Cournot’s model of oligopoly

• Single good produced by \( n \) firms

• Cost to firm \( i \) of producing \( q_i \) units: \( C_i(q_i) \), where \( C_i \) is nonnegative and increasing

• If firms’ total output is \( Q \) then market price is \( P(Q) \), where \( P \) is nonincreasing

Profit of firm \( i \), as a function of all the firms’ outputs:

\[
\pi_i(q_1, \ldots, q_n) = q_i P \left( \sum_{j=1}^{n} q_j \right) - C_i(q_i).
\]

Strategic game:

• players: firms

• each firm’s set of actions: set of all possible outputs

• each firm’s preferences are represented by its profit
Example

- two firms
- inverse demand:

\[ P(Q) = \max\{0, \alpha - Q\} = \begin{cases} 
\alpha - Q & \text{if } Q \leq \alpha \\
0 & \text{if } Q > \alpha 
\end{cases} \]

- constant unit cost: \( C_i(q_i) = cq_i \), where \( c < \alpha \).
Nash equilibrium

Payoff functions

Firm 1’s profit is

\[ \pi_1(q_1, q_2) = q_1(P(q_1 + q_2) - c) \]

\[ = \begin{cases} 
q_1(\alpha - c - q_2 - q_1) & \text{if } q_1 \leq \alpha - q_2 \\
-cq_1 & \text{if } q_1 > \alpha - q_2 
\end{cases} \]

Best response functions

Firm 1’s profit as a function of \( q_1 \):

Up to \( \alpha - q_2 \) this function is a quadratic that is zero when \( q_1 = 0 \) and when \( q_1 = \alpha - c - q_2 \).
So when \( q_2 \) is small, optimal output of firm 1 is \((\alpha - c - q_2)/2\).

As \( q_2 \) increases this output decreases until it is zero.

It is zero when \( q_2 = \alpha - c \).

Best response function is:

\[
b_1(q_2) = \begin{cases} 
  (\alpha - c - q_2)/2 & \text{if } q_2 \leq \alpha - c \\
  0 & \text{if } q_2 > \alpha - c.
\end{cases}
\]

Same for firm 2: \( b_2(q) = b_1(q) \) for all \( q \).
**Nash equilibrium**

Pair \((q_1^*, q_2^*)\) of outputs such that each firm’s action is a best response to the other firm’s action or

\[ q_1^* = b_1(q_2^*) \quad \text{and} \quad q_2^* = b_2(q_1^*). \]

Solution:

\[ q_1^* = q_2^* = (\alpha - c)/3. \]
Conclusion

Game has unique Nash equilibrium:

\[(q_1^*, q_2^*) = \left(\frac{\alpha - c}{3}, \frac{\alpha - c}{3}\right)\]

At equilibrium, each firm’s profit is \((\alpha - c)^2/9\).

Note: Total output \(2(\alpha - c)/3\) lies between monopoly output \((\alpha - c)/2\) and competitive output \(\alpha - c\).
Bertrand’s model of oligopoly

Strategic variable price rather than output.

- Single good produced by \( n \) firms
- Cost to firm \( i \) of producing \( q_i \) units: \( C_i(q_i) \), where \( C_i \) is nonnegative and increasing
- If price is \( p \), demand is \( D(p) \)
- Consumers buy from firm with lowest price
- Firms produce what is demanded

Firm 1’s profit:

\[
\pi_1(p_1, p_2) = \begin{cases} 
    p_1 D(p_1) - C_1(D(p_1)) & \text{if } p_1 < p_2 \\
    \frac{1}{2}p_1 D(p_1) - C_1(\frac{1}{2}D(p_1)) & \text{if } p_1 = p_2 \\
    0 & \text{if } p_1 > p_2 
\end{cases}
\]

Strategic game:

- players: firms
- each firm’s set of actions: set of all possible prices
- each firm’s preferences are represented by its profit
Example

- 2 firms
- $C_i(q_i) = cq_i$ for $i = 1, 2$
- $D(p) = \alpha - p$.

