Outline for Static Games of Complete Information

I. Definition of a game
II. Examples
III. Definition of Nash equilibrium
IV. Examples, continued
V. Iterated elimination of dominated strategies
VI. Mixed strategies
VII. Existence theorem on Nash equilibria
VIII. The Hotelling model and extensions

Definition: An n-player, **static game** of complete information consists of an n-tuple of strategy sets and an n-tuple of payoff functions, denoted by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$

S_i, the **strategy set** of player i, is the set of all permissible moves for player i. We write $s_i \in S_i$ for one of player i's strategies.

u_i, the **payoff function** of player i, is the utility, profit, etc. for player i, and depends on the strategies chosen by all the players: $u_i(s_1, \ldots, s_n)$.

Example: Prisoners’ Dilemma

		Prisoner II	
		Remain Silent	Confess
Prisoner I	Remain Silent	–1 , –1	–5 , 0
	Confess	0 , –5	–4 ,–4

Example: Battle of the Sexes

		F	
		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

<u>Definition</u>: A **<u>Nash equilibrium</u>** of G (in pure strategies) consists of a strategy for every player with the property that no player can improve her payoff by unilaterally deviating:

$(s_1^*, \dots, s_n^*)$ with the property that, for every player i:

$$u_i(s_1^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_n^*)$$
$$\geq u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$$

for all $s_i \in S_i$.

Equivalently, a Nash equilibrium is a mutual best response. That is, for every player i, s_i^* is a solution to:

$$s_i^* \in \underset{s_i \in S_i}{\arg\max} \left\{ u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*) \right\}$$

Example: Prisoners' Dilemma

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Example: Battle of the Sexes

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Cournot (1838) Model of Oligopoly

(a) n firms

(b) Each firm i has a constant marginal (and average) cost of c_i

(c) Inverse aggregate demand function of $P(Q)$

(d) Each firm simultaneously and independently selects a strategy consisting of a ***quantity*** $q_i \in [0, a]$ (where $P(a) = 0$)

Then, with two firms, the payoff functions are:

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - c_1 q_1$$
$$\pi_2(q_1, q_2) = q_2 P(q_1 + q_2) - c_2 q_2 \; .$$

and the strategy sets are:

$$S_1 = [0, a] \qquad\qquad S_2 = [0, a]$$

It is often also convenient to assume a common marginal cost (i.e., $c_1 = c = c_2$) and a linear demand curve $P(Q) = a - Q$.

Solution of Cournot Model with Two Firms

(q_1^*, q_2^*) is a Nash equilibrium if and only if:

$$q_1^* \text{ solves } \max_{q_1}\{q_1\,[P(q_1 + q_2^*) - c]\}$$

and

$$q_2^* \text{ solves } \max_{q_2}\{q_2\,[P(q_1^* + q_2) - c]\}.$$

With $P(Q) = a - Q$, we get first order conditions:

$$q_1(-1) + a - q_1 - q_2^* - c|_{q_1 = q_1^*} = 0$$

(1) $a - 2q_1^* - q_2^* = c$

and: $q_2(-1) + a - q_1^* - q_2 - c|_{q_2 = q_2^*} = 0$

(2) $a - q_1^* - 2q_2^* = c$

Subtracting (1) – (2) gives:

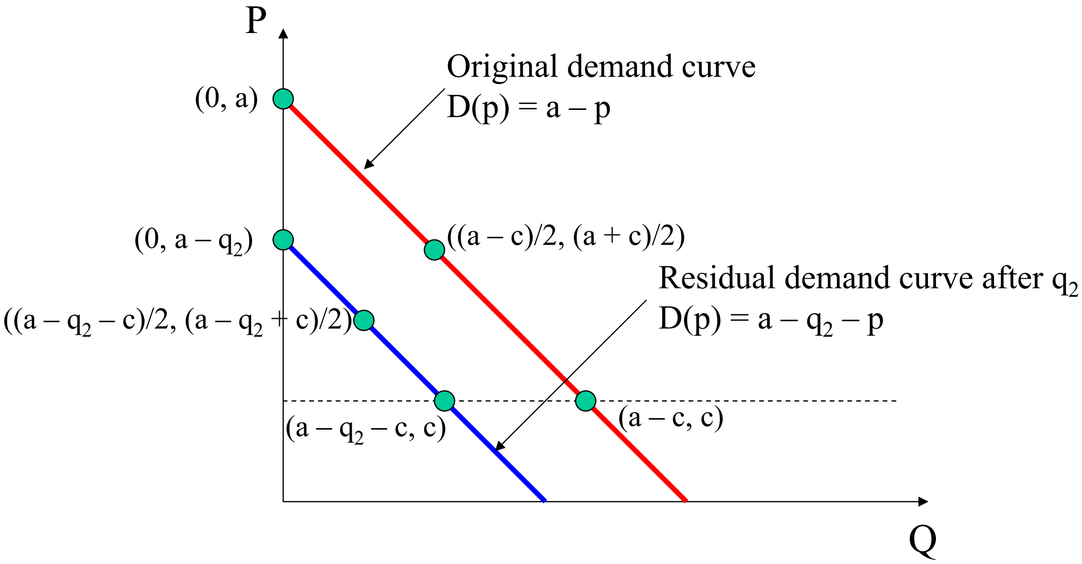
$$q_2^* - q_1^* = 0$$

Substituting $q_2^* = q_1^*$ into (1) gives:

$$a - 2q_1^* - q_1^* = c$$

$$\mathbf{q_1^* = (a - c) / 3\ ;\ \ q_2^* = (a - c) / 3\ .}$$

Best Response for Firm 1 to q_2



$R_1(q_2) = (a - q_2 - c)/2$

Similarly, the best response for firm 2 to q_1 is:

$R_2(q_1) = (a - q_1 - c)/2$

Bertrand (1883) Model of Oligopoly

(a) n firms
(b) Each firm i has a constant marginal (and average) cost of c_i
(c) Aggregate demand function of $Q(P)$
(d) Each firm simultaneously and independently selects a strategy consisting of a ***price*** $p_i \in [0, a]$ (where $Q(a) = 0$)

Then, with two firms, the payoff functions are:

$$\pi_1(p_1, p_2) = \begin{cases} Q(p_1)[p_1 - c_1], & \text{if } p_1 < p_2 \\ \frac{1}{2}Q(p_1)[p_1 - c_1], & \text{if } p_1 = p_2 \\ 0, & \text{if } p_1 > p_2 \end{cases}$$

and

$$\pi_2(p_1, p_2) = \begin{cases} Q(p_2)[p_2 - c_2], & \text{if } p_2 < p_1 \\ \frac{1}{2}Q(p_2)[p_2 - c_2], & \text{if } p_2 = p_1 \\ 0, & \text{if } p_2 > p_1 \end{cases}$$

Bertrand (1883) Model of Oligopoly

As in the Cournot game, the strategy sets are:

$$S_1 = [0, a] \qquad\qquad S_2 = [0, a]$$

and it is again usually convenient to assume a common marginal cost (i.e., $c_1 = c = c_2$).

Solution of Bertrand game with two firms and common marginal cost $c_1 = c = c_2$:

Observation 1: In any Nash equilibrium (p_1^*, p_2^*), it must be the case that $p_1^* \geq c$ and $p_2^* \geq c$.

Proof: Suppose otherwise. Without loss of generality, say $p_1^* \leq p_2^*$ and $p_1^* < c$. Then firm 1 is currently earning strictly negative profits and could profitably deviate to $p_1^* \geq c$ (thereby instead earning nonnegative profits).

Bertrand (1883) Model of Oligopoly

Observation 2: In any Nash equilibrium (p_1^*, p_2^*), it must be the case that $p_1^* = p_2^*$.

Proof: Suppose otherwise. Without loss of generality, say $p_1^* < p_2^*$ (and $p_1^* \geq c$). Then firm 2 is currently earning zero profits and, if $p_1^* > c$, firm 2 can profitably deviate to $p_2^* = p_1^* - \varepsilon$. Meanwhile, if $p_1^* = c$, firm 1 can profitably deviate to $p_1^* = p_2^* - \varepsilon$.

Observation 3: The unique Nash equilibrium is (p_1^*, p_2^*) = (c, c).

Proof: By Observations 1 and 2, the only remaining possibility is $p_1^* = p^* = p_2^* > c$. Then each firm is currently earning profits of:

$$\tfrac{1}{2}D(p^*)[p^* - c]$$

and either firm could profitably deviate to $p^* - \varepsilon$ and thereby come arbitrarily close to earning:

$$D(p^*)[p^* - c].$$

Q.E.D.

