Decision Making Under Uncertainty

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Economics 300

Uncertainty

- Consumers and firms are usually uncertain about the payoffs from their choices
- Example 1: A farmer chooses to cultivate either apples or pears
  - When he makes the decision, he is uncertain about the profits that he will obtain. He does not know which is the best choice
  - This will depend on rain conditions, world prices…

Uncertainty

- Example 2: playing with a fair die
  - You win $2 if 1, 2, or 3,
  - You neither win nor lose if 4, or 5
  - You lose $6 if 6

- Example 3: John’s monthly consumption:
  - $3000 if he does not get ill
  - $500 if he gets ill (so he cannot work)

Vocabulary

- A <u>lottery</u> is a situation which involves uncertain payoffs:
  - Cultivating apples
  - Cultivating pears
  - Playing with a fair die
  - Monthly consumption
- Each lottery results in a prize

Introduction to Random Variables

Let's play a game (in theory):

To play the game, you must pay $5 to select a card from a standard deck. If the card you select is the ace of hearts, you will win $100. For any other ace, you will win $10 and for any other heart, you win $5.

Is anyone willing to play this game?

Possible Outcomes and Corresponding Probabilities

Net Payoff	Probability	
$95	1/52	A♥
$5	3/52	Other Ace
$0	12/52	Other ♥
-5	36/52	Otherwise

What can you expect to win if you were to play this game?

Possible Outcomes and Corresponding Probabilities

Net Payoff	Probability	
$95	1/52	A♥
$5	3/52	Other Ace
$0	12/52	Other ♥
-5	36/52	Otherwise

What can you expect to win if you were to play this game?

$$E(X) = 95\left(\frac{1}{52}\right) + 5\left(\frac{3}{52}\right) + 0\left(\frac{12}{52}\right) + (-5)\left(\frac{36}{52}\right) = -\$1.35$$

Random Variables

A random variable, X, is a quantitative variable that has values that vary according to the rules of probability.

Elements of the sample space are all the possible values that the variable can have.

Example: Experiment of rolling a single die

$$S = \{1,2,3,4,5,6\}$$

The outcomes are numerical and the exact value that will turn up varies. From the rules of probability, each of the values is equally likely, or has an equal probability of happening.

Probability Distribution

The probability distribution of a random variable, X, written as p(x), gives the probability that the random variable will take on each of its possible values.

p(x) = P(X=x) for all possible values of X.

Probability model - the collection of all the possible values and the probabilities that they occur

Rules for Probability Distributions

For each value of the random variable X:

1. The probability must be between 0 and 1 inclusive

$$0 \leq p(x) \leq 1$$

2. The probabilities of all values of X must sum to 1.

$$\sum_{x \in S} p(x) = 1$$

Writing the Probability Distribution

Suppose we were to toss two coins. Using the rules of probability, identify the probability distribution of X, the number of heads in two tosses of a coin:

S = {TT, HT, TH, HH}

p(0) = P(X=0) corresponds to only one outcome, TT, so p(0) = 1/4

p(1) = P(X=1) corresponds to two outcomes, HT, TH, so p(1)=2/4

p(2) = P(X=2) corresponds to only one outcome, HH, so p(2) = 1/4

Display the results in a table:

x	0	1	2
p(x)	0.25	0.5	0.25

Finding Probabilities from a Probability Distribution

The number of copies of *USA Today* that are sold daily by a convenience store is a random variable X that has the following probability distribution.

x	0	1	2	3	4	5
p(x)	0.1	0.12	0.25	0.3	0.2	0.03

The store manager wants to know the probability that on any given day he will sell exactly two copies of *USA Today*.

$p(2) = 0.25$

Finding Probabilities (continued)

x	0	1	2	3	4	5
p(x)	0.1	0.12	0.25	0.3	0.2	0.03

Now what is the probability that two or three copies are sold on any given day?

$$P(X=2 \text{ or } X=3) = p(2) + p(3)$$

$$= 0.25 + 0.30 = 0.55$$

Since the values of a random variable are mutually exclusive, we can simply add the probabilities to obtain the answer.