Nash equilibrium

Best response functions

To find best response function of firm 1, look at its payoff as a function of its output, given output of firm 2.
The diagrams depict the profit function \((p_1 - c)D(p_1)\) for different price points. In the first diagram, the profit is shown as a function of \(p_1\) with price points \(c\) and \(p_2\) on the x-axis. In the second diagram, the profit function is shown again with the price points \(c\), \(p_2\), and \(a\) on the x-axis. The third diagram introduces a new price point \(p^m\) and shows the profit function with a dashed line indicating the price point at which the profit is maximized. The diagrams illustrate the relationship between price and profit, highlighting the optimal pricing strategy for maximizing profit.
Any price greater than \( p_2 \) is a best response to \( p_2 \):

\[
B_1(p_2) = \{ p_1 : p_1 > p_2 \}.
\]

Note: a price between \( p_2 \) and \( c \) is a best response!

\( p_2 = c \)

Any price greater than or equal to \( p_2 \) is a best response to \( p_2 \):

\[
B_1(p_2) = \{ p_1 : p_1 \geq p_2 \}.
\]
$c < p_2 \leq p^m$

There is no *best* response! (a bit less than $p_2$ is *almost* a best response).

$p^m < p_2$

$p^m$ is the unique best response to $p_2$:

$B_1(p_2) = \{p^m\}$. 
Summary

\[ B_i(p_j) = \begin{cases} 
\{p_i : p_i > p_j\} & \text{if } p_j < c \\
\{p_i : p_i \geq p_j\} & \text{if } p_j = c \\
\emptyset & \text{if } c < p_j \leq p^m \\
\{p^m\} & \text{if } p^m < p_j.
\end{cases} \]
Nash equilibrium

\((p_1^*, p_2^*)\) such that \(p_1^* \in B_1(p_2^*)\) and \(p_2^* \in B_2(p_1^*)\)

I.e. intersection of the graphs of the best response functions

So: unique Nash equilibrium, \((p_1^*, p_2^*) = (c, c)\).
“Direct” argument for Nash equilibrium

If each firm charges the price of \( c \) then the other firm can do no better than charge the price of \( c \) also (if it raises its price is sells no output, while if it lowers its price is makes a loss), so \((c, c)\) is a Nash equilibrium.

No other pair \((p_1, p_2)\) is a Nash equilibrium since

- if \( p_i < c \) then the firm whose price is lowest (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price to \( c \)

- if \( p_i = c \) and \( p_j > c \) then firm \( i \) is better off increasing its price slightly

- if \( p_i \geq p_j > c \) then firm \( i \) can increase its profit by lowering \( p_i \) to some price between \( c \) and \( p_j \) (e.g. to slightly below \( p_j \) if \( D(p_j) > 0 \) and to \( p^m \) if \( D(p_j) = 0 \)).

Note: to show a pair of actions is not a Nash equilibrium we need only find a better response for one of the players—not necessarily the best response.
Equilibria in Cournot’s and Bertrand’s models generate different economic outcomes:

- equilibrium price in Bertrand’s model is $c$
- price associated with an equilibrium of Cournot’s model is $\frac{1}{3}(\alpha + 2c)$, which exceeds $c$ since $\alpha > c$.

Does one model capture firms’ strategic reasoning better than the other?

Bertrand’s model: firm changes its behavior if it can increase its profit by changing its price, on the assumption that the other firm will not change its price but the other firm’s output will adjust to clear the market.

Cournot’s model: firm changes its behavior if it can increase its profit by changing its output, on the assumption that the output of the other firm will not change but the price will adjust to clear the market.

If prices can easily be changed, Cournot’s model may thus better capture firms’ strategic reasoning.
Hotelling’s model of electoral competition

- Several candidates vie for political office
- Each candidate chooses a policy position
- Each citizen, who has preferences over policy positions, votes for one of the candidates
- Candidate who obtains the most votes wins.

Strategic game:

- Players: candidates
- Set of actions of each candidate: set of possible positions
- Each candidate gets the votes of all citizens who prefer her position to the other candidates’ positions; each candidate prefers to win than to tie than to lose.