The Pollution Game

Consumers have a choice of three different models of cars, which are identical in all respects except for price and emissions:

Model A: p_A = \$15,000; e_A = 100 units
Model B: p_B = \$16,000; e_B = 10 units
Model C: p_C = \$17,000; e_C = 0 units

A consumer's utility from using a car is given by:

$$U = v - p - E$$

where v = reservation value of a car;
p = price paid for model bought;
$E = \sum_{i=1}^{N} e_i$ = aggregate emissions (over all consumers)
where e_i = 100 or 10 or 0 , depending on which model is purchased by consumer *i*.

Guess 2/3 of Average

The Problem

Each of you have to choose an integer between 0 and 100 in order to guess "2/3 of the average of the responses given by all students in the course".

Each student who guesses the integer which is 2/3 of the average of all the responses rounded up to the nearest integer, wins.

What is your guess?

Guess 2/3 of Average

Statistics		**All Courses**
# of answers:	**50**	**2370**
Average:	**38.46**	**35.79**

Answer	**%**	**all%**
0-1	8%	12%
2-13	6%	8%
14-15	0%	2%
16-21	10%	6%
22	2%	4%
23-32	6%	10%
33-34	14%	11%
35-49	14%	11%
50	14%	16%
51-100	26%	20%

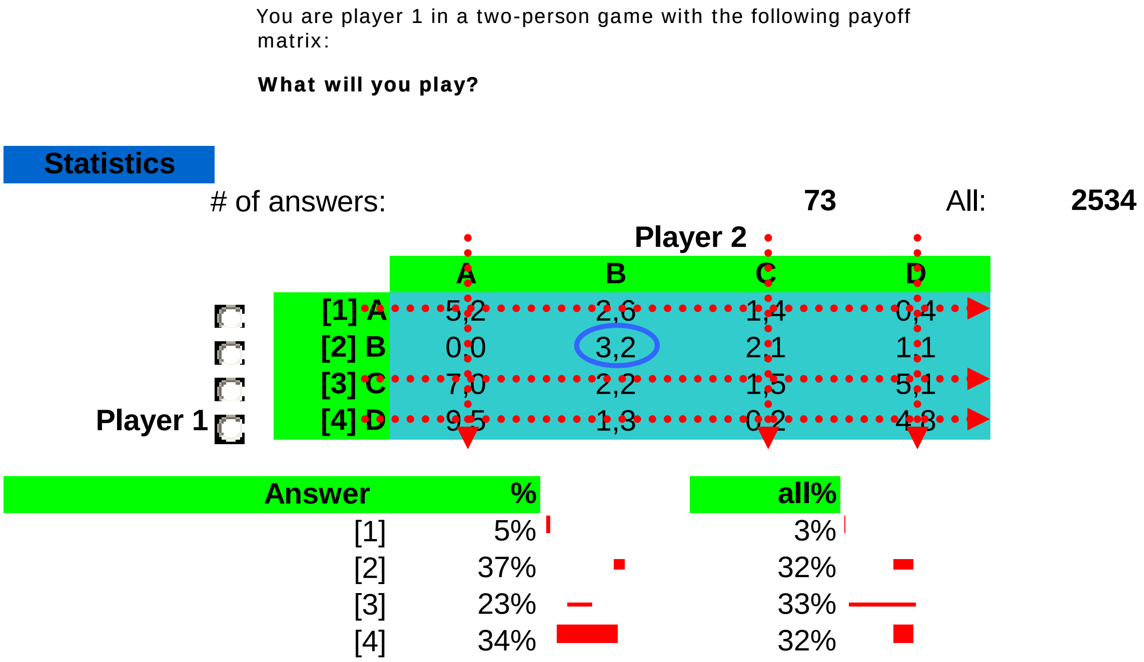
Dominated strategies:

Strategy s_i (strictly) **dominates** strategy s_i' if, for ***all*** possible strategy combinations of opponents, s_i yields a (strictly) higher payoff than s_i' to player i.

Iterated elimination of strictly dominated strategies:

Eliminate all strategies which are dominated, relative to opponents' strategies which have not yet been eliminated.

Successive Elimination



You are player 1 in a two-person game with the following payoff matrix:

What will you play?

