Econ 300, Problem Set 1
Professor Cramton

2.1.2.

Determine which of the following relationships represent functions. Assume that the interval is the set of real numbers unless otherwise indicated.

(a) $y = 5x$

(b) $y \leq x$

(c) $y = a + \sqrt{x}; (0, \infty)$

(d) $y = -x^2$

(e) $y^2 = x$

(f) $y = \frac{1}{x-3}; (0, \infty)$

(g) $y^2 = x^4; (0, \infty)$

2.1.8.

Which functions are continuous over the given intervals?

(a) $y = 8 + \frac{1}{x-7}; (0, \infty)$

(b) $y = \frac{4-ax}{7}; (-\infty, \infty)$

(c) $y = -3 + \frac{1}{x+7}; (0, \infty)$

(d) $y = |2x - 4|; [-3, 3]$

2.2.8.

Show that the average rate of change of a strictly increasing function is positive and that the average rate of change of a strictly decreasing function is negative.

2.2.10.

Sketch the function $y = 8 + 10x - x^2$ over the domain [0,7].

(a) Assume that $x_A = 1, x_B = 4$, and $\lambda = 0.4$. Using the formula $x' = \lambda x_A + (1 - \lambda)x_B$, determine the value of x'. What is the value of $f(x')$?

(b) Calculate the value of y', which is the value of the secant line at x', using the formula $y' = \lambda f(x_A) + (1 - \lambda)f(x_B)$.

(c) Prove that the above function is strictly concave by demonstrating that $f(x') > y'$.

2.3.2.

Condense the following expressions.

(a) $x^a \cdot x^b \cdot x^c \div x^d$

(b) $x^{1/2}x^{3/2} \div x^{1/3}$

(c) $((x^{1/3})^8)^{1/2} \cdot x^2 \div x^{3/4}$

(d) $x^2y^3 \cdot x^3 \div xy^2 \cdot x^{-2}y$

2.3.12.

Consider a function that relates tax revenues R, in billions of dollars, to the average tax rate t such that

$$R = 350t - 500t^2$$

(a) What tax rate(s) is consistent with raising tax revenues equal to \$60 billion?

(b) What tax rate(s) is consistent with raising tax revenues equal to \$61.25 billion?

Econ 300, Problem Set 2
Professor Cramton

3.1.6.

Assume a firm's net profits are \$50 million in 2000 and are expected to grow at a steady rate of 6% per year through the end of the decade. How much would you expect the firm to earn in 2001? In 2003? Now assume that the firm's profits have been growing at 6% since 1997. If a negative value of n can be interpreted as the number of time periods before period t, how much did the company earn in 1998? Graph the path of income growth between 1998 and 2003 and explain why the curve gets steeper over time.

3.2.8.

While you are a senior in high school, your parents decide to invest in the bond market to help pay for your college tuition. They purchase a bond that will pay \$15,000 in one year plus an interest coupon payment of 7% of the bond's value when the interest rate on comparable assets is also 7%. What is the present value of this bond? If interest rates fall to 5% after the bond purchase, what is the present value of that same investment? What is the present value if interest rates rise to 9.5%?

3.3.4.

On a secretive search through your grandfather's basement you uncover an old, dust-covered stamp collection. You take the collection to a stamp dealer who tells you that it currently is worth \$1,000, but in ten years it will triple its value. Unfortunately, you will have to store the collection in a special warehouse during this time. The storage fee is \$200, compounded continuously at a rate of 5% per year but is not paid until you withdraw your collection from the warehouse. Should you store the stamp collection or sell it immediately and invest the proceeds in the stock market, where you expect to make 9.5% continuously compounded return over the coming ten years?

3.3.10.

Consider the production function, $Q = 15L^{4/5}K^{1/5}$, where Q is output, L is labor input, and K represents capital input. Using natural logarithms, transform this exponential function into a linear function. Now assume that $L = 10$ and $K = 5$. What is the value of $\ln(Q)$? Remembering that $\exp(\ln(Q)) = Q$, determine the value of Q.

4.1.2.

