Outline for Dynamic Games of Complete Information

I. Examples of ***dynamic*** games of complete info:
   A. Sequential version of Battle of the Sexes
   B. Sequential version of Matching Pennies
II. Definition of subgame-perfect equilibrium
III. Backward induction in finitely-repeated games:
   A. The finitely-repeated Prisoner's Dilemma / finitely-repeated Bertrand Oligopoly
   B. The Chain-Store Paradox
IV. Stackelberg Model of Duopoly

Outline for Dynamic Games of Complete Information

V. Bargaining:
   A. Alternating-offer bargaining (Rubinstein, 1982)
   B. Digression on the Nash bargaining solution
   C. Digression on the "Coase Theorem"
VI. Infinitely-repeated games and trigger strategies
VII. The Folk Theorem
VIII. Maximally-collusive equilibria

Subgame-Perfect Equilibrium

Definition: In an *n*-player dynamic game of complete information, an *n*-tuple of strategies is said to form a ***subgame-perfect equilibrium*** (SPE) if the strategies constitute Nash equilibria in every subgame.

("Game": An inverted tree.)

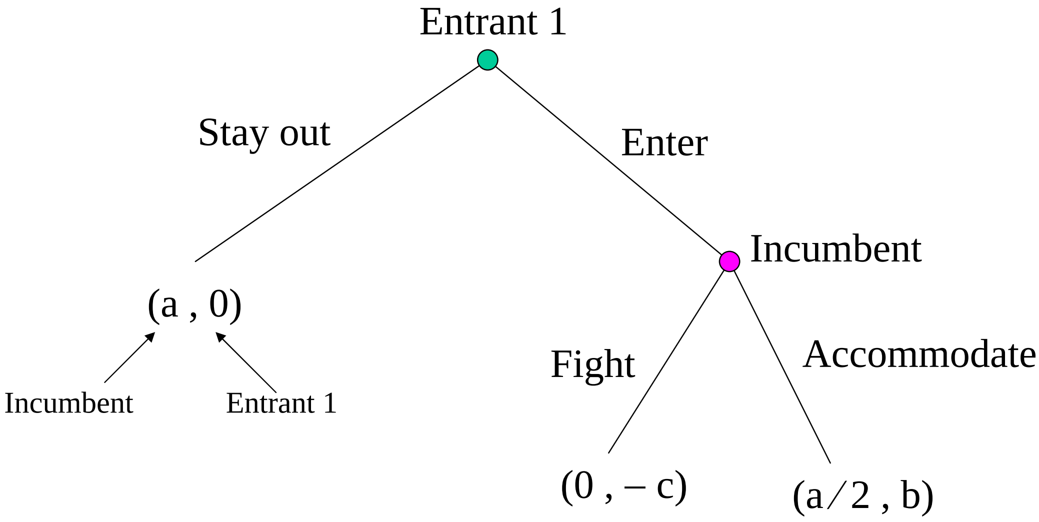
("Subgame": What is left of the original game after some moves have been played.)

Finitely Repeated Prisoners' Dilemma

	Cooperate	Defect
Cooperate	3 , 3	0 ,4
Defect	4 , 0	1 , 1

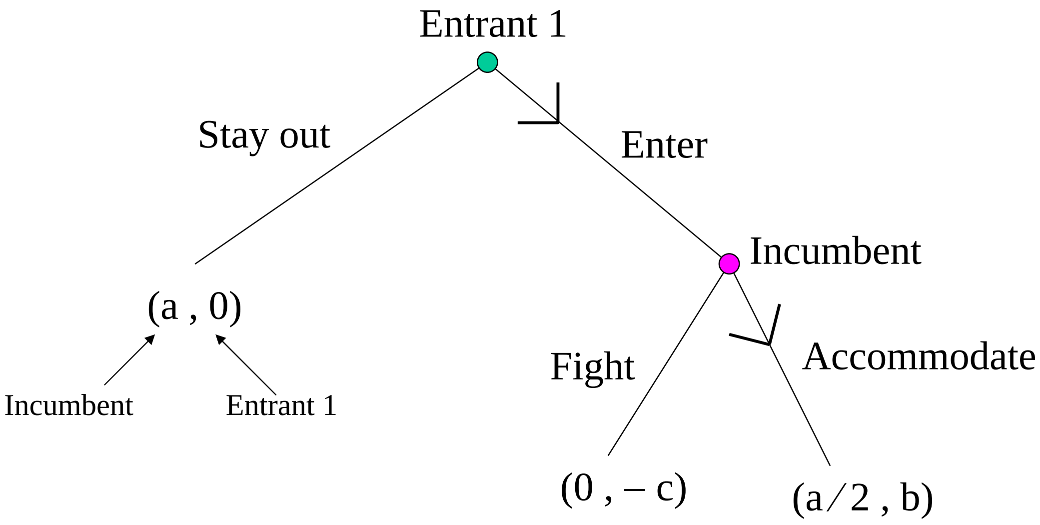
Finitely Repeated
Bertrand Price Competition

