Chapter 4

Mixed Strategies and Mixed Strategy Equilibrium

**Mixed Strategies**

- Two kinds of strategies:
  - pure
  - mixed
- Two kinds of equilibrium:
  - pure strategy
  - mixed strategy
- Two games with mixed strategy equilibria:
  - Matching Pennies
  - Market Niche

Matching Pennies: The payoff matrix (All payoffs in cents)

<table>
<thead>
<tr>
<th></th>
<th>Heads</th>
<th>Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>+1, -1</td>
<td>-1, +1</td>
</tr>
<tr>
<td>Tails</td>
<td>-1, +1</td>
<td>+1, -1</td>
</tr>
</tbody>
</table>

Matching Pennies: No equilibrium in pure strategies

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Heads</td>
<td>+1, -1</td>
<td>-1, +1</td>
</tr>
<tr>
<td>Tails</td>
<td>-1, +1</td>
<td>+1, -1</td>
</tr>
</tbody>
</table>

**Computing Mixed Strategy Equilibria in 2×2 Games**

- Solution criterion: each pure strategy in a mixed strategy equilibrium pays the same at equilibrium
- Each pure strategy not in a mixed strategy equilibrium pays less
- Detailed calculations for Matching Pennies and Market Niche
- An appealing condition on equilibria: payoff dominance

Matching Pennies: What about mixed strategies?

<table>
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<tr>
<td>Heads</td>
<td>+1, -1</td>
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</tr>
<tr>
<td>Tails</td>
<td>-1, +1</td>
<td>+1, -1</td>
</tr>
</tbody>
</table>

**Probability Assignment**

- \[ x \] is the probability of Heads
- \[ 1 - x \] is the probability of Tails
- \[ y \] is the probability of Heads
- \[ 1 - y \] is the probability of Tails

\[
\begin{array}{ccc}
\text{probability} & y & 1-y \\
1 & 2 & h \\
x & H & +1, -1 \\
-1-x & T & -1, +1 \\
\end{array}
\]

That is, \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \)
Need to calculate player 1’s expected utility from player 2’s mixed strategy

\[
\begin{array}{ccc}
\text{probability} & y & 1-y \\
\text{1} & \text{2} & \\
\text{H} & +1, -1 & -1, +1 \\
\text{T} & -1, +1 & +1, -1 \\
\end{array}
\]

EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1
EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y

Need to calculate player 2’s expected utility from player 1’s mixed strategy

\[
\begin{array}{ccc}
\text{probability} & y & 1-y \\
\text{1} & \text{2} & \\
\text{H} & +1, -1 & -1, +1 \\
\text{T} & -1, +1 & +1, -1 \\
\end{array}
\]

EU_2(h) = x \times -1 + (1-x) \times 1 = 1 - 2x
EU_2(t) = x \times 1 + (1-x) \times -1 = 2x - 1

In equilibrium, Player 1 is willing to randomize only when he is indifferent between H and T

EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1
EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y

In equilibrium: EU_1(H) = EU_1(T)
∴ 2y - 1 = 1 - 2y
⇒ 4y = 2
⇒ y = \frac{1}{2}
⇒ 1 - y = 1 - \frac{1}{2} = \frac{1}{2}
∴ y = 1 - y = \frac{1}{2}

Similarly, Player 2 is willing to randomize only when she is indifferent between h and t

Player 1’s Conditions:
EU_1(H) = EU_1(T)

Player 2’s Conditions:
EU_2(h) = x \times -1 + (1-x) \times 1 = 1 - 2x
EU_2(t) = x \times 1 + (1-x) \times -1 = 2x - 1

In equilibrium: EU_2(h) = EU_2(t)
∴ 1 - 2x = 2x - 1
⇒ x = \frac{1}{2} and 1 - x = 1 - \frac{1}{2} = \frac{1}{2}
∴ x = 1 - x = \frac{1}{2}

Matching Pennies:
Equilibrium in mixed strategies

\[
\begin{array}{ccc}
\text{probability} & \frac{1}{2} & \frac{1}{2} \\
\text{1} & \text{2} & \\
\text{h} & \text{t} & \\
\frac{1}{2} & \text{H} & +1, -1 & -1, +1 \\
\frac{1}{2} & \text{T} & -1, +1 & +1, -1 \\
\end{array}
\]

EU_1: 0 = 0
EU_2: 0 = 0

Each is playing a best response to the other!