Statistics

# of answers: **73** All: **2534**

		Player 2			
		A	**B**	**C**	**D**
☐	**[1] A**	5,2	2,6	1,4	0,4
☐	**[2] B**	0,0	3,2	2,1	1,1
☐	**[3] C**	7,0	2,2	1,5	5,1
Player 1 ☐	**[4] D**	9,5	1,3	0,2	4,8

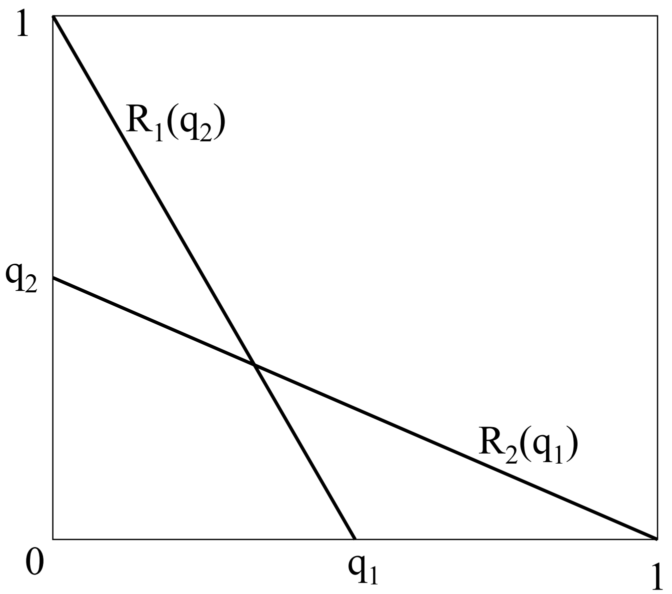
Answer	%	all%
[1]	5%	3%
[2]	37%	32%
[3]	23%	33%
[4]	34%	32%

Results on Iterated Elimination of Strictly Dominated Strategies

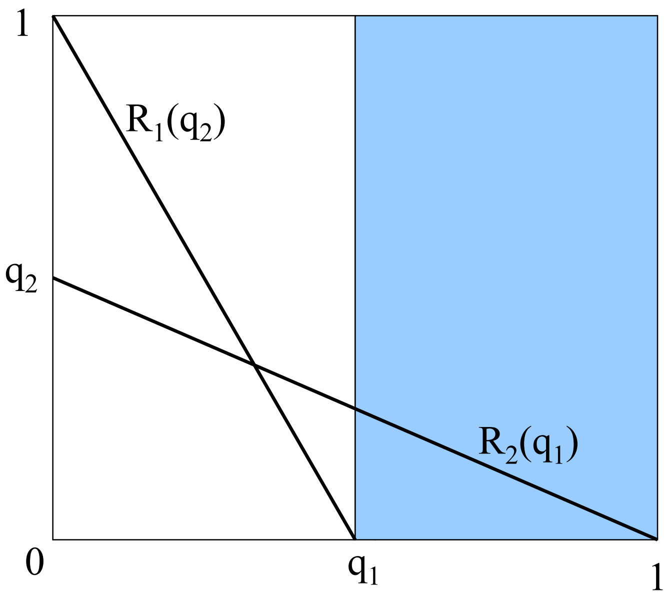
Proposition 1: If iterated elimination of strictly dominated strategies yields a ***unique*** strategy *n*-tuple, then this strategy *n*-tuple is the ***unique*** Nash equilibrium (and it is a ***strict*** Nash equilibrium).

(Definition: A ***strict*** Nash equilibrium is a strategy n-tuple with the property that every unilateral deviation makes the deviator ***strictly*** worse off.)

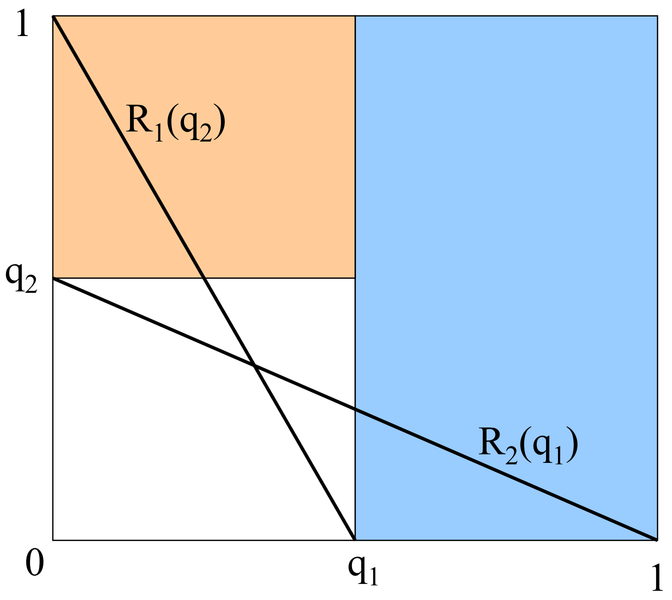
Proposition 2: Every Nash equilibrium survives iterated elimination of strictly dominated strategies.