Chain-Store Paradox (Selten, 1978)



a, b, c > 0

Chain-Store Paradox (Selten, 1978)



Entrant 1

Stay out

Enter

Incumbent

(a , 0)

Incumbent

Entrant 1

Fight

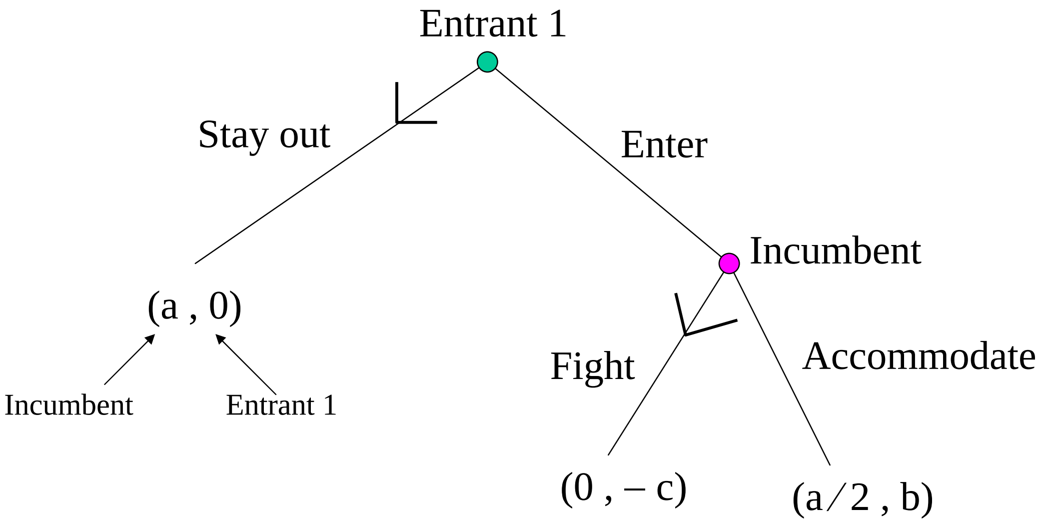
Accommodate

(0 , – c)

(a ⁄ 2 , b)

a, b, c > 0

Chain-Store Paradox (Selten, 1978)



Entrant 1

Stay out

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Entrant 1

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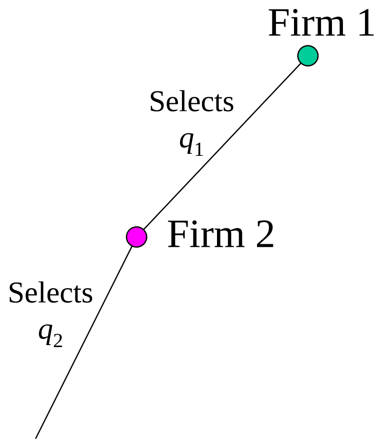
(a ⁄ 2 , b)

a, b, c > 0

Stackelberg Model of Duopoly



$$(q_1[P(q_1 + q_2) - c], \mathrm{q}_2[P(q_1 + q_2) - c])$$

Solution of Stackelberg Model of Duopoly

First, observe that firm 2 solves:

$$q_2^* = \arg\max\{q_2\,[P(q_1 + q_2) - c]\}.$$

Consequently, given q_1, can calculate q_2 by:

$$q_2{*}(q_1) = \frac{a-q_1-c}{2}\ .$$

Hence, firm 1 solves:

$$\max_{q_1}\left\{q_1\left[a - q_1 - q_2{*}(q_1) - c\right]\right\} =$$

$$= \max_{q_1}\left\{q_1\left[a - c - q_1 - \frac{a-q_1-c}{2}\right]\right\}$$

$$= \max_{q_1}\left\{q_1\left[\frac{a-q_1-c}{2}\right]\right\}.$$

Giving a first-order condition of:

$$-\tfrac{1}{2}q_1 + \frac{a-q_1-c}{2} = 0$$

Thus, we conclude:

$$q_1^* = \frac{a-c}{2}\ ; \quad q_2{*}(q_1{*}) = \frac{a-c}{4}\ .$$

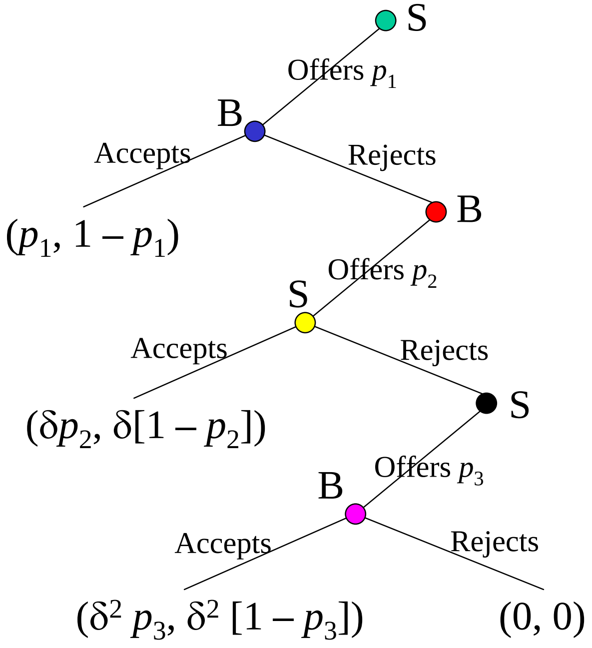
Bargaining

	# of Troops	# of Troops
NATO Proposal	9,000	14,000
Warsaw Pact Proposal	16,000	20,000

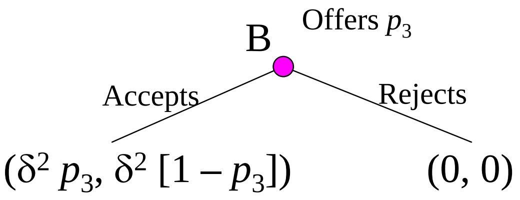
Bargaining

	# of Troops	# of Troops
NATO Proposal	9,000	14,000
Warsaw Pact Proposal	16,000	20,000
Final Agreement	13,000	17,000