Mixed strategies are not intuitive:
You randomize to make me indifferent.

Row randomizes to make Column indifferent.
Column randomizes to make Row indifferent.

Then each is playing a best response to the other.
**Market Niche: The payoff matrix**

```
Firm 2
  Enter     Stay Out
Firm 1
  Enter   -50, -50   100, 0
  Stay Out 0, 100     0, 0
```

**Market Niche: Two pure strategy equilibria**

```
Firm 2
  Enter     Stay Out
Firm 1
  Enter   -50, -50   100, 0
  Stay Out 0, 100     0, 0
```

**Market Niche: What about mixed strategies?**

```
probability  y  1-y
1  2  e  s
Firm 2
  x E -50, -50 100, 0
  1-x S 0, 100 0, 0
```

That is, 0 ≤ x ≤ 1 and 0 ≤ y ≤ 1

**Need to calculate firm 1’s expected utility from firm 2’s mixed strategy**

```
EU1(E) = y × -50 + (1- y) × 100 = 100 - 150y
EU1(S) = y × 0 + (1- y) × 0 = 0
```

**Need to calculate firm 2’s expected utility from firm 1’s mixed strategy**

```
EU2(e) = x × -50 + (1- x) × 100 = 100 - 150x
EU2(s) = x × 0 + (1- x) × 0 = 0
```

**In equilibrium, Firm 1 is willing to randomize only when it is indifferent between E and S**

```
EU1(E) = EU1(S)
∴ 100 - 150y = 0
⇒ 150y = 100
⇒ y = 2/3
⇒ 1 - y = 1 - 2/3 = 1/3
∴ y = 2/3 and 1 - y = 1/3
```
Similarly, Firm 2 is willing to randomize only when it is indifferent between h and t.

**Firm 1’s Conditions:**
\[
EU_1(E) = EU_1(S)
\]

**Firm 2’s Conditions:**
\[
EU_2(e) = x \times -50 + (1-x) \times 100 = 100 - 150x \\
EU_2(s) = x \times 0 + (1-x) \times 0 = 0
\]

In equilibrium: \( EU_2(e) = EU_2(s) \)

\[
\therefore \quad 100 - 150x = 0 \\
\Rightarrow \quad 150x = 100 \\
\therefore \quad x = \frac{2}{3} \text{ and } 1 - x = \frac{1}{3}
\]

**Mixed Strategies and bluffing: Liar’s Poker**
- Mixed strategies as a way to be unpredictable
- Bluffing and mixed strategies
- Liar’s poker, a game where bluffing pays

**Market Niche:**
**Equilibrium in mixed strategies**

**Probability:**
1 2
\
\begin{array}{c|cc|c}
& E & S & EU_1 \\
\hline
2/3 & -50, -50 & 100, 0 & 0 \\
1/3 & 0, 100 & 0, 0 & 0 \\
\end{array}

**Probability:**
\[
EU_2: 0 = 0
\]

Each firm is playing a best response to the other!

**Liar’s Poker: extensive form**

**Liar’s Poker: normal form**

**Liar’s Poker: No pure strategy equilibrium**
Liar’s Poker:
What about mixed strategies?

Each player calculates his expected utility from other’s mixed strategy

In equilibrium, player 1 is willing to randomize only when he is indifferent between A and K

Similarly, Player 2 is willing to randomize only when she is indifferent between c and f

Mixed Strategy Equilibria of Coordination Games and Coordination Problems

Games with mixed strategy equilibria which cannot be detected by the arrow diagram

The mixed strategy equilibrium of Video System Coordination is not efficient
Correlated Equilibrium

- Mixed strategy Nash equilibria tend to have low efficiency
- Correlated equilibria
  - public signal
  - Nash equilibrium in game that follows

Asymmetric Mixed Strategy Equilibria

- Making a game asymmetric often makes its mixed strategy equilibrium asymmetric
- Asymmetric Market Niche is an example

Asymmetrical Market Niche: The payoff matrix

Asymmetrical Market Niche: Two pure strategy equilibria

Asymmetrical Market Niche: What about mixed strategies?