Consider the following system of equations where the endogenous variables are z, x, and y and the exogenous variables are h and a:

$$\begin{aligned}x &= 6z + 3h - 4a + 10\\ y &= 4z - h + 6\\ x &= y\end{aligned}$$

(a) Use repeated substitution to solve for the equilibrium values of the endogenous variables, $\bar{z}, \bar{x}$, and $\bar{y}$.

(b) Using comparative static analysis, determine the impact on $\bar{z}, \bar{x}$, and $\bar{y}$ due to a change in a, *ceteris paribus*.

(c) Now assume that $\Delta a = 2$. Determine the impact on the above endogenous variables due to this change.

6.2.8.

Calculate the difference quotient for the function

$$y = 4x^2 - 2x + 7,$$

where $x_0 = 3$ and $\Delta x = 3$.

(a) Holding x_0 constant, determine the impact on the difference quotient when $\Delta x = 1.5$ and $\Delta x = 0.5$.

(b) Based on your answers, what would the value of the limit of the difference quotient be as Δx approaches zero; that is, what is $\lim_{\Delta x \to 0} \Delta y / \Delta x$?

Econ 300, Problem Set 3
Professor Cramton

7.2.2.

Use the quotient rule to differentiate each function.

(a) $f(x) = \frac{2x+7}{x^2-1}$

(b) $f(x) = \frac{bx^3+cx^2+x-4}{x}$

(c) $f(x) = \frac{e^{2x}}{x^2}$

(d) $f(x) = \frac{(3x+2)^2}{x}$

7.2.8.

A standard tool of economic analysis is the Cobb-Douglas production function. This function shows how much output (Q) is produced with a given amount of labor (L) and capital (K) as follows: $Q = AK^\alpha L^{1-\alpha}$. The parameter A represents the efficiency of the economy, so A increases with technological change. Suppose that, over time, efficiency, capital, and labor each grow at the rates

$$A(t) = A_0e^{ct}, \quad K(t) = K_0e^{ft}, \quad \text{and} \quad L(t) = L_0e^{gt},$$

where A_0, K_0, and L_0 are initial values for technology, capital, and labor, respectively, and c, f, and g are their respective rates of growth. What is the percentage growth rate of output in terms of the production parameters and the growth parameters c, f, and g? (*Hint*: Recall that the derivative of a natural logarithm is close to a percentage change.)

7.2.14.

A person's productivity at a particular job may change with experience. Consider the following model of effective labor input, L,

$$L = 10(1 - e^{-0.1t}) + 2,$$

where L is a measure of the "effective labor input" of a particular worker and t is the worker's tenure (that ism years spent in a particular job). Use differentiation to determine the marginal change in the effective labor input with respect to time on the job, t. Provide a brief economic interpretation of the differences in these values of dL/dt.

7.3.4.

Consider the function $y = 1/3x^3 - 4x^2 + 6x + 8$ over the interval [0,4]. Use the second derivative to determine whether the function is concave or convex. Confirm your results with the definitions for concavity and convexity found in Chapter 2, where a function is

$$\begin{aligned} \text{Concave: } & if\ f(\lambda x^A + (1-\lambda)x^B \geq \lambda f(x^A) + (1-\lambda)f(x^B) \text{ and} \\ \text{Convex: } & if\ f(\lambda x^A + (1-\lambda)x^B \leq \lambda f(x^A) + (1-\lambda)f(x^B). \end{aligned}$$

Assume that $\lambda = 0.4, x^A = 0$, and $x^B = 4$.

7.3.8.

Suppose each equation below is a utility function for the consumption of cookies, c. What restrictions on the parameters (for example, positive, negative, less than 1, and so forth) are required for the utility function to reflect that utility rises at a decreasing rate with the number of cookies consumed? (Assume that $c > 0$.)

(a) $U(c) = \Psi + ae^{\beta c}$, where the parameters are a, β, and Ψ

(b) $U(c) = \theta c^{\lambda}$, where the parameters are θ and λ

7.3.10.

