Auctioning a Single Item

- Auctions and Competitive Bidding
  - McAfee and McMillan (*Journal of Economic Literature*, 1987)
  - Milgrom and Weber (*Econometrica*, 1982)

Auctions

- 450% of the world GNP is traded each year by auction.
- Understanding auctions should help us understand the formation of markets by modeling the competition on one side of the market.
- Auctions represent an excellent application of game theory, since in an auction the rules of the game are made explicit.

Simple Auctions

Auctions typically take one of four simple forms:

Oral		Sealed Bid
English (↑ price)		2nd Price
Dutch (↓ price)	≡	1st Price

Simple Auctions

- *English*: price increases until only one bidder is left; the remaining bidder gets the good and pays the highest bid.
- *Dutch*: prices decreases until a bidder accepts the price; this bidder gets the good and pays the price at acceptance.
- *Second Price*: each bidder submits a bid in a sealed envelope; the highest bidder gets the good and pays the second highest bid.
- *First Price*: each bidder submits a bid in a sealed envelope; the highest bidder gets the good and pays the amount of his bid.

Models of Private Information

(1) Independent Private Value:

$v_i \sim F_i$ independently of v_j for $j \neq i$.

(2) Common Value:

$e_i = v + \varepsilon_i$, $\varepsilon_i \sim F_i$ w/ mean 0.

(3) Affiliated Value:

$v_i(x,s)$, my value depends on private information $x = (x_1,...,x_n)$ and state of world s.

Models of Private Information

- Independent private value model: It makes sense if differences in value arise from heterogeneous preferences over the attributes of the item
- Common Value: It makes sense if the bidders have homogeneous preferences, so they value the item the same ex post, but have different estimates of this true value.
- Affiliated value model: In this model, each bidder has private information that is positively correlated with the bidder's value of the good.

Auction Exercise

- Bid for single object
- Common value = $1 per bean
- On slip of paper write:
  - Name
  - Estimate (# of beans × $1)
  - Bid in first-price sealed-bid auction
  - Bid in second-price sealed-bid auction

Benchmark Model

Independent Private Values, Symmetric, Risk Neutral Bidders

- buyer values $v_1,\ldots,v_n \sim F$ on $[0,\infty)$
- seller value v_0 (common knowledge)
- order statistics $v_{(1)} \geq v_{(2)} \geq \ldots \geq v_{(n)}$.
- Unique equilibrium in dominant strategies:
  - *English*: bid up to your value or until others stop.
  - *2nd Price*: bid your value.
- The bidder with the highest value wins and pays the second highest value.

Benchmark Model

- The winner gets $v_{(1)} - v_{(2)}$ ex post and expects in the interim state to get:

$$E_{v_{(1)}}[v_{(1)} - v_{(2)}] = E\left(\frac{1-F(v_{(1)})}{f(v_{(1)})}\right).$$

This represents the information rent received by the winner of the auction.

- The seller's expected revenue:

$$E[MR(v_{(1)})] = E\left(v_{(1)} - \frac{1-F(v_{(1)})}{f(v_{(1)})}\right), \quad \text{with } J'>0.$$

First Price (≡ Dutch)

Symmetric Equilibrium Bidding Strategy

- Bidder's expected profit:

$$\pi(v,b(v)) = (v - b(v))\Pr(Win|b(v)).$$

- By the envelope theorem,

$$\frac{d\pi}{dv} = \frac{\partial\pi}{\partial b}\frac{\partial b}{\partial v} + \frac{\partial\pi}{\partial v} = \frac{\partial\pi}{\partial v}$$

- But then $d\pi/dv = \Pr(Win|b(v)) = \Pr(\text{highest bid})$
  $= \Pr(\text{highest value}) = F(v)^{n-1}$ a.e.