Need to calculate each firm’s expected utility from the firm’s mixed strategy
In equilibrium, Firm 1 is willing to randomize only when it is indifferent between E and S

\[ EU_1(E) = y \times -50 + (1- y) \times 150 = 150 - 200y \]
\[ EU_1(S) = y \times 0 + (1- y) \times 0 = 0 \]

In equilibrium: \( EU_1(E) = EU_1(S) \)
\[ \Rightarrow 150 - 200y = 0 \]
\[ \Rightarrow 200y = 150 \]
\[ \Rightarrow y = \frac{3}{4} \]
\[ \Rightarrow 1 - y = \frac{1}{4} \]
\[ \therefore y = \frac{3}{4} \text{ and } 1 - y = \frac{1}{4} \]

Similarly, Firm 2 is willing to randomize only when it is indifferent between h and t

Firm 1’s Conditions:
\[ EU_1(E) = EU_1(S) \]
Firm 2’s Conditions:
\[ EU_2(e) = x \times -50 + (1- x) \times 100 = 100 - 150x \]
\[ EU_2(s) = x \times 0 + (1- x) \times 0 = 0 \]

In equilibrium: \( EU_2(e) = EU_2(s) \)
\[ \Rightarrow 100 - 150x = 0 \]
\[ \Rightarrow 150x = 100 \]
\[ \Rightarrow x = \frac{2}{3} \text{ and } 1 - x = \frac{1}{3} \]

Asymmetrical Market Niche:
Equilibrium in mixed strategies

\[ \begin{array}{ccc}
   & e & s \\
1/3 & 0 & 100 \\
2/3 & -50 & 150 \\
\end{array} \]
\[ EU_1: 0 = 0 \]
\[ EU_2: 0 = 0 \]

Each firm is playing a best response to the other!

Asymmetrical Market Niche:
Equilibrium in mixed strategies

Although the two pure strategy equilibria (E,s) and (S,e) did not change in Asymmetrical Market Niche, the mixed strategies equilibrium did change.

Chicken

- Two drivers race toward a cliff
- Strategy choice:
  - swerve
  - straight ahead
- More general version of the game:
  - back down
  - do not back down
- Solution as in Market Niche Game

Chicken:
The payoff matrix
Chicken:
strategy for player 1

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Drive Straight Ahead</th>
<th>Swerve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive Straight Ahead</td>
<td>-10, -10</td>
<td>1, -1</td>
</tr>
<tr>
<td>Swerve</td>
<td>-1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Chicken:
strategy for player 2

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Drive Straight Ahead</th>
<th>Swerve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive Straight Ahead</td>
<td>-10, -10</td>
<td>1, -1</td>
</tr>
<tr>
<td>Swerve</td>
<td>-1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Chicken:
two pure strategy Nash equilibria

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Drive Straight Ahead</th>
<th>Swerve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive Straight Ahead</td>
<td>-10, -10</td>
<td>1, -1</td>
</tr>
<tr>
<td>Swerve</td>
<td>-1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Chicken:
The payoff matrix

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Probability</th>
<th>1 - y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive Straight Ahead</td>
<td>y</td>
<td>1 - y</td>
</tr>
<tr>
<td>Swerve</td>
<td>1 - x</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

That is, 0 ≤ x ≤ 1 and 0 ≤ y ≤ 1

In equilibrium, player 1 is willing to randomize only when she is indifferent between “swerve” and “straight”

\[
EU_1(\text{straight}) = y \times (-10) + (1-y) \times 1 = 1 - 11y \\
EU_1(\text{swerve}) = y \times (-1) + (1- y) \times 0 = -y
\]

In equilibrium: \(EU_1(\text{swerve}) = EU_1(\text{straight})\)

\[
\therefore \quad 1 - 11y = - y \\
\Rightarrow \quad 1 = 10y \\
\Rightarrow \quad y = 1/10 \\
\Rightarrow \quad 1 - y = 1 - 1/10 = 9/10
\]

\[
\therefore \quad y = 1/10 \text{ and } 1 - y = 9/10
\]
Similarly, player 2 is willing to randomize only when he is indifferent between “swerve” and “straight”