Find the coefficient of absolute risk aversion

$$\theta = -\frac{U''(c)}{U'(c)}$$

for the square root function $U(c) = a\sqrt{c}$ and the logarithmic function $U(c) = \beta \ln(c)$.

Econ 300, Problem Set 4
Professor Cramton

8.4.6.

Find the derivative of each implicit function, where $dY/dX = -F_X/F_Y$, provided that $F_Y \neq 0$.

(a) $F(x, y) = x^2 + y^2 + (xy)^{1/3} = 0$

(b) $F(x, y) = x^2 y + y^2 x + xy = 0$

(c) $F(x, y) = \ln x^3 + (xy)^2 - 4y = 0$

(d) $F(x, y, w) = w^3 y^3 + x^3 + wxy + 7 = 0$ (Find $\partial y/\partial x$.)

(e) $F(x, y) = xy^2 e^y$

9.1.6.

Show that a quadratic function does not have an inflection point. What is the geometric interpretation of this?

9.2.10.

The compensation firms pay workers typically includes benefits as well as salary. The cost of one particular benefit, health insurance, has been rising rapidly. Consider a firm that uses only labor as its input and has the production function

$$f(L) = \alpha L^\beta,$$

where L is the number of workers employed, $\alpha > 0$, and $1 > \beta > 0$. The profits of this firm, Π, equal

$$\Pi = p \cdot f(L) - (w + b)L,$$

where p is the price of the firm's product, w is the wage paid to a worker, and b is the cost of providing a worker with benefits.

(a) Find the number of workers the firm will demand in order to maximize its profits. Assume that the firm takes as given p, w, and b. How does the optimal number of workers demanded vary with the value of b?

(b) Show that your solution represents the maximization of profits as supposed to the minimization of profits.

10.1.2.

Find the stationary point(s) of each function.

(a) $g(u, v) = 10 + 20u - 2u^2 + 16v - v^2 - 2uv$

(b) $g(u, v) = 100 - 5u + 4u^2 - 9v + 5v^2 + 8uv$

(c) $g(u, v) = \frac{1}{3}u^3 + 3uv + 2u - \frac{3}{2}v^2$

(d) $g(u, v) = 72u^{1/3}v^{1/3} - 6u - 3v$

10.2.2.

Determine whether the stationary points (u^*, v^*) for each function $g(u, v)$ in parts (a) through (d) of question 2 in Section 10.1 represent maxima, minima, or saddle points.

Econ 300, Problem Set 5
Professor Cramton

10.1.8.

A firm uses inputs of labor, L, and capital, K, to produce its outputs, Q, according to the production function

$$Q = f(K, L) = 9L^{1/3}K^{1/3}.$$

The firm is a price-taker in the inputs markets. Labor is paid an hourly wage of $w = 12$, and the price of capital is $r = 6$. The firm sells its output at a price of $P = 4$ per unit. Maximize the profit function

$$\Pi(K, L) = Pf(K, L) - wL - rK$$

to determine the optimum level of each input the firm should use.

10.1.10.

The publishing firm America's Western Lore (AWL) sells its cowboy novels in two regions of the country, Texas and Massachusetts. The demand for these books in Texas is given by the inverse demand function

$$P_T = \alpha - \beta Q_T,$$

while the inverse demand function in Massachusetts is

$$P_M = \gamma - \theta Q_M,$$

where Q refers to quantity, P refers to price, and the subscripts refer to Texas or Massachusetts. We initially assume that books cannot be shipped between Texas and Massachusetts. We also assume that $\beta > \theta$ and $\alpha > \gamma$.

(a) Find the optimal quantities to sell in Massachusetts and in Texas under the assumption that the total cost function for AWL is linear and

$$TC = \Psi + c(Q_M + Q_T),$$

where c is a parameter and Ψ represents fixed costs. Determine the profits earned by AWL.

(b) Find the optimal quantities to sell in Massachusetts and in Texas under the assumption that the total cost function for AWL is quadratic and

$$TC = \Psi + c(Q_M + Q_T)^2,$$

where c is a parameter and Ψ represents fixed costs.