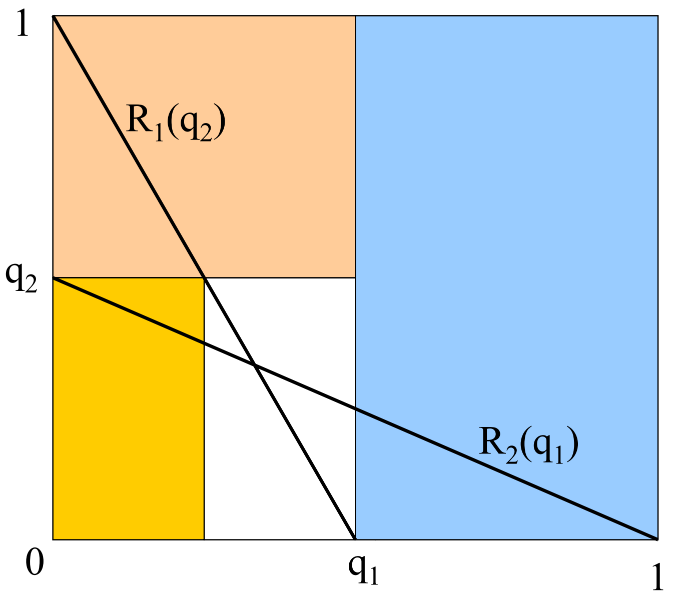
Cournot Duopoly: Best Response Functions



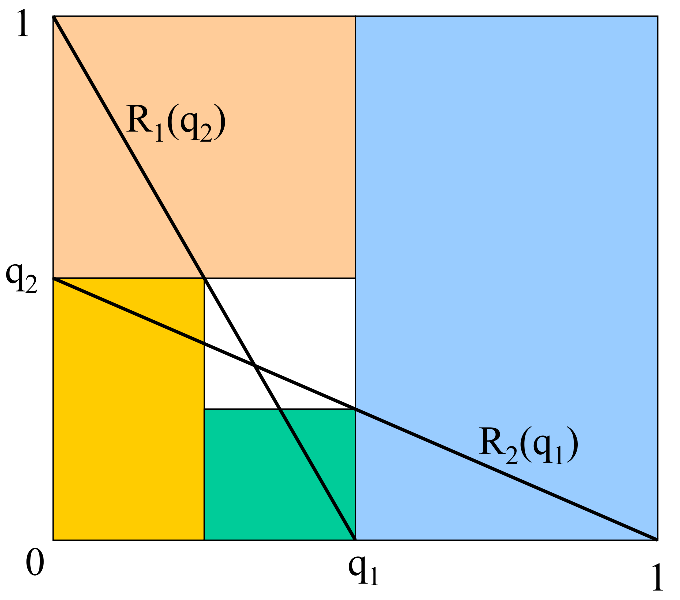
$q_1 > ½$ is strictly dominated by $q_1 = ½$



$q_1 > ½$ is strictly dominated by $q_1 = ½$
$q_2 > ½$ is strictly dominated by $q_2 = ½$



$q_1 > ½$ is strictly dominated by $q_1 = ½$
$q_2 > ½$ is strictly dominated by $q_2 = ½$
$q_1 < ¼$ is strictly dominated by $q_1 = ¼$



$q_1 > ½$ is strictly dominated by $q_1 = ½$
$q_2 > ½$ is strictly dominated by $q_2 = ½$
$q_1 < ¼$ is strictly dominated by $q_1 = ¼$
$q_2 < ¼$ is strictly dominated by $q_2 = ¼$

Example: Matching Pennies

		II	
		Heads	Tails
I	Heads	1 , –1	–1 , 1
	Tails	–1 , 1	1 , –1

Definition: Let player *i* have *K* pure strategies available. Then a **mixed strategy** for player *i* is a probability distribution over those *K* strategies.