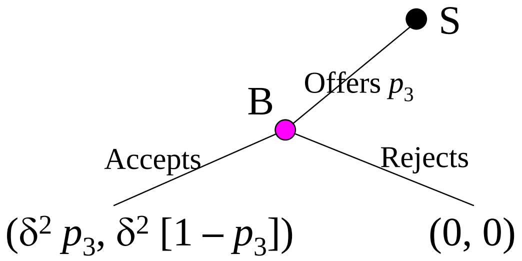
Alternating–Offer Bargaining



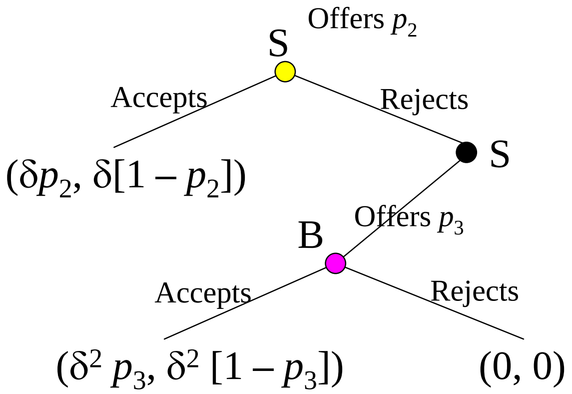
Alternating–Offer Bargaining



Alternating–Offer Bargaining



Alternating–Offer Bargaining



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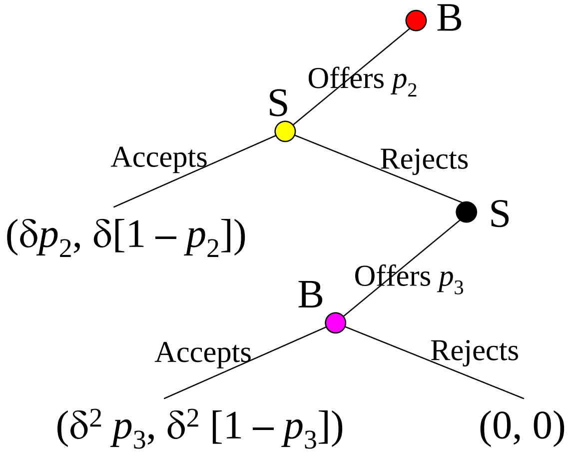
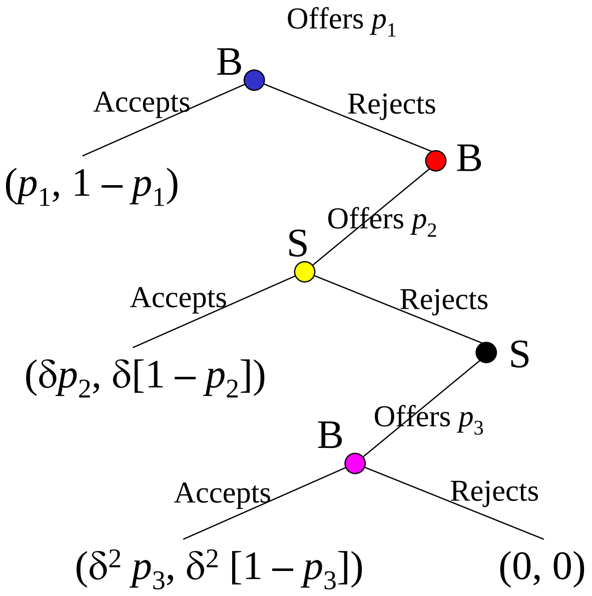
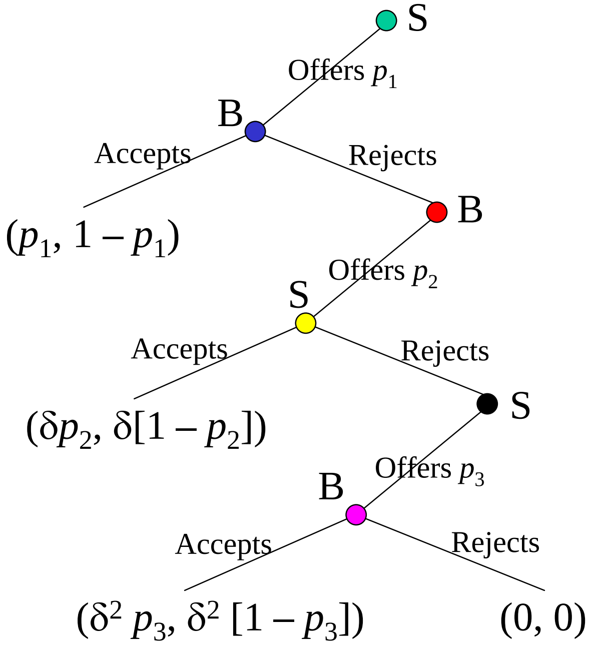


Alternating–Offer Bargaining



Alternating–Offer Bargaining



The unique SPE of the infinite-horizon, alternating-offer bargaining game

In any period in which it is the seller's turn to make an offer:

- The seller offers a price of $\frac{1}{1+\delta}$.
- The buyer accepts any price $p \leq \frac{1}{1+\delta}$ and rejects any price $p > \frac{1}{1+\delta}$.