(c) Now suppose some residents of Texas and Massachusetts begin transporting books such that $P_T = P_M$, and then AWL faces the overall inverse demand function

$$P = \left(\frac{\alpha\theta + \gamma\beta}{\beta + \theta} \right) - \left(\frac{\beta\theta}{\beta + \theta} \right) Q,$$

where P is the common price in Texas and Massachusetts and $Q = Q_M + Q_T$. Determine the profits AWL would have earned in this case.

11.1.2.

You own a farm that produces two types of wheat, type x and type z. You are under contract to deliver 12 tons of wheat, in any combination of your choosing, to the bread manufacturer operating in your farm district. Find the combination of crops that minimize the cost of fulfilling this contract given your cost function $C = 3x^2 - 4xz + 9z^2 - 8z + 36$.

11.2.6.

The Cobb-Douglas production function takes the form

$$Q = AK^\alpha L^{1-\alpha},$$

where Q is the amount of output, K is the amount of capital input, and L is the amount of labor input. Suppose that a firm faces a linear cost-of-inputs function

$$C = wL + rK,$$

where C is the cost of inputs, w is the wage rage, and r is the rental rate on capital.

(a) Set up a Lagrangian function reflecting the constrained optimization problem of obtaining the most output given a budget $\bar{C}$ to spend on inputs. Solve this for the optimal levels of capital and labor.

(b) Set up a Lagrangian function reflecting the constrained optimization problem of spending the least amount on inputs given that the level of output must equal the amount $\bar{Q}$. Solve this for the optimal levels of capital and labor.

Econ 300, Problem Set 6
Professor Cramton

Question 1. Kuhn-Tucker Optimization

Find the maximum value of the function

$$r(a,b) = 2a^2 + b^2$$

subject to the constraints

$$\begin{aligned} 2a + b &\leq 9 \\ a^2 + b^2 &\geq 16 \end{aligned}$$

Question 2. Probability

A hospital has two rooms for surgery. The probability that a given room is free is 0.5. Show your work in answering the following questions:

(a) What is the probability that both rooms are free?

(b) What is the probability that both rooms are occupied?

(c) What is the probability that at least one room is free?

Question 3. Probability

Among the students registered in Econ 200, 30% are social science majors, 20% are business majors, and the remaining 50% are other majors. Among each of these groups, 40% are first year students, 30% are second year students and the remaining 30% are upper year students.

(a) Define which events are mutually exclusive.

(b) What is the probability that a randomly chosen student is a second year business major?

(c) What is the probability that a randomly chose student is either a second year, or a business major (he/she could be both)?

Question 4. Decision making under uncertainty

The price of some stock today is \$300. Assume that the stock's value in one year is a random variable X with the following probability distribution: $P(X = 400) = 0.1, P(X = 350) = 0.4, P(X = 300) = 0.3, P(X = 270) = 0.2$.

(a) Compute the expected value $E(X)$, which represents the expected price of the stock in one year.

(b) What is the expected return of investing into this stock? (Hint, ignoring dividends, the expected return on the stock is given by $\mu_R = \frac{E(X)}{\text{Price today}}$).

(c) Assume there is a riskless bond in the economy that yields a (certain) return of 4% a year. This is lower than the expected return from the stock you found above. Will some people in this economy choose to hold the bond? Why would they do so? Explain.

Question 5. Decision making under uncertainty

Consider an individual facing the risk of becoming ill and her willingness to insure herself. The individual has a probability of 5% of getting seriously ill in a given year. If she gets ill, her yearly net income will amount to \$20,000 (this amount takes into account the forgone income during her period of illness as well as the cost of hospitalization). With a probability of 95%, however, she will not get ill and will make a yearly income of \$45,000. Assume her utility function is given by $U(x) = \sqrt{x}$.

(a) Compute the individual's expected utility.

(b) Find the certain level of income that would make the individual as well off as being in the risky situation described above (this amount is called the "certainty equivalent").

(c) Use this to find the maximum premium she would be willing to pay to get rid of this risk, i.e. to insure herself against the economic consequences of illness.