Notation:

Strategy space:

$$S_i = \{s_{i1}, \dots , s_{iK}\}$$

Mixed strategy:

$$p_i = (p_{i1}, \dots , p_{iK})$$

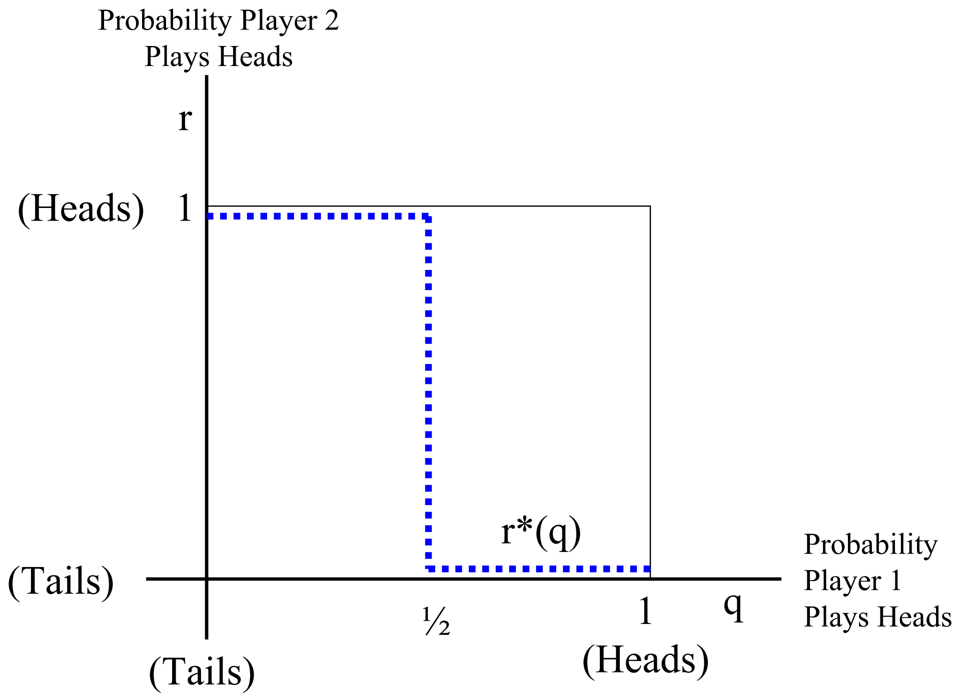
such that $\sum_{k=1}^{K} p_{ik} = 1$

and each p_{ik} is between zero and one ($0 \leq p_{ik} \leq 1$).

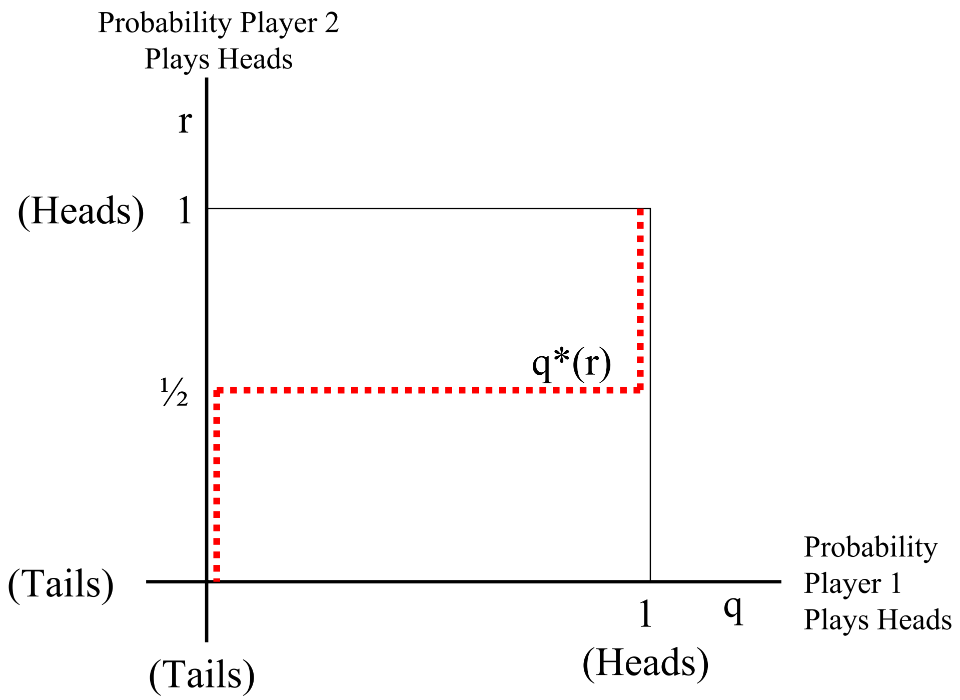
Facts:

