Strategic games

A strategic game is a model of interaction in which each player chooses an action not having been informed of the other players' actions.

We can think of the players' actions as being taken *simultaneously*.

Even though players are not informed of others' actions, they may "know" what they will be, from their past experience.

Need to specify:

- who is involved
- what they can do
- their preferences over the possible outcomes

Definition *A* **strategic game** *(with ordinal preferences) consists of*

- *a set of* **players**
- *for each player, a set of* **actions**
- *for each player, a* **preference relation** *over the set of action profiles.*

Action profile: list of actions, one for each player.

Frequently work with **payoff functions** that represent preferences, rather than directly with preferences.

Example: Prisoner's Dilemma

- Two suspects in a crime, held in separate cells.
- Enough evidence to convict each on minor, but not major offense unless one informs on the other.
- Each can be *quiet* or can *fink* (inform).
- Both quiet: each convicted of minor offense—1 year in prison.
- One and only one finks: one who finks is used as a witness against the other, and goes free; other gets 4 years.
- Both fink: each gets 3 years in prison.

Strategic game: two players; each player's actions $\{Q, F\}$; preference orderings

$$(F,Q) \succ_1 (Q,Q) \succ_1 (F,F) \succ_1 (Q,F)$$
$$(Q,F) \succ_2 (Q,Q) \succ_2 (F,F) \succ_2 (F,Q).$$

Choose payoff functions that represent preferences.

		Suspect 2	
		Quiet (C)	*Fink* (D)
Suspect 1	*Quiet* (C)	$2, 2$	$0, 3$
	Fink (D)	$3, 0$	$1, 1$

This game models a situation in which there are gains from cooperation, but each player prefers to be a "free rider".

Working on a joint project

- Each of two people can either work hard or goof off.
- Each person prefers to goof off if the other works hard—project would be better if worked hard, but not worth the extra effort.

		Person 2	
		Work hard	*Goof off*
Person 1	*Work hard*	$2, 2$	$0, 3$
	Goof off	$3, 0$	$1, 1$

This game is the Prisoner's Dilemma!

Duopoly

- Two firms producing same good.
- Each firm can charge high price or low price.
- Both firms charge high price: each gets profit of \$1,000
- One firm charges high price, other low: firm charging high price gets no customers and loses \$200, while one charging low price makes profit of \$1,200
- Both firms charge low price: each earns profit of \$600

	high	*low*
high	1,000, 1,000	−200, 1,200
low	1,200, −200	600, 600

This is the Prisoner's Dilemma.
But not every duopoly is necessarily a Prisoner's Dilemma.

Nuclear arms race

Early development of game theory was in 1950s, when USA and USSR were involved in arms race.

Suppose

- two countries
- what matters is relative strength, and bombs are costly, so that for each country:

  have bombs, other doesn't $\succ$ neither country has bombs $\succ$

  both countries have bombs $\succ$ don't have bombs, other does

Then game is Prisoner's Dilemma, with C means don't build bombs and D means build bombs.

Tragedy of the commons

Each of two farmers can allow their sheep to graze a little, or a lot. If they both graze a lot, the commons is depleted.

This may also be modeled as a Prisoner's Dilemma.

Example: BoS

In the *Prisoner's dilemma* the players agree that (C, C) is a desirable outcome, though each has an incentive to deviate from this outcome.

In *BoS* the players disagree about the outcome that is desirable.

- Two people wish to go out together to a concert of music by either Bach or Stravinsky.
- Their main concern is to go out together, but one person prefers Bach and the other person prefers Stravinsky.
- If they go to different concerts then each of them is equally unhappy listening to the music of either composer.

Choosing payoff representations, game is:

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

(Game is also referred to as the "Battle of the Sexes".)

BoS models a wide variety of situations:

- Two officials of a political party deciding the stand to take on an issue.

  Suppose they disagree about the best stand but agree that they will be better off if they both take the same stand than if they take different stands, and are indifferent between the two cases in which they take different stands.

- Two merging firms choosing between the two computer technologies they each used in the past.

  Because of their different experiences the firms may have different preferences as to which technology is used when they merge, but agree that it is better for them both to adopt the same technology than different technologies (in which case they cannot operate effectively, whatever the assignment of technologies).

Example: Matching Pennies

In the *Prisoner's dilemma* and *BoS* there are aspects of both conflict and cooperation. Matching Pennies is purely conflictual.