Player 1’s Conditions:
\[ EU_1(\text{swerve}) = EU_1(\text{straight}) \]

Player 2’s Conditions:
\[ EU_2(\text{straight}) = x \times (-10) + (1-x) \times 1 = 1 - 11x \]
\[ EU_2(\text{swerve}) = x \times (-1) + (1- x) \times 0 = -x \]

In equilibrium: \[ EU_2(\text{swerve}) = EU_2(\text{straight}) \]
\[ 1 - 11x = - x \]
\[ \Rightarrow \quad x = 1/10 \]
\[ \therefore \quad x = 1/10 \text{ and } 1 - x = 9/10 \]

**Chicken:**

The payoff matrix

<table>
<thead>
<tr>
<th>player 2 probability</th>
<th>1/10</th>
<th>9/10</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1 probability</td>
<td>straight</td>
<td>swerve</td>
</tr>
<tr>
<td>1/10</td>
<td>-10, -10</td>
<td>1, -1</td>
</tr>
<tr>
<td>9/10</td>
<td>-1, 1</td>
<td>0, 0</td>
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**Everyday Low Prices**

- Sales are mixed strategies
- Sears’ marketing campaign to do away with sales, called Everyday Low Prices
- Two types of buyers:
  - informed
  - uninformed
- A mixed strategy equilibrium tells how often to run sales

**Evenday Low Pricing:**

The payoff matrix

<table>
<thead>
<tr>
<th>Retailer 2 Normal price</th>
<th>Retailer 1 Sale price</th>
</tr>
</thead>
<tbody>
<tr>
<td>NP</td>
<td>7500, 7500</td>
</tr>
<tr>
<td>SP</td>
<td>8500, 7500</td>
</tr>
</tbody>
</table>

**Everyday Low pricing:**

What about mixed strategies?

\[ x, y \text{ between } 0 \text{ and } 1 \]
That is, \[ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \]
Each retailer calculates its expected utility from other’s mixed strategy.

In equilibrium, Retailer 1 is willing to randomize only when it is indifferent between NP and SP.

In equilibrium, Retailer 2 is willing to randomize only when it is indifferent between c and f.

Everyday Low Pricing: Equilibrium in mixed strategies.

Appendix: Bluffing in 1-card Stud Poker.

Mixed strategies are not intuitive: You randomize to make me indifferent.

Row randomizes to make Column indifferent.
Column randomizes to make Row indifferent.

Then each is playing a best response to the other.
One-card Stud Poker
Payoff matrix, player 1

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: Bet AKQ</td>
<td>0, 0</td>
</tr>
<tr>
<td>II: Bet AK</td>
<td>0, 0</td>
</tr>
<tr>
<td>III: Bet AQ</td>
<td>(2b-4a)/9, (4a-2b)/9</td>
</tr>
<tr>
<td>IV: Bet A</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Player 2
I: Bet AKQ | 0, 0 | (2b-a)/9, (a-2b)/9 |
II: Bet AK | (a+b)/9, (b-a)/9 | (2b-4a)/9, (4a-2b)/9 |
III: Bet AQ | (b-2a)/9, (2a-b)/9 | 0, 0 |
IV: Bet A | 0, 0 | 0, 0 |

One-card Stud Poker.
Payoff matrix, player 1, a=$1, b=$1

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: Bet AKQ</td>
<td>0, 0</td>
</tr>
<tr>
<td>II: Bet AK</td>
<td>1/9, -1/9</td>
</tr>
<tr>
<td>IV: Bet A</td>
<td>-2/9, 2/9</td>
</tr>
</tbody>
</table>

One-card Stud Poker.
Payoff matrix, player 1, a=$1 b=$2

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: Bet AKQ</td>
<td>0, 0</td>
</tr>
<tr>
<td>II: Bet AK</td>
<td>3/9, -3/9</td>
</tr>
<tr>
<td>III: Bet AQ</td>
<td>-3/9, 3/9</td>
</tr>
<tr>
<td>IV: Bet A</td>
<td>0, 0</td>
</tr>
</tbody>
</table>