First Price (≡ Dutch)

- By the Fundamental Theorem of Calculus,

$$\pi(v) = \pi(0) + \int_0^v F(u)^{n-1}du = \int_0^v F(u)^{n-1}du,$$

- Substituting into $\pi(v,b(v)) = (v - b(v))\Pr(Win|b(v))$ yields

$$b(v) = v - \frac{\pi(v)}{\Pr(Win)} = v - F(v)^{-(n-1)}\int_0^v F(u)^{n-1}du.$$

Example

- $v \sim U$ on $[0,1]$
- Then $F(v) = v$, so

$$b(v) = v - v/n = v(n-1)/n.$$

- The optimal bid converges to the value as $n\to\infty$, so in the limit the seller is able to extract the full surplus.
- In equilibrium, the bidder bids the expected value of the second highest value given that the bidder has the highest value.

Benchmark Auction: Revenue Equivalence

- Seller's revenue:

  English = 2nd = 1st = Dutch.
- This follows since all four have the same probability of winning,

  $d\pi/dv = \Pr(\text{Win})$, and $\pi(0) = 0$,

  so the bidder gets the same profit in each, and hence so must the seller. [This follows from the analysis of the direct revelation game.]

Optimal Auction

- In the optimal auction, the seller sets a reserve price r s.t. $MR(r)=v_0$, where $MR(v)=v-[1-F(v)]/f(v)$, so $r=MR^{-1}(v_0)$.
- The purpose of r is to reduce information rents (no rents for $v < r \Rightarrow$ less rents $v > r$).
- Example: Single bidder.
  - By setting r, the seller gets $\pi(r)= r\,[1- F(r)] + v_0F(r)$.
  - The F.O.C. for maximization is

    $1 - F(r) - rf(r) + v_0f(r) = 0$, so $v_0 = r - [1-F(r)]/f(r)$.
- The optimal r does not change with the number of bidders.

Asymmetric Bidders
Myerson (1982)

- If $F_i \neq F_j$, $i \neq j$, the seller should employ an asymmetric auction by awarding the good to the bidder i with the highest value of

$$MR_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}.$$

- For the special case where the distributions are the same except the means differ, then the optimal auction favors bidders with the lower expected values ex ante.

Royalty

Should the seller use a royalty?

- Suppose $\tilde{v}$ is an ex post observation about the true value. Then under a linear royalty scheme the winner's payment is:

$$p = b + r\tilde{v}.$$

Royalty

- If $\tilde{v}$ is exogenous (no moral hazard), then r should be 100%.
- If $\tilde{v}$ is influenced by the costly effort of the winner, then a lower royalty rate must be used to preserve the correct incentives for the winning bidder to develop the good.
- With moral hazard, the optimal contract is linear in $\tilde{v}$.
- If the bidders are risk averse, a royalty can also serve to shift risk from the bidders to the seller.

Risk Aversion

The revenue equivalence theorem no longer applies. The seller's revenue from the 1st Price > English.

- English auction: it still is a dominant strategy to bid your value, so the outcome is the same as in the risk neutral case.
- First price auction: a bidder has an incentive to increase his bid from the risk neutral bid, since by increasing the bid, his risk is reduced: he gets a higher probability of winning a smaller prize. This lottery with reduced risk is preferable for the risk averse bidder.

- Competition is greater with the first price auction, as bidders attempt to limit risk by bidding higher.

Risk Aversion

- Optimal Auction:

  The seller increases bidder competition by subsidizing high bidders that lose and penalizing low bidders.

- Number of Bidders

  Keeping the number of bidders secret increases competition ex ante by reducing the bid dispersion and the informational rents.

Collusion

- Bidding ring: n bidders colluding as 1 (and essentially submitting a single bid).
- The seller should set a higher reserve r using the distribution $F(v)^n$. The seller's profit as a function of the reserve is

$$\pi(r) = r[1 - F(r)^n] + v_0 F(r)^n.$$

- The first-order-condition for maximization is satisfied if r is such that

$$v_0 = r - \frac{1 - F(r)^n}{nF(r)^{n-1} f(r)}.$$

Many Items

What if the seller is not selling a single item, but many items?