- Each of two people chooses either the Head or Tail of a coin.
- If the choices differ, person 1 pays person 2 a dollar.
- If they are the same, person 2 pays person 1 a dollar.
- Each person cares only about the amount of money that she receives.

	Head	*Tail*
Head	$1, -1$	$-1, \ \ 1$
Tail	$-1, \ \ 1$	$1, -1$

In this game the players' interests are diametrically opposed (such a game is called "strictly competitive"): player 1 prefers to take the same action as the other player, while player 2 prefers to take the opposite action.

Matching pennies may, for example, model the choice of appearance for a product by an established producer of a good and a new firm in a market of fixed size:

- Each firm can choose between two different appearances for the product.
- The established producer prefers the newcomer's product to look different from its own (so that its customers will not be tempted to buy the newcomer's product)
- The newcomer prefers that the products look alike.

Nash equilibrium

What actions will players in strategic game choose?

Suppose each player is experienced: has played game, or similar games, many times in the past, and has a good idea of actions other players will choose

What action profiles are *steady states* (situations from which there is no tendency for any player to deviate)?

In a steady state in which each player has a good idea of the other players' actions, each player's action must be optimal, given the other players' actions.

That is: the action profile a^* must be such that no player i has any action b_i such that she prefers (b_i, a^*_{-i}) to a^*.

Equivalently:

Definition *The action profile a^* in a strategic game with ordinal preferences is a* **Nash equilibrium** *if, for each player i and every action b_i of player i, a^* is at least as good for player i as the action profile (b_i, a^*_{-i}):*

$$a^* \succsim_i (b_i, a^*_{-i}) \text{ for every action } b_i \text{ of player } i.$$

The action profile a^* is a **Nash equilibrium** if

$$a^* \succsim_i (b_i, a^*_{-i}) \text{ for every action } b_i \text{ of player } i.$$

Alternative definition

For each player i, let the function u_i represent i's preferences.

Then a^* is a Nash equilibrium if and only if

$$u_i(a^*) \geq u_i(b_i, a^*_{-i}) \text{ for every action } b_i \text{ of player } i.$$

Note

The definition does not imply either that a strategic game necessarily has a Nash equilibrium, or that it has at most one. Some games possess no equilibrium and others have many equilibria.

Prisoner's Dilemma

	Quiet	*Fink*
Quiet	$2, 2$	$0, 3$
Fink	$3, 0$	$1, 1$

In the *Prisoner's Dilemma*, $(Fink, Fink)$ is the unique Nash equilibrium (and is strict).

No other action profile satisfies the condition for a Nash equilibrium:

- $(Quiet, Quiet)$ does not satisfy the condition since $($*Quiet*,*Quiet*$) \prec_1 ($*Fink*,*Quiet*$)$
- $(Fink, Quiet)$ and $(Quiet, Fink)$ do not satisfy the condition since $($*Fink*,*Quiet*$) \prec_2 ($*Fink*,*Fink*$)$ (player 2 prefers to fink if player 1 finks) and $($*Quiet*,*Fink*$) \prec_1 ($*Fink*,*Fink*$)$ (player 1 prefers to fink if player 2 finks).

Thus the outcomes predicted by the notion of Nash equilibrium for the situations modeled as a *Prisoner's Dilemma* are:

- working on a joint project: both people goof off
- duopoly: both firms charge a low price
- arms race: both countries build bombs
- grazing on the commons: both farmers graze their sheep a lot.

	Quiet	*Fink*
Quiet	$2, 2$	$0, 3$
Fink	$3, 0$	$1, 1$

Note

In this game, the Nash equilibrium action of each player (*Fink*) is the best action for each player not only if the other player chooses her equilibrium action (*Fink*), but also if she chooses *Quiet*.

That is, it is optimal for a player to choose *Fink regardless* of the action she anticipates her opponent will choose.

In most of the games we study, a player's best action depends on the other players' actions.

BoS

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Two Nash equilibria: $(Bach, Bach)$ and $(Stravinsky, Stravinsky)$.

Both of these outcomes are compatible with a steady state; both outcomes are stable social norms.

If, in every encounter, both players choose *Bach*, then no player has an incentive to deviate; if, in every encounter, both players choose *Stravinsky*, then no player has an incentive to deviate.