Note: Citizens are not players in this game.
Example

- Two candidates
- Set of possible positions is a (one-dimensional) interval.
- Each voter has a single favorite position, on each side of which her distaste for other positions increases equally.

![Diagram of favorite positions](attachment://favorite_positions.png)

- Unique median favorite position $m$ among the voters: the favorite positions of half of the voters are at most $m$, and the favorite positions of the other half of the voters are at least $m$.

Note: $m$ may not be in the center of the policy space.
**Positions and votes**

Candidate who gets most votes wins.

\[
\frac{(x_1 + x_2)}{2}
\]

\[x_1 \quad m \quad x_2\]

\[\text{vote for 1} \quad \text{vote for 2}\]

In this case, candidate 1 wins.

**Best responses**

Best response of candidate \(i\) to \(x_j\):

- \(x_j < m\):

\[x_j \quad m\]

any position for \(i\) in here wins

candidate \(i\) wins if \(x_i > x_j\) and \(\frac{1}{2}(x_i + x_j) < m\), or in other words \(x_j < x_i < 2m - x_j\). Otherwise she either ties or loses. Thus every position between \(x_j\) and \(2m - x_j\) is a best response of candidate \(i\) to \(x_j\).

- \(x_j > m\): symmetrically, every position between \(2m - x_j\) and \(x_j\) is a best response of candidate \(i\) to \(x_j\).

- \(x_j = m\): if candidate \(i\) choose \(x_i = m\) she ties for first place; if she chooses any other position she loses. Thus \(m\) is the unique best response of candidate \(i\) to \(x_j\).
Summary

Candidate $i$’s best response function:

$$B_i(x_j) = \begin{cases} 
\{x_i : x_j < x_i < 2m - x_j\} & \text{if } x_j < m \\
\{m\} & \text{if } x_j = m \\
\{x_i : 2m - x_j < x_i < x_j\} & \text{if } x_j > m.
\end{cases}$$
Unique Nash equilibrium, in which both candidates choose the position $m$.

Outcome of election is tie.

Competition between the candidates to secure a majority of the votes drives them to select the same position.
Direct argument for Nash equilibrium

$(m, m)$ is an equilibrium: if either candidate chooses a different position she loses.

No other pair of positions is a Nash equilibrium:

- if one candidate loses then she can do better by moving to $m$ (where she either wins outright or ties for first place)
- if the candidates tie (because their positions are either the same or symmetric about $m$), then either candidate can do better by moving to $m$, where she wins outright.
The War of Attrition

• Two parties involved in a costly dispute

• E.g. two animals fighting over prey

• Each animal chooses time at which it intends to give up

• Once an animal has given up, the other obtains all the prey

• If both animals give up at the same time then they split the prey equally.

• Fighting is costly: each animal prefers as short a fight as possible.

Also a model of bargaining between humans.

Let time be a continuous variable that starts at 0 and runs indefinitely.

Assume value to party \( i \) of object in dispute is \( v_i > 0 \); value of half of object is \( v_i/2 \).

Each unit of time that passes before one of parties concedes costs each party one unit of payoff.
Strategic game

- players: the two parties
- each player’s set of actions is \([0, \infty)\) (set of possible concession times)
- player \(i\)’s preferences are represented by payoff function

\[
u_i(t_1, t_2) = \begin{cases} 
-t_i & \text{if } t_i < t_j \\
\frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\
v_i - t_j & \text{if } t_i > t_j,
\end{cases}
\]

where \(j\) is the other player.
Best responses

Suppose player $j$ concedes at time $t_j$:

Intuitively: if $t_j$ is small then optimal for player $i$ to wait until after $t_j$; if $t_j$ is large then player $i$ should concede immediately.