- Consider the extreme case where the seller can produce an unlimited amount of the good at a marginal cost of c.
- Optimal selling mechanism:

  The seller should not use an auction at all, but instead should post a fixed price r such that

$$c = r - [1 - F(r)]/f(r).$$

Costs of Entering Bidding

In many real auctions it is costly to bid. How can this affect the bidding behavior?

- Fishman (1988) considers a model of preemptive bidding in takeovers, where it is costly for bidder i to acquire his private value information v_i.
- If the bidders enter sequentially, then the initial bidder may make a preemptive bid if his value is high enough, to signal that he has a high value, and thereby, discourage other bidders from entering.

Correlated Values

- Correlated values can be modeled as each bidder i having private information x_i with higher x_i implying a higher value v.

- In this case, $E(v|x_i) \geq E(v|x_i, x_i > x_j \; \forall \; j \neq i)$, so that winning is bad news about value (winner's curse).

Results

- If the bidders have private values, but these values are not statistically independent, then Crémer & McLean (1985) show that the seller can extract all rents in the optimal auction for the case of a finite number of values.
- This is accomplished by offering each bidder a choice between a lottery and an second price auction.
- This works whenever i's value can be recovered from the values of the other bidders.

Milgrom and Weber
(*Econometrica*, 1982)

Revenue Equivalence (ipv and risk neutrality):
- English = 2nd Price = 1st Price = Dutch.

If the bidders are risk averse then:
- Dutch = 1st Price ≥ English = 2nd Price.

How then can we explain the frequent use of the English auction?
- Values are positively correlated (resell of the object is possible).

Common Value

- Each bidder has a private estimate $x_i = v + \varepsilon_i$ of the value v, where ε_i represents the noise in the estimate.

Example:
- A glass jar filled with coins. Each bidder in the room estimates the value of the coins in the jar, but the estimate is imperfect: some overestimate the value of the coins, others underestimate the value.

Common Value

- In a first price auction, you would expect each bidder to make a bid that is an increasing function of the estimate.
- If everyone adopts the same bidding strategy, then the winner of the auction is going to be the bidder that overestimated the value of the coins the most.
- A bidder that does not condition his bid on the assumption that his estimate is the most optimistic among all the bidders will lose money quickly as a result of the winner's curse.

Results: Winner's Curse

I won. Therefore, I overestimated the most. My bid only matters when I win, so I should condition my bid on winning (i.e., that I overestimated the most).

- Winning is bad news about my estimate of value. This is a form of adverse selection that arises in any exchange setting: if you want to trade with me, it must be that no one else offered more, because they did not think that the item is worth what I am willing to pay.

Results:
Value of Private Information

- Rents to the bidders come solely from the privacy of the information and not the <u>quality</u> of the information.
- For example, in an auction with three bidders, if two have the same information and a third has poorer but independent information, then the two with the same information will get a payoff of zero in equilibrium, whereas the one with poorer information gets a positive payoff.
- One implication of this result is that the seller should reveal all his private information.

Results:
Price and Information

- In common value auctions, price tends to aggregate information: as $n \to \infty$, the price converges to the true value if the monotone-likelihood-ratio-property is satisfied.

General Symmetric Model with Affiliated Values

Milgrom and Weber's model:

- n bidders, each with private info x_i, which can be thought of as i's estimate of the value.
- Let $x = \{x_1,...,x_n\}$ be the vector of estimates.
- Bidder i's value of the good depends on the state of world s and the private information x.
- Let f(s,x) be the joint density, which is symmetric in x.
- Bidder i's value $v_i = u(s,x_i,x_{-i})$ is assumed to be symmetric in x, increasing, and continuous.
- Assume values are affiliated: if you have high value, it is more likely that I have high value.

