Introduction to Probability

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Economics 300

1) Which of the following is more likely to be a Powerball winner?

A) 32 11 5 24 18 3 B) 1 2 3 4 5 6

2) Here are the results of 9 coin flips: HHHHHHHHH
What is the most likely outcome for the 10^{th} coin?

3) Why is a royal flush in spades the highest hand in poker?

(10, Jack, Queen, King and Ace, all in spades)

52 choose 5 = 2,598,960

Definition of Probability

The probability that an event A will occur is written as P(A) and is read "the probability of A".

Three Different Types of Probability:

1. Theoretical 2. Empirical 3. Subjective

Vocabulary of Probability Theory

Experiment - anything that produces an outcome

Examples: flipping a coin, rolling a die

Outcome - a particular result of an experiment

Examples: coin lands heads, die roll is 5

Sample space - set of all possible outcomes of an experiment

Examples: S = {Heads, Tails} S= {1,2,3,4,5,6}

Event - an outcome or a set of outcomes that are of interest to the experimenter

Probability Rules for Equally Likely Events

In S where outcomes are equally likely, then the probability of event A, P(A) is

$$P(A) = \frac{n_A}{N} = \frac{\text{Number of ways that A can occur}}{\text{Total \# of possible outcomes}}$$

Approach is called Classical Theoretical Probability

A Simple Experiment

Suppose we were to roll a die:

What is the sample space?

$$S = \{1, 2, 3, 4, 5, 6\}$$

Event 3 = die lands on 3, $P(3) = 1/6$

Event Even = die lands on even number, $P(\text{Even}) = 3/6 = 1/2$

Another Simple Experiment

Selecting a card from a deck of cards

Event Q♥ = the Queen of Hearts is selected, P(Q♥) = 1/52

Event Q = a Queen is selected, P(Q) = 4/52 = 1/13

Event ♥ = a Heart is selected, P(♥) = 13/52 = 1/4

Event KQJ = a face card is selected, P(KQJ) = 12/52 = 3/13

Rules about Probabilities

- $0 \leq P(A) \leq 1$, the probability of an event must be a number between 0 and 1, inclusive

- $P(S) = 1$, the probability of the entire sample space must be equal to 1

- If an event must happen, $P(A) = 1$

- If an event is impossible, $P(A) = 0$

Complement Rule

The complement, A^C, of an event A consists of exactly the outcomes that are not in A.

$$P(A) + P(A^C) = 1$$

$$P(A) = 1 - P(A^C)$$

Example: Rolling a die, event A = the number face up is greater than 2. A^C = all numbers that are not larger than 2.

$$P(A^C) = 2/6$$

$$P(A) = 1 - P(A^C) = 1 - 2/6 = 4/6$$

Empirical Probabilities

An empirical probability is one that is calculated from sample data and is an estimate for the true probability.

Collect sample data and calculate relative frequencies.

Find the exact probability that an item taken from the sample will have some characteristic of interest.

If the data are a good representation of the population, then you can use the relative frequencies as estimates of the true probabilities for the population.

Example of Empirical Probabilities

Frequency distribution of annual income for U.S. families:

Income	Frequency (1000s)
Under $10,000	5,216
$10,000 - $14,999	4,507
$15,000 - $24,999	10,040
$25,000 - $34,999	9,828
$35,000 - $49,999	12,841
$50,000 - $74,999	14,204
$75,000 and over	12,961
Total	69,597

Example of Empirical Probabilities Continued

Find the probability that a U.S. family selected at random has an income under $10,000

Income	Frequency (1000s)
Under $10,000	5,216
$10,000 - $14,999	4,507
$15,000 - $24,999	10,040
$25,000 - $34,999	9,828
$35,000 - $49,999	12,841
$50,000 - $74,999	14,204
$75,000 and over	12,961
Total	69,597

Probability = 5,216/69,597 = 0.075

Example of Empirical Probabilities Continued

Find the probability that a U.S. family selected at random has an income between $35,000 and $49,999.

Income	Frequency (1000s)
Under $10,000	5,216
$10,000 - $14,999	4,507
$15,000 - $24,999	10,040
$25,000 - $34,999	9,828
$35,000 - $49,999	12,841
$50,000 - $74,999	14,204
$75,000 and over	12,961
Total	69,597

Probability = 12,841/69,597 = 0.186

Subjective Probabilities

The probability of the event, is found by simply guessing or estimating its value based on knowledge of the relevant circumstances.

Example: What is the probability that the Dow Jones Industrial Average will exceed 17,000 today?
financial analysts use their expert knowledge to develop estimates of the probability.

Compound Events

Combining of two or more events from the experiment.

P (this or that)

Addition Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example of the Addition Rule

Suppose that 40 % of cars registered on campus are manufactured in the United States, 30 % in Japan, 10 % in Germany, and 20% in other countries.