1. Theorem (Nash, 1950):
   Every finite game has at least one Nash equilibrium (when mixed strategies are permitted).

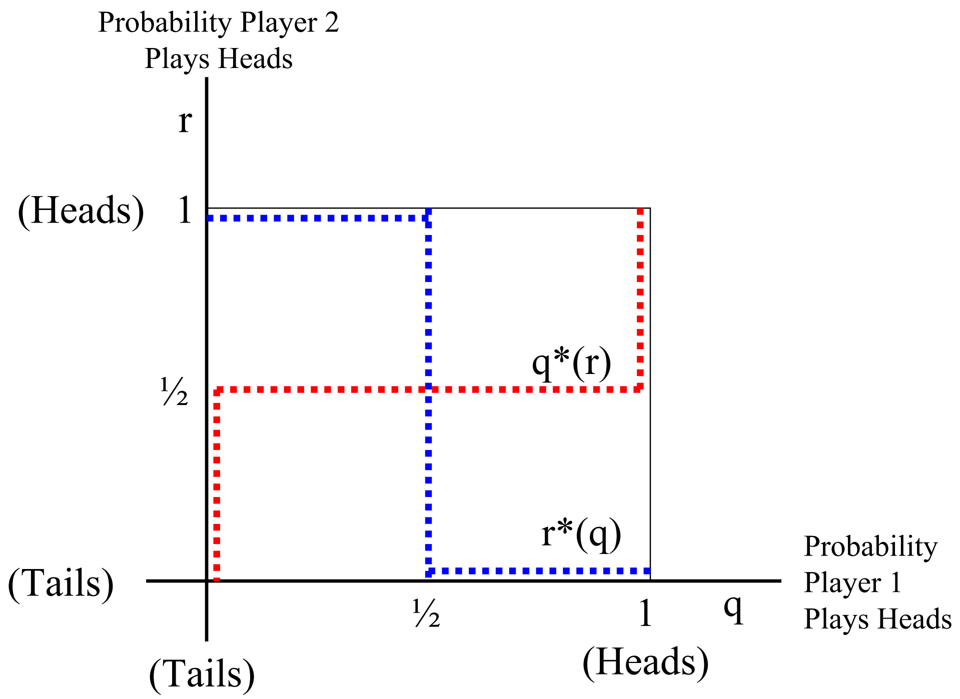
2. If, in a mixed-strategy Nash equilibrium, player i places positive probability on each of two strategies, then player i must be indifferent between these two strategies (i.e., they yield player i the same expected payoff).



Best response correspondence of Player 2



Best response correspondence of Player 1



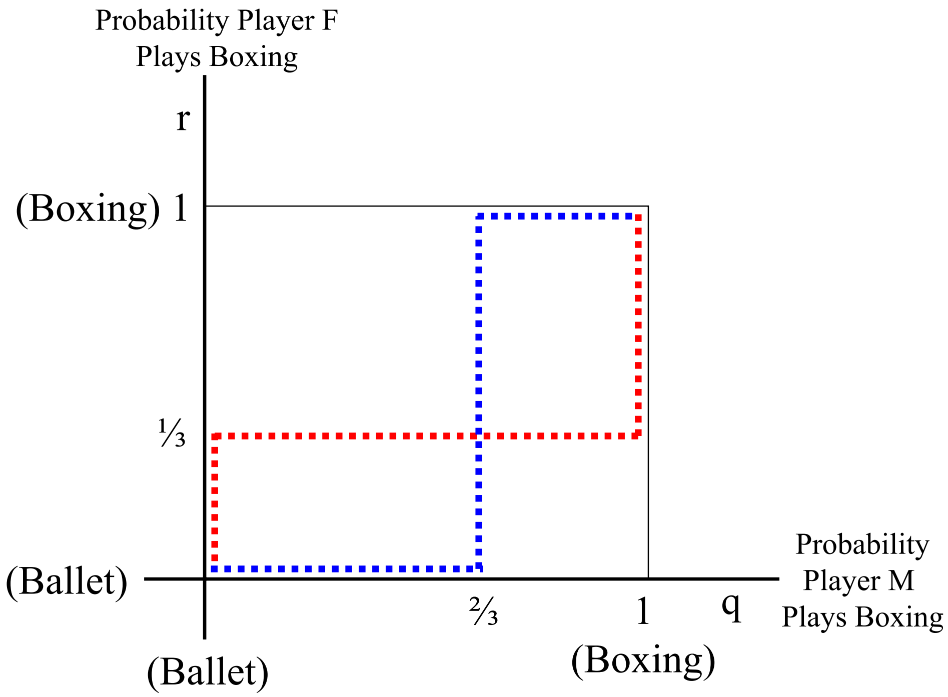
Matching Pennies

Example: Battle of the Sexes

F

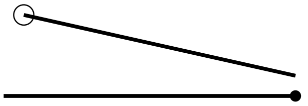
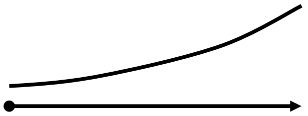
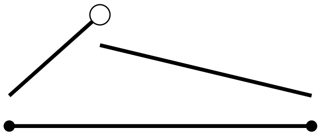
M