Precisely: if $t_j < v_i$:

so any time after $t_j$ is a best response to $t_j$
If $t_j = v_i$:

so conceding at 0 or at any time after $t_j$ is a best response to $t_j$

If $t_j > v_i$:

so best time for player $i$ to concede is 0.
So player $i$'s best response function:

\[
B_i(t_j) = \begin{cases} 
\{t_i : t_i > t_j\} & \text{if } t_j < v_i \\
\{0\} \cup \{t_i : t_i > t_j\} & \text{if } t_j = v_i \\
\{0\} & \text{if } t_j > v_i.
\end{cases}
\]

Best response function of player 1:
Nash equilibrium

Nash equilibria: \((t_1, t_2)\) such that either

\[ t_1 = 0 \text{ and } t_2 \geq v_1 \]

or

\[ t_2 = 0 \text{ and } t_1 \geq v_2. \]

That is: either player 1 concedes immediately and player 2 concedes at the earliest at time \(v_1\), or player 2 concedes immediately and player 1 concedes at the earliest at time \(v_2\).
Note: in no equilibrium is there any fight

Note: there is an equilibrium in which either player concedes first, regardless of the sizes of the valuations.

Note: equilibria are asymmetric, even when $v_1 = v_2$, in which case the game is symmetric.

E.g. could be a stable social norm that the current owner of the object concedes immediately; or that the challenger does so.

Single population case: only symmetric equilibria are relevant, and there are none!
Auctions

Common type of auction:

- people sequentially bid for an object
- each bid must be greater than previous one
- when no one wishes to submit a bid higher than current one, person making current bid obtains object at price she bid.

Assume everyone is certain about her valuation of the object before bidding begins, so that she can learn nothing during the bidding.

Model

- each person decides, before auction begins, maximum amount she is willing to bid
- person who bids most wins
- person who wins pays the second highest bid.

Idea: in a dynamic auction, a person wins if she continues bidding after everyone has stopped—in which case she pays a price slightly higher than the price bid by the last person to drop out.
Strategic game:

• players: bidders

• set of actions of each player: set of possible bids (nonnegative numbers)

• preferences of player $i$: represented by a payoff function that gives player $i$ $v_i - p$ if she wins (where $v_i$ is her valuation and $p$ is the second-highest bid) and 0 otherwise.

This is a **sealed-bid second-price auction**.

How to break ties in bids?

Simple (but arbitrary) rule: number players $1, \ldots, n$ and make the winner the player with the lowest number among those that submit the highest bid.

Assume that $v_1 > v_2 > \cdots > v_n$. 
Nash equilibria of second-price sealed-bid auction

One Nash equilibrium

\((b_1, \ldots, b_n) = (v_1, \ldots, v_n)\)

Outcome: player 1 obtains the object at price \(v_2\); her payoff is \(v_1 - v_2\) and every other player’s payoff is zero.

Reason:

- Player 1:
  - if she changes her bid to some \(x \geq b_2\) the outcome does not change (remember she pays the second highest bid)
  - if she lowers her bid below \(b_2\) she loses and gets a payoff of 0 (instead of \(v_1 - b_2 > 0\)).

- Players 2, \ldots, \(n\):
  - if she lowers her bid she still loses
  - if she raises her bid to \(x \leq b_1\) she still loses
  - if she raises her bid above \(b_1\) she wins, but gets a payoff \(v_i - v_1 < 0\).
Another Nash equilibrium

$(v_1, 0, \ldots, 0)$ is also a Nash equilibrium:

Outcome: player 1 obtains the object at price 0; her payoff is $v_1$ and every other player’s payoff is zero.

**Reason:**

- **Player 1:**
  - any change in her bid has no effect on the outcome

- **Players 2, \ldots, n:**
  - if she raises her bid to $x \leq v_1$ she still loses
  - if she raises her bid above $v_1$ she wins, but gets a negative payoff $v_i - v_1$. 
Another Nash equilibrium

$(v_2, v_1, 0, \ldots, 0)$ is also a Nash equilibrium:

Outcome: player 2 gets object at price $v_2$; all payoffs 0.

