Econ 300, Problem Set 6, Suggested Answers
Professor Cramton

Question 1

The solution will follow the lecture notes style of solving Kuhn-Tucker problems:

One important point is to notice that we have to reverse one of the constraints to be able to write the Lagrangian in the usual form. So, instead of $a^2 + b^2 \geq 16$, we use $-a^2 - b^2 \leq -16$. The Lagrangian is as follows:

$$\mathcal{L}(a, b, \lambda, \mu) = 2a^2 + b^2 - \lambda(2a + b - 9) - \mu(-a^2 - b^2 + 16)$$

Optimality Conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a} &= 4a - 2\lambda + 2a\mu = 0 \\
\frac{\partial \mathcal{L}}{\partial b} &= 2b - \lambda + 2b\mu = 0 \\
2a + b - 9 &\leq 0 \\
-a^2 - b^2 + 16 &\leq 0 \\
\lambda(2a + b - 9) &= 0 \\
\mu(-a^2 - b^2 + 16) &= 0 \\
\lambda \geq 0,\ u &\geq 0
\end{aligned}$$

There are four possible cases:

(a) $\lambda = 0, \mu = 0$

The first order conditions become:

$$\begin{aligned}
4a &= 0 \\
2b &= 0 \\
a = b &= 0
\end{aligned}$$

For this to be a solution it must satisfy each and every one of the optimality conditions. We can see that the second constraint ($a^2 + b^2 \geq 16$) is not satisfied. Hence, $a = b = 0$ is not a solution.

(b) $\lambda \neq 0, \mu = 0$

First order conditions become:

$$\begin{aligned}
4a &= 2\lambda \\
2b &= \lambda
\end{aligned}$$

Dividing each side of both equations with each other we get $a = b$. Substitute this result into the constraint which we are considering as binding: $2a + b = 9 \rightarrow 2a + a = 9 \rightarrow a = b = 3$. For this to be a solution one must check all optimality conditions and in fact all are satisfied. We conclude that $a = b = 3$ is a solution to the problem.

(c) $\lambda = 0, \mu \neq 0$

First order conditions become:

$$4a = -2\mu a \rightarrow \mu = -2$$
$$2b = -2\mu b \rightarrow \mu = -1$$

The first order conditions yield to possible values for the multiplier which is cannot be part of a solution.

(d) $\lambda \neq 0, \mu \neq 0$

First order conditions become:

$$\begin{aligned} 4a - 2\lambda + 2a\mu &= 0 \\ 2b - \lambda + 2b\mu &= 0 \\ 2a + b &= 9 \\ a^2 + b^2 &= 16 \end{aligned}$$

Here we have a system of four equations and four unknowns. As an easier way, we can solve the last two equations since they involve only a, b. Hence, we can solve the subsystem of 2 equations and 2 unknowns:

$$\begin{aligned} a^2 + (9 - 2a)^2 &= 16 \\ a^2 + 81 - 36a + 4a^2 - 16 &= 0 \\ 5a^2 - 36a + 65 &= 0 \end{aligned}$$

Using the quadratic formula to find the roots of this equation: $a = \frac{36 \pm \sqrt{36^2 - 4(5)(65)}}{10}$ has no real roots. So, we conclude that there is no solution for this case among real numbers.

As a result the only solution to this problem is $a = b = 3$ when $\lambda \neq 0, \mu = 0$.

Question 2

$$P(room = free) = 0.5, P(room = occupied) = 0.5$$

(a)

$$P(Both\ rooms = free) = 0.5 * 0.5 = 0.25$$

(b)

$$P(Both\ rooms\ are\ occupied) = 0.5 * 0.5 = 0.25$$

(c)

$$\begin{aligned} P(at\ least\ one\ room = free) &= P(room1 = free\ and\ room2 = occupied) \\ &+ P(room1 = occupied\ and\ room2 = free) \\ &+ P(Both\ rooms = free) \\ &= 0.5 * 0.5 + 0.5 * 0.5 + 0.5 * 0.5 = 3 * 0.25 = 0.75 \end{aligned}$$

Question 3

$$P(Major = SocialSciences) = 0.3$$
$$P(Major = Business) = 0.2$$
$$P(Major = Other) = 0.5$$
$$P(Year = 1) = 0.4$$
$$P(Year = 2) = 0.3$$
$$P(Year > 2) = 0.3$$

(a) Mutually exclusive events are the ones that have a zero probability of occurring at the same time.

Here mutually exclusive events are : $\{Year = 1, Year = 2, Year > 2\}$ and any possible combination of events that involves the student being in the 1st, 2nd or higher years at the same time.

(b) $P(Year = 2 \ and \ Major = Business) = 0.3 * 0.2 = 0.06$

(c) $P(Year = 2 \ or \ Major = Business) = P(Year = 2) + P(Major = Business) - P(Year = 2 \ and \ Major = Business) = 0.3 + 0.2 - 0.3 * 0.2 = 0.44$

Question 4

Let's denote the current price of the stock as $P_t = \$300$.

Stock's price in the next period is P_{t+1} and has the following probability distribution:

P_{t+1}	$Prob(P_{t+1})$
\$400	0.1
\$350	0.4
\$300	0.3
\$270	0.2

(a) Expected price of the stock $= E(P_{t+1}) = 0.1 * 400 + 0.4 * 350 + 0.3 * 300 + 0.2 * 270 = \$324$.

(b) What is the expected return on the stock?

We denote the expected return as

$$Return = \frac{E(P_{t+1})}{P_t} = \frac{324}{300} = 1.08$$

This means that every dollar invested in this stock will yield \$1.08 in expected terms, which means that there is an expected rate of 8%.

(c) Despite the fact that the return on the riskless bond is 4% (less than 8%) there might still be people willing to hold the riskless bond instead of the risky stock depending on their coefficient of risk aversion. Hence, in order to answer this question precisely we would need an assumption on the utility function.

Question 5

Event A : becoming ill
Event B : staying healthy

$$P(A) = 0.05$$
$$P(B) = 0.95$$

Income in case event A happens = \$20,000
Income in case event B happens = \$45,000

(a) Expected utility:

$$\begin{aligned} EU &= U(20000) * P(A) + U(45000) * P(B) \\ &= 20000^{1/2} * 0.05 + 45000^{1/2} * 0.95 \\ &= 209 \end{aligned}$$

(b) Find the "certainty equivalent" level of income:

The question can be formulated as follows: Find the amount of income that would make the individual indifferent between staying in the risky situation or receiving \$X with certainty.

$$\begin{aligned} EU &= U(X) \\ 209 &= X^{1/2} \\ X &= \$43,681 \end{aligned}$$

(c) Find the maximum premium:

The question can be formulated as follows: Find x such that

$$\begin{aligned} EU &= U(45000 - x) \\ 209 &= (45000 - x)^{1/2} \\ x &= \$1,329 \end{aligned}$$