	Boxing	Ballet
Boxing	2 , 1	0 , 0
Ballet	0 , 0	1 , 2



Battle of the Sexes

For best response to exist need maximum to exist

- Continuous function on compact set has a maximum; hence, require:
  - closed  or no max
  - bounded  or no max
  - continuous  or no max

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Brouwer Fixed Point Theorem:

Suppose that X is a nonempty, compact, convex set in $\mathbb{R}^n$. Also suppose that the ***function*** $f : X \to X$ is continuous. Then there exists a ***fixed point*** of f, i.e., a point $x \in X$ such that $x = f(x)$.

Kakutani Fixed Point Theorem:

Suppose X as above. Also suppose that the ***correspondence*** $F : X \to X$ is nonempty and convex-valued, and that $F(\cdot)$ has a closed graph. Then there exists a ***fixed point*** of F, i.e., a point $x \in X$ such that $x \in F(x)$.

Notes:

(1) The correspondence $F(\cdot)$ is said to have a ***closed graph*** if, simply, the graph of $F(\cdot)$ is a closed set. That is, $F(\cdot)$ has a closed graph if it has the property that whenever the sequence $(x^n, y^n) \to (x, y)$, with $y^n \in F(x^n)$ for every n, then $y \in F(x)$.

Essentially the same as upper hemicontinuity (u.h.c.).

(2) The best-response correspondence $BR_i(\cdot)$ of each player i has a closed graph, by the following argument.

Suppose that there is a sequence $(x^n, y^n) \to (x, y)$ such that $y^n \in BR_i(x^n)$ for every n, but $y \notin BR_i(x)$. Then there exists $\varepsilon > 0$ and $y' \neq y$ such that:

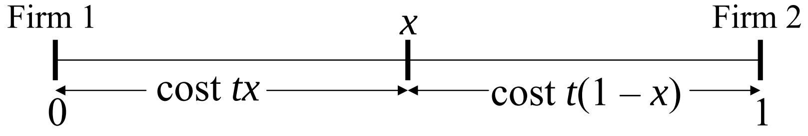
$$u_i(y', x) > u_i(y, x) + \varepsilon .$$

But this contradicts:

$$u_i(y', x^n) \leq u_i(y^n, x^n), \text{ for every } n.$$

Product Differentiation: The Hotelling Model

Consumers are uniformly distributed on the interval [0, 1].
There are two firms, located at $x = 0$ and $x = 1$, which each produce the same physical good at marginal cost of c.
Consumers have transportation cost t per unit of distance.



Each consumer consumes 0 or 1 units of the good:

$$u(0) = 0; \quad u(1) = v.$$

If firm 1 charges p_1 and firm 2 charges p_2, the consumer located at x gets $v - p_1 - tx$ from purchasing at firm 1 and gets $v - p_2 - t(1 - x)$ from purchasing at firm 2.

Let $\tilde{x}$ denote the customer who is indifferent between purchasing at firm 1 and firm 2. Then:

$$v - p_1 - t\tilde{x} = v - p_2 - t(1 - \tilde{x})$$

$$2t\tilde{x} = t + p_2 - p_1$$

$$\tilde{x} = \tfrac{1}{2} + \tfrac{p_2 - p_1}{2t} .$$

The profits of firm 1 are given by:

$$\pi_1(p_1, p_2) = [p_1 - c]\,\tilde{x} = [p_1 - c][\tfrac{1}{2} + \tfrac{p_2 - p_1}{2t}] .$$

The profits of firm 2 are given by:

$$\pi_2(p_1, p_2) = [p_2 - c][1 - \tilde{x}] = [p_2 - c][\tfrac{1}{2} - \tfrac{p_2 - p_1}{2t}] .$$

These imply the first-order conditions of:

(1) $t + c + p_2^* - 2p_1^* = 0$

(2) $t + c + p_1^* - 2p_2^* = 0$.

Solving yields:

$\boldsymbol{p_1^* = t + c}$; $\boldsymbol{p_2^* = t + c}$ **.**