Reason:

- Player 1:
  - if she raises her bid to $x < v_1$ she still loses
  - if she raises her bid to $x \geq v_1$ she wins, and gets a payoff of 0

- Player 2
  - if she raises her bid or lowers it to $x > v_2$, outcome remains same
  - if she lowers her bid to $x \leq v_2$ she loses and gets 0

- Players 3, $\ldots$, $n$:
  - if she raises her bid to $x \leq v_1$ she still loses
  - if she raises her bid above $v_1$ she wins, but gets a negative payoff $v_i - v_1$.

Player 2’s may seem “risky”—but isn’t if the other players adhere to their equilibrium actions.

Nash equilibrium requires only that each player’s action be optimal, given the other players' actions.

In a dynamic setting, player 2’s bid isn’t credible (why would she keep bidding above $v_2$?) [Will study this issue later.]
**Distinguishing between equilibria**

For each player $i$ the action $v_i$ *weakly dominates* all her other actions.

That is: player $i$ can do no better than bid $v_i$ *no matter what the other players bid*.

*Argument*:

- If the highest of the other players’ bids is at least $v_i$, then
  - if player $i$ bids $v_i$ her payoff is 0
  - if player $i$ bids $x \neq v_i$ her payoff is either zero or negative.

- If the highest of the other players’ bids is $\bar{b} < v_i$, then
  - if player $i$ bids $v_i$ her payoff is $v_i - \bar{b}$ (she obtains the object at the price $\bar{b}$)
  - if player $i$ submits some other bid then she either obtains the good and gets the same payoff, or does not obtain the good and gets the payoff of zero.

*Summary*

Second-price auction has many Nash equilibria, but the equilibrium $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ is the only one in which every players’ action weakly dominates all her other actions.
First-price auction

Another auction form:

- auctioneer begins by announcing a high price
- price is gradually lowered until someone indicates a willingness to buy the object at that price.

Model

Strategic game:

- players: bidders
- actions of each player: set of possible bids (nonnegative numbers)
- preferences of player $i$: represented by a payoff function that gives player $i$ $v_i - p$ if she wins (where $v_i$ is her valuation and $p$ is her bid) and 0 otherwise.

This is a first-price sealed-bid auction.

One Nash equilibrium

$(v_2, v_2, v_3, \ldots, v_n)$

Outcome: player 1 obtains the object at the price $v_2$.

Why is this a Nash equilibrium?
Property of all equilibria

In all equilibria the object is obtained by the player who values it most highly (player 1)

Argument:

• If player $i \neq 1$ obtains the object then we must have $b_i > b_1$.

• But there is no equilibrium in which $b_i > b_1$:
  • if $b_i > v_2$ then $i$’s payoff is negative, so she can do better by reducing her bid to 0
  • if $b_i \leq v_2$ then player 1 can increase her payoff from 0 to $v_1 - b_i$ by bidding $b_i$.

Another equilibrium

$(v_1, v_1, v_3, \ldots, v_n)$

Outcome: player 1 obtains the object at the price $v_1$.

As before, player 2’s action may seem “risky”: if there is any chance that player 1 submits a bid less than $v_1$ then there is a chance that player 2’s payoff is negative.


**Domination**

As in a second-price auction, any player $i$’s action of bidding $b_i > v_i$ is weakly dominated by the action of bidding $v_i$:

- if the other players’ bids are such that player $i$ loses when she bids $b_i$, then it makes no difference to her whether she bids $b_i$ or $v_i$.

- if the other players’ bids are such that player $i$ wins when she bids $b_i$, then she gets a negative payoff bidding $b_i$ and a payoff of 0 when she bids $v_i$.

*However*, in a first-price auction, unlike a second-price auction, a bid by a player less than her valuation is not weakly dominated.

*Reason*: if player $i$ bids $v'_i < v_i$ and the highest bid of the other players is $< v'_i$, then player $i$ is better off than she is if she bids $v_i$.

**Revenue equivalence**

The price at which the object is sold, and hence the auctioneer’s revenue, is the same in the equilibrium $(v_1, \ldots, v_n)$ of the second-price auction as it is in the equilibrium $(v_2, v_3, \ldots, v_n)$ of the first-price auction.