For example: suppose a population of men interacts with a population of women.

Then two social norms are stable: both players choose the action associated with the outcome preferred by women, and both players choose the action associated with the outcome preferred by men.

Matching Pennies

	Head	*Tail*
Head	$1,-1$	$-1,\ \ 1$
Tail	$-1,\ \ 1$	$1,-1$

No Nash equilibrium! No action profile has the property that each player's action is optimal given the other player's action.

Strict versus weak Nash equilibrium

In a Nash equilibrium, each player's equilibrium action has to be *at least as good as* every other action, not necessarily better.

A player may be indifferent between her equilibrium action and some other action (given the other players' actions).

Lifting an object

Two people want to move an object; need to cooperate to lift it.

	Lift	*Don't lift*
Lift	$1,1$	$0,0$
Don't lift	$0,0$	$0,0$

The game has two Nash equilibria: $(\textit{Lift}, \textit{Lift})$ and $(\textit{Don't lift}, \textit{Don't lift})$.

If each player's Nash equilibrium action is *better* than all her other actions then we say that the equilibrium is strict: an action profile a^* is a **strict Nash equilibrium** if for every player i we have

$$a^* \succ_i (b_i, a^*_{-i}) \text{ for every action } b_i \neq a^*_i \text{ of player } i.$$

	Lift	*Don't lift*
Lift	$1, 1$	$0, 0$
Don't lift	$0, 0$	$0, 0$

$(Lift, Lift)$: strict Nash equilibrium

$(Don't\ lift, Don't\ lift)$: Nash equilibrium, but not strict Nash equilibrium

Best response functions

For any list of actions a_{-i} of all the players other than i, let $B_i(a_{-i})$ be the set of player i's *best* actions, given that every other player j chooses a_j:

$$B_i(a_{-i}) = \{a_i \in A_i \colon (a_i, a_{-i}) \succsim_i (b_i, a_{-i}) \text{ for all } b_i \in A_i\}.$$

B_i is the **best response function** of player i.

B_i is a *set-valued* function: its values are sets, not points.

Every member of the set $B_i(a_{-i})$ is a **best response** of player i to a_{-i}: if each of the other players adheres to a_{-i} then player i can do no better than choose a member of $B_i(a_{-i})$.

Prisoner's Dilemma

		Suspect 2: *Quiet* (C)	Suspect 2: *Fink* (D)
Suspect 1	*Quiet* (C)	$2, 2$	$0, 3$
Suspect 1	*Fink* (D)	$3, 0$	$1, 1$

$B_i(\mathit{Quiet}) = \{\mathit{Fink}\}$ for $i = 1$, 2

$B_i(\mathit{Fink}) = \{\mathit{Fink}\}$ for $i = 1$, 2

BoS

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

$B_i(Bach) = \{Bach\}$ for $i = 1$, 2

$B_i(Stravinsky) = \{Stravinsky\}$ for $i = 1$, 2

Lifting game

	Lift	*Don't lift*
Lift	$1, 1$	$0, 0$
Don't lift	$0, 0$	$0, 0$

Here there is more than one best response to one of the actions:

$B_i(\text{Don't lift}) = \{Lift, \text{Don't lift}\}$ for $i = 1$, 2.

$B_i(Lift) = \{Lift\}$ for $i = 1$, 2.

Alternative definition of Nash equilibrium

Proposition *The action profile a^* is a Nash equilibrium of a strategic game with ordinal preferences if and only if every player's action is a best response to the other players' actions:*

$$a_i^* \in B_i(a_{-i}^*) \text{ for every player } i. \tag{1}$$

The case of single-valued best response functions

Suppose each player i has a unique best response to each list a_{-i} of the other players' actions.

For each player i and each a_{-i}, denote the single member of $B_i(a_{-i})$ by $b_i(a_{-i})$ (that is, $B_i(a_{-i}) = \{b_i(a_{-i})\}$).

Then (1) is equivalent to

$$a_i^* = b_i(a_{-i}^*) \text{ for every player } i, \tag{2}$$

a collection of n equations in the n unknowns a_i^*, where n is the number of players in the game.