Find the probability that a randomly selected car was manufactured in Japan or Germany.

P (J or G) = P(J)   + P(G)   – P(J and G)

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= 0.30 + 0.10 – 0

= 0.40

Mutually Exclusive

We call events, A and B, mutually exclusive if

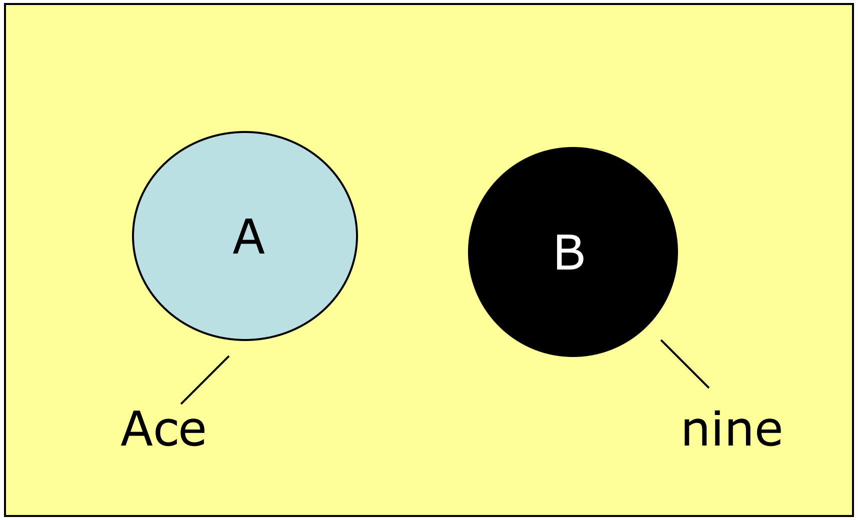
P(A and B) = 0 , in other words event A and event B cannot occur at the same time.

- Hence, for two mutually exclusive events A and B, the probability that one or the other occurs is the sum of the probabilities of the two events.

$$P(A \text{ or } B) = P(A) + P(B)$$

Venn Diagram: Mutually Exclusive Events

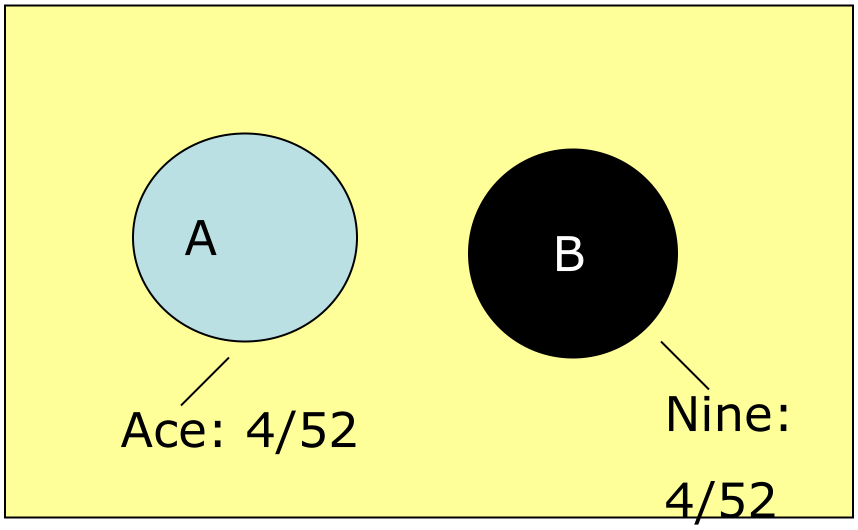
One card is drawn from a standard deck of cards. What is the probability that it is an ace or a nine?



Events A and B are mutually exclusive. A card can be either an Ace or a nine, but can not be both

Venn Diagram:
Mutually Exclusive Events

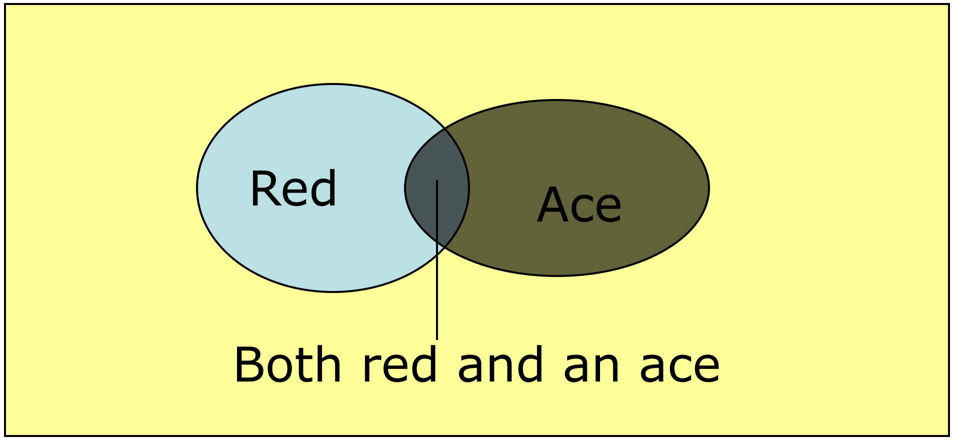
One card is drawn from a standard deck of cards. What is the probability that it is an ace or a nine?