Definition

- The random variables $Z = \{S,X_1,...,X_n\}$ are *affiliated* if the joint density f(z) is such that

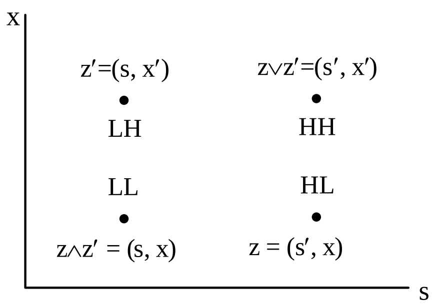
$$\forall\ z,z' \quad f(z \vee z')f(z \wedge z') \geq f(z)f(z')$$

where $z \vee z' = \max\{z,z'\}$ and $z \wedge z' = \min\{z,z'\}$.

Affiliation

Pr(all high)Pr(all low) ≥ Pr(high,low)Pr(low,high)

Pr(HH)Pr(LL) ≥ Pr(HL)Pr(LH)



Definition

The conditional density g(x|s) satisfies the monotone likelihood ratio property, <u>MLRP</u>, if $\forall$ s < s' and x < x'

$$\frac{g(x|s)}{g(x|s')} \geq \frac{g(x'|s)}{g(x'|s')}$$

(i.e., the likelihood ratio is decreasing in x).

Properties of Affiliated RVs

- $z = \{z_1,...,z_n\} \in A$ (affiliated)
  1. $E[g(z)h(z)|s] \geq E[g(z)|s]E[h(z)|s]$
  2. $f \in A \Leftrightarrow \partial^2 \ln(f)/\partial z_i \partial z_j \geq 0$.
  3. $f = g \cdot h$, $g,h \geq 0$, $g,h \in A \Rightarrow f \in A$.
  4. $z \in A$, $g_1,...,g_k \uparrow \Rightarrow g_1(z_1),...,g_k(z_k) \in A$.
  5. $z_1,...,z_k \in A \Rightarrow z_1,...,z_{k-1} \in A$.
  6. $z \in A$, $H \uparrow \Rightarrow E[H(z)|a \leq z \leq b] \uparrow$ as a,b $\uparrow$.
  7. $E[V_i \mid X_i = x, Y_1=y_1,...,Y_{n-1}=y_{n-1}] \uparrow$ in x, where $Y_1 \geq ... \geq Y_{n-1}$ are the order statistics of the n-1 other bidders.

Analysis of Auctions with Affiliated Information

- Throughout the analysis, assume
  1. No collusion
  2. The choice of auction doesn't reveal info
  3. The choice of auction doesn't affect who plays

Analysis of Auctions with Affiliated Information

- For each auction, M&W do the following:
  1. Find the symmetric equilibrium bidding function
  2. Determine how the seller should use any private information
  3. Find the order of the simple auctions with respect to the seller's revenue

Results

- The main result is that in terms of seller revenue:

  English ≥ 2nd Price ≥ 1st Price = Dutch

- Intuition: The equilibrium bid function depends on everyone's information. The more (affiliated) information you condition on, the higher the bid.

Results

- How does the price depend on the bids in the simple auctions?
  1. 1st Price: only 1st bid
  2. 2nd Price: 1st and 2nd bids
  3. English: all bids
- Hence, the English auction does best because it involves conditioning on the most information. Here it is assumed that the bidders in an English auction observe the point at which each bidder drops out of the auction.

Second Price

- Let $v(x,y) = E[V_i \mid X_i = x, Y_1 = y]$, which is increasing in x,y.
- *Claim*. In a second price auction, the optimal strategy is to bid $b(x) = v(x,x)$; that is, bid your expected value given your signal is the same as the second highest.

Second Price

Proof Sketch.

- Maximize the probability of winning whenever it is profitable, since your bid does not affect your payment.
- Hence, b(x) is chosen to maximize $E[(V_i - P)1_{\{P<b\}}|x]$, where $P = \max_{j\neq i} b(x_j)$.
- The solution to this maximization can be found by applying the revelation principle.