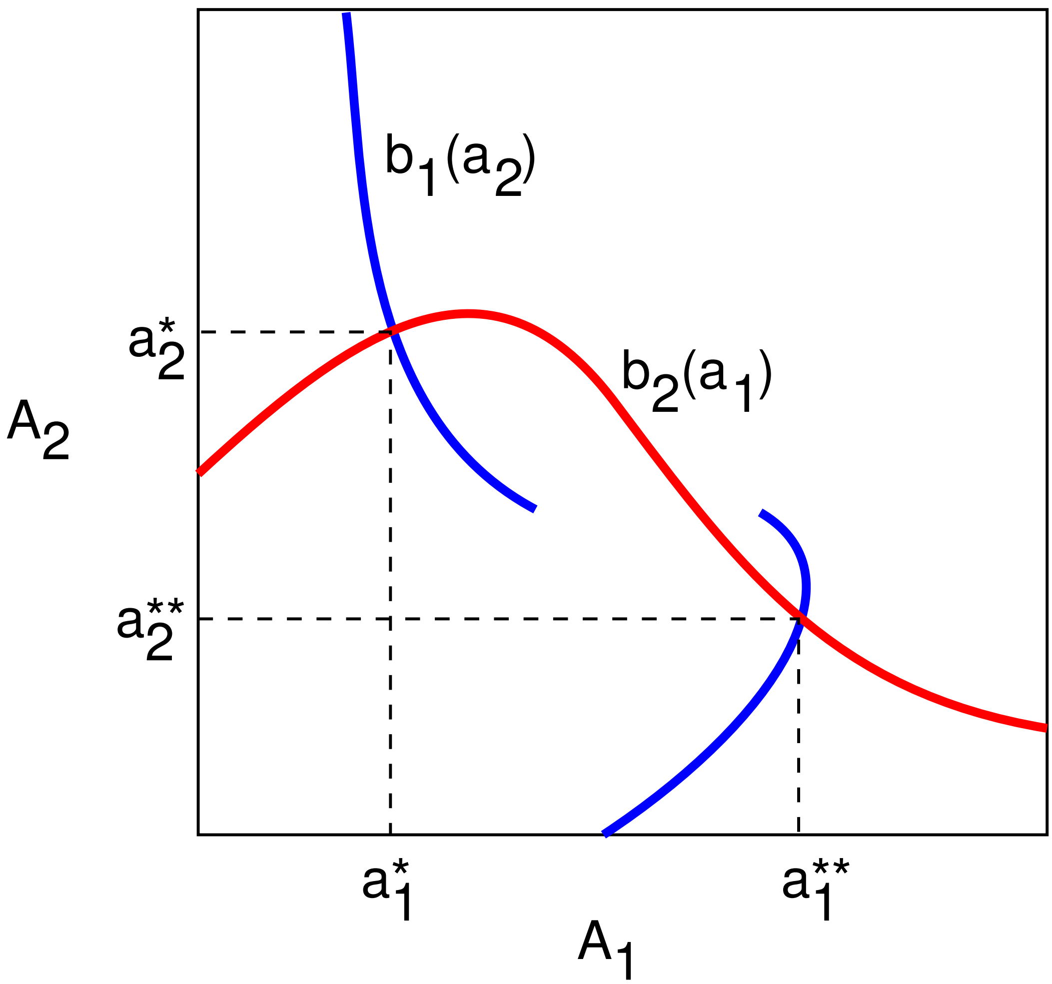
The case of single-valued best response functions and two players