P(A or B) = P(A) + P(B) = 8/52 = 2/13

Venn Diagram:
Events that are Not Mutually Exclusive

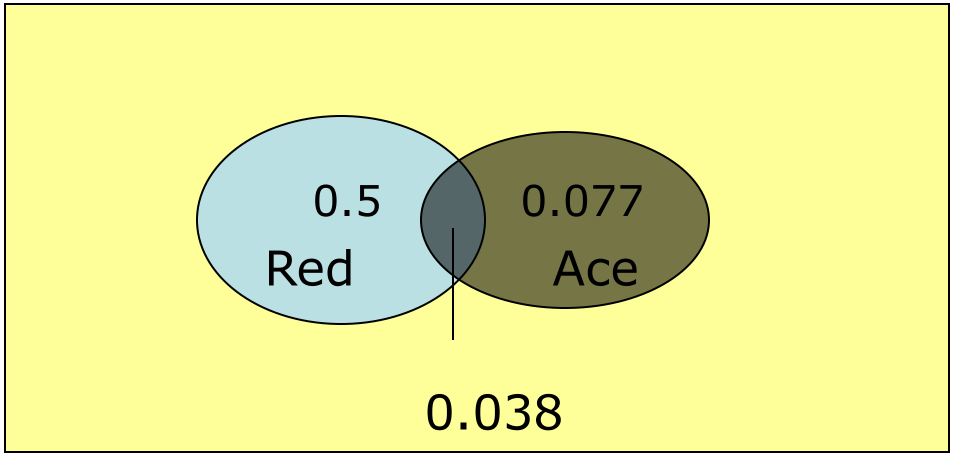
One card is drawn from a standard deck of cards. What is the probability that it is red or an ace?



These events are not mutually exclusive as it is possible for a card to be both red and an ace (ace of hearts, ace of diamonds)

Probabilities of Events that Are Not Mutually Exclusive

Find P(Red or Ace):



= P(Red) + P(Ace) – P(Both Red and Ace)

= 0.5 + 0.077 -0.038 = 26/52 + 4/52 – 2/52

=0.539 = 28/52

Example

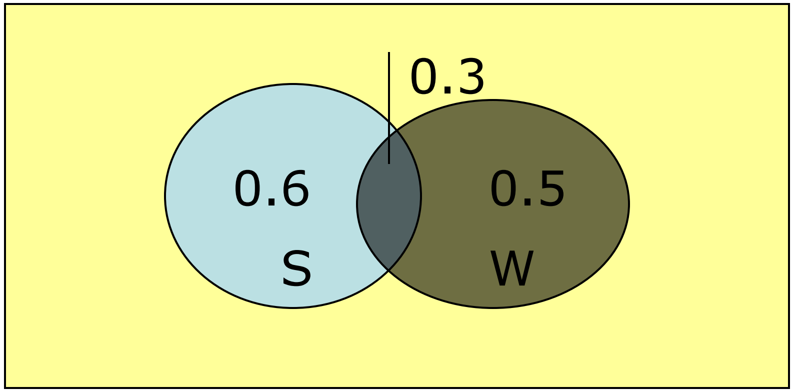
The Illinois Tourist Commission selected a sample of 200 tourists who visited Chicago during the past year. The survey revealed that 120 tourists went to the Sears Tower, 100 went to Wrigley Field and 60 visited both sites. What is the probability of selecting a tourist at random who visited the Sears Tower or Wrigley Field?

Example (continued)

P(Sears Tower) = 120/200 = 0.6

P(Wrigley Field) = 100/200 = 0.5

P(Both) = 60/200 = 0.3



P(S or W) = P(S) + P(W) – P(S and W)

= 0.6 + 0.5 - 0.3

= 0.8

Example (continued)

What is the probability that a randomly selected person visited either the Sears Tower or Wrigley Field but NOT both?

P(S or W but NOT both) = P(S or W) – P(S and W)

= 0.8 – 0.3 = 0.5

Example (continued)

What is the probability that a randomly selected tourist went to neither location?

$$P(\text{neither location}) = 1 - P(\text{either location})$$

$$= 1 - P(S \text{ or } W)$$

$$= 1 - 0.8 = 0.2$$

Law of Large Numbers (LLN)

- The long-run relative frequency of repeated independent events gets closer and closer to the true relative frequency as the number of trials increases
- Example – an infinite number of coin flips should produce heads 50% of the time
- If probabilities remain the same and the events are independent, the probability of the next trial is always the same, no matter what has happened up to then