In any period in which it is the buyer's turn to make an offer:

- The buyer offers a price of $\frac{\delta}{1+\delta}$.
- The seller accepts any price $p \geq \frac{\delta}{1+\delta}$ and rejects any price $p < \frac{\delta}{1+\delta}$.

Shaked-Sutton (84) Proof of Uniqueness of Subgame-Perfect Equilibrium in Alternating-Offer Bargaining

Suppose there exists at least one SPE in: (a) the alternating-offer game where the seller moves first; (b) the alternating-offer game where the buyer moves first

Let M be the supremum of seller payoffs over all SPEs in the game where the seller goes first
Let m be the infimum of buyer payoffs over all SPEs in the game where the buyer goes first

Observe, for every $n \geq 1$, $\exists\ \sigma_n$ (SPE) s.t. $\Pi_s(\sigma_n) \geq M - 1/n$ (game where seller goes first)

The following is easily seen to be an SPE of the game where the buyer goes first:

The buyer offers a division $(\delta\Pi_s(\sigma_n), 1 - \delta\Pi_s(\sigma_n))$. The seller accepts (in period 0)

$\sigma_n' \equiv$

Given any deviation, play σ_n beginning in period 1.

Observe that: $\Pi_B(\sigma_n') = 1 - \delta\Pi_S(\sigma_n) \leq 1 - \delta M + \delta(1/n)$, for all n
Hence $m \leq 1 - \delta M$

But if the buyer offers the seller a division $(\delta M + \varepsilon, 1 - \delta M - \varepsilon)$, the seller must accept for any $\varepsilon > 0$ since $\delta M + \varepsilon$ today is better than M tomorrow.

Hence $m \geq 1 - \delta M - \varepsilon$, for all $\varepsilon > 0$.

Conclude $m = 1 - \delta M$.

Now observe that the following is an equilibrium of the game where the seller goes first.

The seller offers $[1 - \delta + \delta^2\Pi_S(\sigma_n), \delta - \delta^2\Pi_S(\sigma_n)]$. The buyer accepts in period 0.

$\sigma_n'' \equiv$ In case of any deviation, the buyer offers $[\delta\Pi_S(\sigma_n), 1 - \delta\Pi_S(\sigma_n)]$. The seller accepts in period 1.

In case of further deviation, the parties play σ_n' in all subsequent periods.

Observe that:

$$\Pi_S(\sigma_n'') = 1 - \delta + \delta^2\Pi_S(\sigma_n) \geq 1 - \delta + \delta^2 M - \delta^2(1/n) \quad \text{for all } n \geq 0$$

Hence $M \geq 1 - \delta + \delta^2 M$.

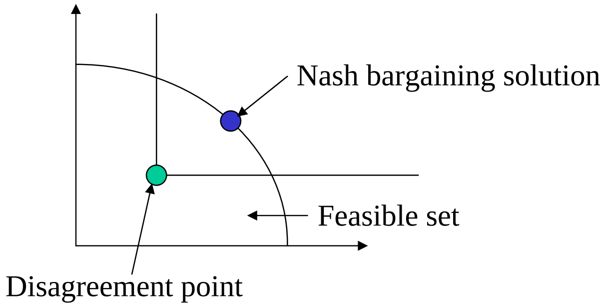
But suppose the seller ever offered the buyer a share smaller than δm. Surely the buyer would reject. [Similarly, the buyer never makes an offer (which is accepted) in which his own share is less than m.] Consequently:

$$M \leq 1 - \delta m = 1 - \delta(1 - \delta M) = 1 - \delta + \delta^2 M.$$

This demonstrates $M = 1 - \delta + \delta^2 M$, and hence $M = (1 - \delta)/(1 - \delta^2) = 1/(1 + \delta)$.
Then $m = 1 - \delta/(1 + \delta) = 1/(1 + \delta)$.

Reversing the roles, this show that the seller's payoff in any SPE is uniquely $1/(1 + \delta)$.

Furthermore this can only be realized if the offer is $1/(1 + \delta)$ in the first period.



Axiom 1: The solution should not depend on linear transformations of players' utility functions.

Axiom 2: The solution should be individually-rational and Pareto-optimal.

Axiom 3: "Independence of Irrelevant Alternatives."

Axiom 4: Symmetry.