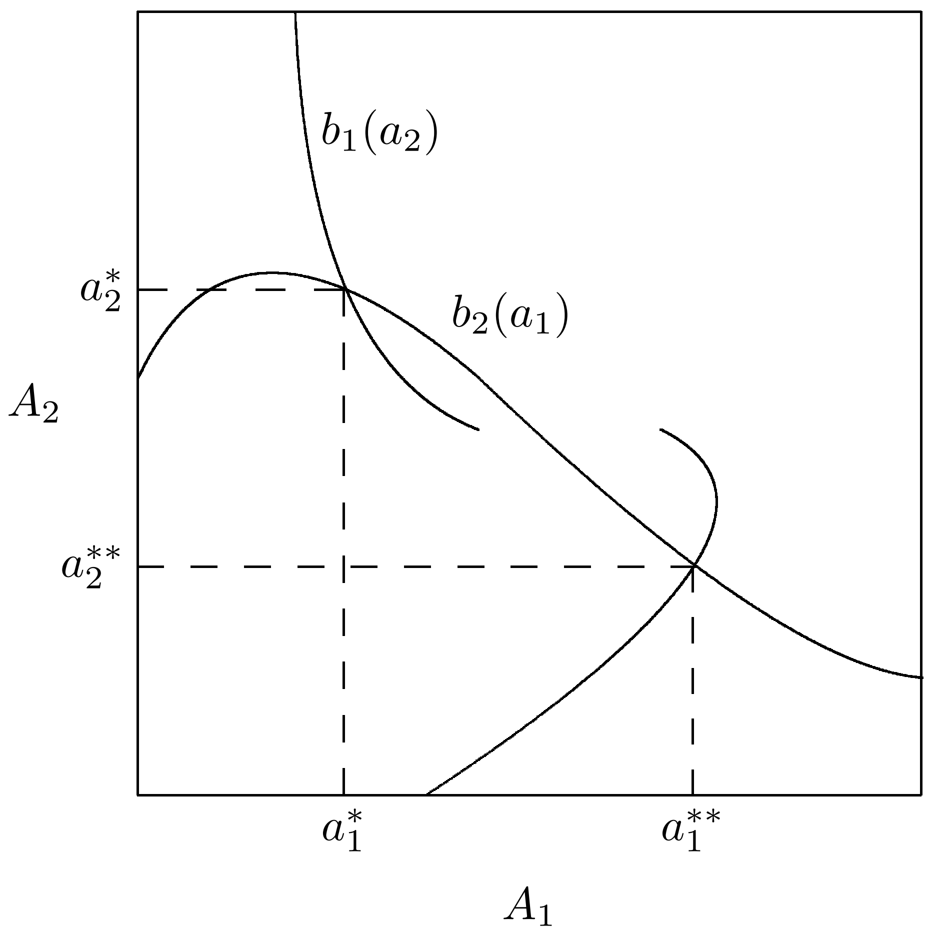
If the game has two players then the pair (a_1^*, a_2^*) of actions is a Nash equilibrium if and only if

$$\begin{aligned} a_1^* &= b_1(a_2^*) \\ a_2^* &= b_2(a_1^*). \end{aligned}$$

Example

Consider a case in which A_1 and A_2 are intervals of real numbers.





Two Nash equilibria: (a_1^*, a_2^*) and (a_1^{**}, a_2^{**}).

Finding Nash equilibrium using best response functions

Method:

- calculate the best response functions
- find an action profile a^* that satisfies

$$a_i^* \in B_i(a_{-i}^*) \text{ for every player } i$$

or, if every player's best response is always unique, find a solution of the n equations

$$a_i^* = b_i(a_{-i}^*) \text{ for every player } i.$$

Example: synergistic relationship

- Two individuals
- Each decides how much effort to devote to relationship
- Amount of effort is nonnegative real number
- If both individuals devote more effort to the relationship, then they are both better off; for any given effort of individual j, the return to individual i's effort first increases, then decreases.
- Specifically, payoff of i: $a_i(c + a_j - a_i)$, where $c > 0$ is a constant.

Note: Infinitely many actions, so we cannot present the game in a table.

To find Nash equilibria, construct players' best response functions.

Player i's payoff function: $a_i(c + a_j - a_i)$.

Given a_j, this function is quadratic in a_i that is zero at $a_i = 0$ and at $a_i = c + a_j$, and reaches a maximum in between.

Symmetry of quadratic functions implies that the best response of each individual i to a_j is

$$b_i(a_j) = \tfrac{1}{2}(c + a_j).$$

Nash equilibria: pairs (a_1^*, a_2^*) that solve the two equations

$$\begin{aligned} a_1 &= \tfrac{1}{2}(c + a_2) \\ a_2 &= \tfrac{1}{2}(c + a_1). \end{aligned}$$

Unique solution, (c, c).

Hence game has a unique Nash equilibrium, $(a_1^*, a_2^*) = (c, c)$.

Dominated actions

A player's action is "strictly dominated" if it is inferior, no matter what the other players do, to some other action.

Definition *In a strategic game with ordinal preferences, player i's action b_i* **strictly dominates** *her action b'_i if*

$(b_i, a_{-i}) \succ_i (b'_i, a_{-i})$ *for every list a_{-i} of the other players' actions,*

where $\succ_i$ is player i's (strict) preference relation.