Second Price

Proof Sketch

- Suppose the bidder reports x'.
- Then P < b iff b(y) < b(x') iff y < x'. Hence,

  select x' to $$\max \int_{-\infty}^{x'} [v(x,y) - b(y)]f(y|x)dy.$$

- The FOC evaluated at x' = x is

  $$[v(x,x) - b(x)]f(x|x) = 0.$$

- Hence, b(x) = v(x,x). QED

Other Results

1. If n+1st bidder's information is a garbling of X_1,Y_1, then n+1st bidder's profit = 0.
2. If $(X_0,S,X_1,\ldots,X_n) \in A$ and the seller announces x_0, then $E[V(Y_1,Y_1) \mid X_i>Y_1] \leq E[V(Y_1,Y_1,X_0) \mid X_i>Y_1]$.

- This says that the expected sale price is greater if the seller announces x_0. The proof uses a property of affiliation: the expected value is increasing when one conditions on something affiliated.

English

- Assume each bidder observes the other bidders dropping out.
- The optimal strategy, after k bidders have dropped out at prices $p_1,\ldots,p_k$ is

$$b_k(x|p_1,\ldots,p_k) = E[V_i \mid x, (y_1,\ldots,y_{n-k-1})=x,\{b_{j-1}(Y_{n-j}|p_1,\ldots,p_{j-1})=p_j\}_{j=1,k}]$$

- The seller's revenue from English ≥ 2nd Price, since the English auction reveals affiliated information (the n - 2 low bidders reveal their signals by the bid at which they drop out).

First Price (Dutch)

- Suppose the bidder reports x'. Then

x' chosen to $$\max \int_{\underline{x}}^{x'} [v(x,y) - b(x')]f(y|x)dy.$$

- The FOC evaluated at x' = x is

$$0 = [v(x,x) - b(x)]f(x,x) - \int_{\underline{x}}^{x} b'(x)f(y|x)dy.$$

- Hence, $$b'(x) = [v(x,x) - b(x)]\frac{f(x|x)}{F(x|x)},$$

  which is a first-order linear differential equation with boundary condition $b(\underline{x}) = v(\underline{x},\underline{x})$.

First Price (Dutch)

- *Claim*. The seller's revenue from 2nd Price ≥ 1st Price.
- *Proof Sketch*. Let $R(x',x) = E[V_i 1_{\{Y1<x'\}}|X_i = x]$. In either auction, the bidder will report x' to max $R(x',x) - P(x',x)F(x'|x)$, where $P(x',x)$ is the expected price and $F(x'|x)$ is the probability of winning given x' is reported and i has information x. The FOC evaluated at x' = x is

$$0 = R_1 - P_1F - Pf.$$

- Let P^1 = 1st price and P^2 = 2nd price, and $P_1 = \partial P/\partial x'$; $P_2 = \partial P/\partial x$.

First Price (Dutch)

Proof Sketch

- Suppose $P^1 > P^2$ (higher rev in 1st price).
- 1st price: $P_2^1 = 0$ (price only depends on x').
- 2nd price: $P_2^2 \geq 0$ by affiliation.
- From the FOC, $P_1 = [R_1 - Pf]/F$. Hence,

$$\frac{dP^2}{dx} = P_1^2 + P_2^2 \geq P_1^2 = \frac{R_1 - P^2 f}{F} \geq \frac{R_1 - P^1 f}{F} = P_1^1 = \frac{dP^1}{dx}$$

- So $P^1 > P^2$ and $dP^2/dx \geq dP^1/dx$, but this is a contradiction, since P^1 and P^2 start at the same point. Hence, $P^1 \leq P^2$. QED

Linkage Principle

A higher price is obtained if the price is linked to more affiliated information.

Auction	Condition on
1st Price	winner's estimate $X_i \geq Y_1$
2nd Price	1st & 2nd estimate $X_i \geq Y_1$ & $Y_1=x_1$
English	all estimates $X_i \geq Y_1$, $Y_1=x_1$, $b(Y_j)=p_j$

- Hence, English ≥ 2nd Price ≥ 1st Price and the seller should always reveal information.
- The more information you condition on the higher is the price.