Theorem: If the feasible set is convex, closed and bounded above, there is a unique solution satisfying Axioms 1-4, and it is given by:

$$\max_{x \geq d} (x_1 - d_1)(x_2 - d_2)$$

where *d* denotes the disagreement point and *x* is an element of the feasible set.

From Coase (1960), “The Problem of Social Cost”:

# of steers in herd	Total crop loss	Marginal crop loss
1	1	1
2	3	2
3	6	3
4	10	4

Value of crop = $1 per unit
Cost of fence = $9

Definition: Let G be a static game. Then the T-period ***repeated game***, denoted $G(T, \delta)$, consists of game G repeated T times. At each period t, the moves from periods 1, . . . $t - 1$ are known to every player. Payoffs are computed by:

$$u_i = \sum_{t=1}^{T} \delta^{t-1} \underset{\nwarrow}{u_{it}}$$

(u_{it} denotes the payoff to player i in period t)

If $T = \infty$, then $G(T, \delta)$ is referred to as the ***infinitely-repeated game***. The ***average payoff*** to player i is then given by:

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_{it}$$

Finitely Repeated Games

- Stage game with multiple NE

2

		x_2	y_2	z_2
	x_1	**1**, **1**	5, 0	0, 0
1	y_1	0, 5	4, 4	0, 0
	z_1	0, 0	0, 0	**3**, **3**

Two pure strategy NE: (x_1,x_2) and (z_1,z_2)

Trigger Strategy Equilibria

First, define a ***main equilibrium path*** to be an action suggested for every player *i* (*i* = 1, … ,*n*) and for every period:

$$\vec{s} = \underbrace{(s_{11}, \ldots, s_{n1})}_{\text{period 1}}, \underbrace{(s_{12}, \ldots, s_{n2})}_{\text{period 2}}, \underbrace{(s_{13}, \ldots, s_{n3})}_{\text{period 3}}, \ldots$$

Second, define $(s_1^*, \ldots, s_n^*)$ to be a Nash equilibrium of the static game *G*.

A ***trigger strategy*** for player *i* in the repeated game $G(T, \delta)$ is given by:

$$\sigma_{it} = \begin{cases} s_{it}, \text{ if } \textbf{\textit{every}} \text{ player has played according to } \vec{s} \text{ in all} \\ \qquad \text{previous periods } 1, \ldots, t-1 \text{ (or if } t=1\text{)}, \\ s_i^*, \text{ if there has been any prior deviation by } \textbf{\textit{any}} \text{ player}. \end{cases}$$

Example: Infinitely-Repeated Cournot Game

Description of trigger strategies:

$$q_{it} = \begin{cases} \frac{a-c}{4}, & \text{if } q_{1s} = \frac{a-c}{4} = q_{2s}, \text{ for all } s = 1, \ldots, t-1, \\ \frac{a-c}{3}, & \text{otherwise}. \end{cases}$$

Payoff along equilibrium path:

$$\sum_{t=0}^{\infty} \delta^t \left(\tfrac{a-c}{4}\right)\left(a - \tfrac{a-c}{4} - \tfrac{a-c}{4} - c\right) = \left(\tfrac{1}{1-\delta}\right)\tfrac{1}{8}(a-c)^2.$$

Payoff from optimally deviating:

$$\tfrac{3}{8}(a-c)\left(a-\tfrac{5}{8}(a-c)-c\right)+\sum_{t=1}^{\infty}\delta^t\left(\tfrac{a-c}{3}\right)\left(a-\tfrac{2}{3}(a-c)-c\right)$$

$$= \tfrac{9}{64}(a-c)^2+\delta\left(\tfrac{1}{1-\delta}\right)\tfrac{1}{9}(a-c)^2.$$

Example: Infinitely-Repeated Cournot Game

This is a trigger-strategy ***equilibrium*** if and only if:

Payoff$_i$(equilibrium path) ≥ Payoff$_i$(optimally deviating)

$$\left(\tfrac{1}{1-\delta}\right)\tfrac{1}{8}\cancel{(a-c)^2} \geq \tfrac{9}{64}\cancel{(a-c)^2} + \delta\left(\tfrac{1}{1-\delta}\right)\tfrac{1}{9}\cancel{(a-c)^2}.$$

$$\tfrac{1}{8} \geq \tfrac{9}{64}(1-\delta)+\tfrac{1}{9}\delta.$$

$$\left[\tfrac{9}{64}-\tfrac{1}{9}\right]\delta \geq \tfrac{9}{64}-\tfrac{1}{8}.$$

$$\left[\frac{9\times 9-64}{64\times 9}\right]\delta \geq \tfrac{1}{64}.$$

$$\left[\tfrac{17}{64\times 9}\right]\delta \geq \tfrac{1}{64}.$$

$$\boldsymbol{\delta \geq \tfrac{9}{17}}.$$

The Folk Theorem

Definition: The n-tuple $(x_1, \ldots , x_n)$ of payoffs to the n players is called ***feasible*** if it arises from the play of pure strategies or if it is a convex combination of payoffs from pure strategies.

Theorem: Let $(e_1, \ldots , e_n)$ be the payoffs from a Nash equilibrium of G and let $(x_1, \ldots , x_n)$ be any feasible payoffs from G.
If $x_i > e_i$ for every player i, then there exists a subgame-perfect equilibrium of $G(\infty, \delta)$ that attains $(x_1, \ldots , x_n)$ as the average payoff, ***provided that*** δ ***is sufficiently close to* 1**.

Example: Prisoners' Dilemma

		Prisoner II	
		Remain Silent	Confess
Prisoner I	Remain Silent	(–1 , –1)	(–5 , 0)
	Confess	(0 , –5)	(–4 ,–4)

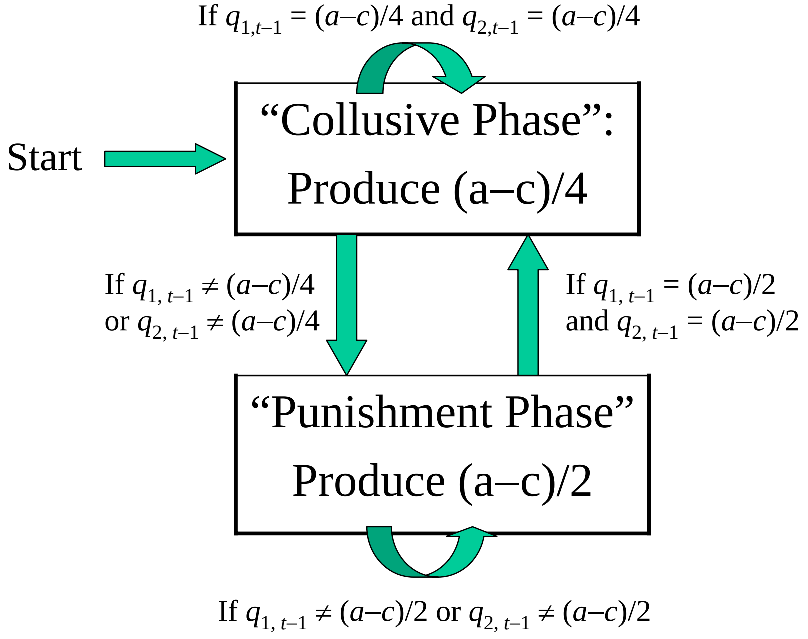
Example: Battle of the Sexes

F

		Boxing	Ballet
M	Boxing	(2 , 1)	(0 , 0)
	Ballet	(0 , 0)	(1 , 2)

Maximally-Collusive Equilibria

Example: Two-Phase Strategies in Repeated Cournot.



If $q_{1,t-1} = (a-c)/4$ and $q_{2,t-1} = (a-c)/4$

Start

"Collusive Phase": Produce (a–c)/4

If $q_{1,\,t-1} \neq (a-c)/4$ or $q_{2,\,t-1} \neq (a-c)/4$

If $q_{1,\,t-1} = (a-c)/2$ and $q_{2,\,t-1} = (a-c)/2$

"Punishment Phase" Produce (a–c)/2

If $q_{1,\,t-1} \neq (a-c)/2$ or $q_{2,\,t-1} \neq (a-c)/2$