In terms of payoffs: player i's action b_i strictly dominates her action b'_i if

$u_i(b_i, a_{-i}) > u_i(b'_i, a_{-i})$ for every list a_{-i} of the other players' actions.

Example: Prisoner's Dilemma

		Suspect 2	
		Quiet (C)	*Fink* (D)
Suspect 1	*Quiet* (C)	$2,2$	$0,3$
	Fink (D)	$3,0$	$1,1$

Fink strictly dominates the action *Quiet*: regardless of her opponent's action, a player prefers the outcome when she chooses *Fink* to the outcome when she chooses *Quiet*.

Example: BoS

	Bach	*Stravinsky*
Bach	$2,1$	$0,0$
Stravinsky	$0,0$	$1,2$

Neither action strictly dominates the other.

If an action strictly dominates the action b_i', we say that b_i' is **strictly dominated**.

A strictly dominated action is not a best response to any actions of the other players (whatever the other players do, some other action is better)

Hence a strictly dominated action is not used in any Nash equilibrium.

Thus when looking for Nash equilibria, we can ignore all strictly dominated actions.

For example, we can deduce that (D, D) is the unique Nash equilibrium of the *Prisoner's Dilemma* by noting that C is strictly dominated for each player.

Weak domination

A player's action is "weakly dominated" if the player has another action that is never worse, and sometimes better, depending on what the other players do.

Definition *In a strategic game with ordinal preferences, player i's action b_i* **weakly dominates** *her action b_i' if*

$(b_i, a_{-i}) \succsim_i (b_i', a_{-i})$ *for every list a_{-i} of the other players' actions*

and

$(b_i, a_{-i}) \succ_i (b_i', a_{-i})$ *for some list a_{-i} of the other players' actions,*

where $\succ_i$ is player i's strict preference relation and $\succsim_i$ is her weak preference relation.

In terms of payoff functions: player i's action b_i weakly dominates her action b_i' if

$u_i(b_i, a_{-i}) \geq u_i(b_i', a_{-i})$ for every list a_{-i} of the other players' actions

with strict inequality for some list a_{-i} of the other players' actions.

Can a weakly dominated action be used by a player in a Nash equilibrium?

Lifting game

	Lift	*Don't lift*
Lift	$1,1$	$0,0$
Don't lift	$0,0$	$0,0$

Don't lift is weakly dominated by the action *Lift* for each player

But $(\textit{Don't lift}, \textit{Don't lift})$ is a Nash equilibrium: neither player can do better by choosing *Lift* if the other player chooses *Don't lift*.

Equilibrium in a single population

Nash equilibrium: steady state of an interaction between the members of several populations, one for each player in the game.

What if members of a *single* homogeneous population interact?

Example: pedestrians approaching each other on a sidewalk

Consider case in which each interaction involves only two participants.

A two-player game is *symmetric* if each player has the same set of actions and each player's evaluation of an outcome depends only on her action and that of her opponent, not on whether she is player 1 or player 2.

Definition *A two-player strategic game with ordinal preferences is* **symmetric** *if the player's sets of actions are the same and*

$$(a_1, a_2) \succsim_1 (b_1, b_2) \text{ if and only if } (a_2, a_1) \succsim_2 (b_2, b_1),$$

where $\succsim_i$ is player i's preference relation (so that there are payoff functions u_1 and u_2 such that $u_1(a_1, a_2) = u_2(a_2, a_1)$ for all (a_1, a_2)).

Want to model a steady state of a situation in which the members of a single population are repeatedly matched in pairs to play a symmetric game.

Only one role in the game, so steady state is characterized by a *single* action used by every participant.

An action a^* corresponds to a steady state if no player can do better by using any other action, given all other players use a^*.

That is, (a^*, a^*) is a Nash equilibrium of the game.

Definition *A pair* (a_1^*, a_2^*) *of actions in a symmetric two-player strategic game is a* **symmetric Nash equilibrium** *if it is a Nash equilibrium and* $a_1^* = a_2^*$*.*

That is: to model interaction in single population:

- use symmetric game
- look for symmetric Nash equilibrium

Example: approaching pedestrians

	Left	*Right*
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

Two symmetric Nash equilibria: $(Left, Left)$ and $(Right, Right)$.

That is, two steady states: every pedestrian steps to the left as she approaches another pedestrian, or both participants step to the right.