:} \\
\text{Produce } &\frac{(a-c)}{4} \\
\text{If } & q_{1,t-1} \neq \frac{(a-c)}{4} \text{ or } q_{2,t-1} \neq \frac{(a-c)}{4} \\
\text{“Punishment Phase”} \\
\text{Produce } & \frac{(a-c)}{2} \\
\text{If } & q_{1,t-1} = \frac{(a-c)}{2} \text{ and } q_{2,t-1} = \frac{(a-c)}{2}
\end{align*}

If \(q_{1,t-1} \neq \frac{(a-c)}{2} \text{ or } q_{2,t-1} \neq \frac{(a-c)}{2}\)