Probability Notation Summary

Finding the probability that X takes on a value that is . . .	What it means	Notation
at least x	All the values of the random variable that are x or greater	$P(X \geq x)$
more than x	All the values of the random variable that are greater than x	$P(X > x)$
at most x	All the values of the random variable that are x or less	$P(X \leq x)$

Probability Notation Summary

Finding the probability that X takes on a value that is . . .	What it means	Notation
less than x	All the values of the random variable that are less than x	$P(X<x)$
between x1 and x2	All the values of the random variable that are greater than x1 and less than x2	$P(x1<x<x2)$
between x1 and x2, inclusive	All the values of the random variable that start with x1 and go up to and include x2	$P(x1\leq x\leq x2)$

Example of Finding Probabilities

x	0	1	2	3	4	5
p(x)	0.1	0.12	0.25	0.3	0.2	0.03

The store needs to sell at least three copies per day to make a profit from the sales. The manager is looking for the probability that the store sells at least three copies, or $P(X \geq 3)$.

These means P(X=3 or X=4 or X=5), which is calculated as

$$p(3) + p(4) + p(5) = 0.30 + 0.20 + 0.03 = 0.53$$

What is the probability that the store will not make a profit?

Expected Value

- For a lottery (*X*) with prizes $x_1, x_2, \ldots, x_n$ and probabilities of winning $p_1, p_2, \ldots, p_n$, the expected value of the lottery is

$$E(X) = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n$$

$$E(X) = \sum_{i=1}^{n} p_i x_i$$

- The expected value is a weighted sum of the prizes
  - the weights are the probabilities
- The symbol for the expected value of X is *E(X)*

Expected Value of monthly consumption (Example 3)

- Example 3: John's monthly consumption:
  - X_1=$4000 if he does not get ill
  - X_2=$500   if he gets ill (so he cannot work)
  - Probability of illness 0.25
  - Consequently, probability of no illness=1-0.25=0.75
  - The expected value is:

$$E(X) = p_1 x_1 + p_2 x_2$$

$$E(X) = 0.75(\$4000) + 0.25(\$500) = 3125$$

Independence

- Two events A and B are *independent* if and only if P(A and B) = P(A)P(B)
- Knowing one happened does not change probability of other
  P(A|B) = P(A); P(B|A) = P(B)
  P(A|B) = probability of A given B occurs
- Example - independent
  - Flips of a coin
  - Rolls of a die
- Example - not independent
  - A = cloudy day; B = rainy day
  - Draws from a deck of cards without replacement

Conditional Probability

- P(A|B) = probability A occurs given B occurs

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

- For independent events, P(A and B) = P(A)P(B), so P(A|B) = P(A); P(B|A) = P(B)

Let's Make a Deal

Curtain 1	**Curtain 2**	**Curtain 3**

How long to get a pattern of flips?

- Pattern HTT
- Pattern HTH
- Flip coins until pattern is reached
- HHTHTTHHHTHTTTH…
  - In experiment above, HTT reached in 6 flips, HTH reached in 4 flips
- Does it take longer to reach HTT or HTH? (on average)
  - HTT takes more flips than HTH
  - HTH takes more flips than HTT
  - Both take the same

Example - Racehorse

A man buys a racehorse for $20,000, and enters it in two races. He plans to sell the horse afterward, hoping to make a profit. If the horse wins both races, its value will jump to $100,000. If it wins one of the races, it will be worth $50,000. If it loses both races, it will be worth only $10,000. The man believes there is a 20% chance that the horse will win the first race and a 30% chance it will win the second one. Assuming that the two races are independent events, find the man's expected profit.

Example - Racehorse

The random variable in this example is the profit from selling the horse.

Profit	**p(x)**
-$10,000	P(Win no races)
30,000	P(Win race 1 or 2, not both)
$80,000	P(Win both races)

First step is to find the probability distribution for this model.

Example - Racehorse

P(Win both races) = P(Win 1 and Win 2)

= P(Win 1) P(Win 2) = $0.2 \times 0.3 = 0.06$

P(Win 1 or Win 2, not both) =

P(Win 1 and Not win 2) or P (Not win 1 and win 2)

=P(Win 1) P(Not win 2) + P(Not win 1) P(Win 2)

= $0.2 \times 0.7 + 0.8 \times 0.30 = 0.38$

P(Win no races) = 1 - P(Win Both) - P(Win 1 or 2)

= $1 - 0.06 - 0.38 = 0.56$

Example - Racehorse

Probability Model:

Profit	p(x)
-$10,000	0.56
30,000	0.38
$80,000	0.06

Find the expected profit: $\mu = E(x) = \sum xp(x)$

E(x) = -$10,000(0.56) + $30,000(0.38) + $80,000(0.06)

= $10,600