Constrained Optimization

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Economics 300

Constraints

- Limited budget
- Limited revenue
- Limited hours to work
- ….

- Scarce resources create constraints

Constrained optimization

- Includes an objective function and constraints

- Choose variables (x_1,x_2) to maximize (or minimize) an objective function $f(x_1,x_2)$ subject to constraints

Consumption problem

- You have \$6.00 to spend on a lunch of soup and veggies
- Restaurant sells both soup and veggies by weight
  - An ounce of soup (S) is \$0.25
  - An ounce of veggies (V) costs \$0.50
- How many ounces of each will you purchase?

Budget constraint

Budget = $6

Price of soup (P_S)= 0.25

Price of veggies (P_V)= 0.50

If you spend all on soup, then you can buy 6/.25 =24 ounces of soup

If you spend all on veggies, then you can buy 6/.5 = 12 ounces of veggies

Budget constraint

$$P_S\, S + P_V\, V = \text{Budget}$$

Budget = 6

Price of soup (P_S)= 0.25

Price of veggies (P_V)= 0.50

$$0.25\, S + 0.5\, V = 6$$

Objective

- Utility you derive from consumption of soup and veggies is

$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

- Choose (S,V) to maximize

$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

Subject to

$$0.25S + 0.5V = 6$$

Substitution method

- Start with the constraint

$$0.25S + 0.5V = 6$$

$$0.25S = 6 - 0.5V$$

$$S = 24 - 2V$$

Substitution method

- S = 24 – 2V
- Substitute into the utility function

$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

$$u(V) = \frac{1}{2}\ln(24 - 2V) + \frac{1}{2}\ln(V)$$

- Now the problem is a univariate maximization
- Choose V to maximize

$$u(V) = \frac{1}{2}\ln(24-2V) + \frac{1}{2}\ln(V)$$

$$u(V) = \frac{1}{2}\ln(24-2V) + \frac{1}{2}\ln(V)$$

- FOC

$$\frac{du}{dV} = \frac{-2}{2(24-2V)} + \frac{1}{2V} = 0$$

$$2V = 24 - 2V$$

$$\boxed{V^* = 6}$$

We know that S = 24 – 2V, so $\boxed{S^* = 12}$

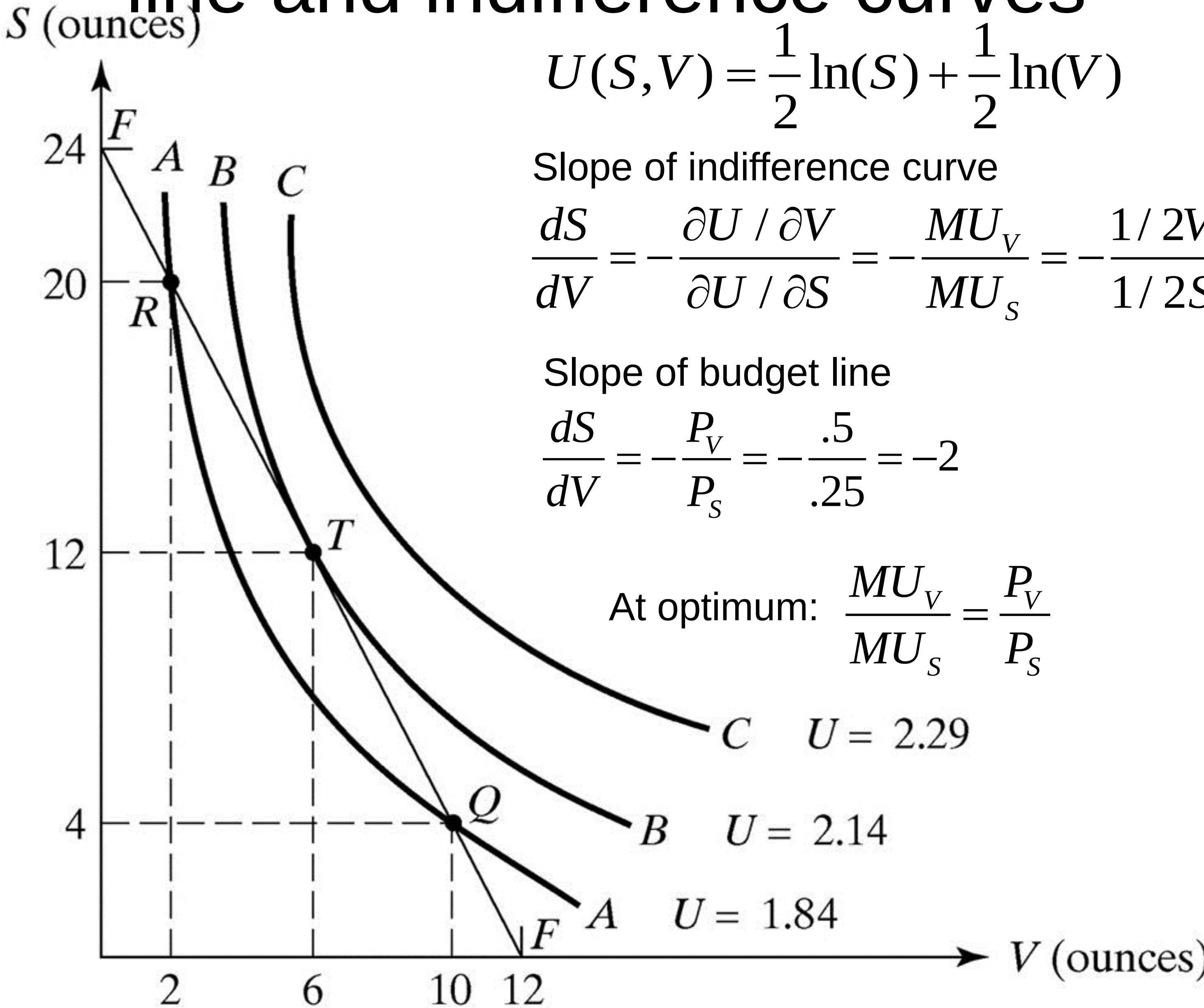
- SOC

$$\frac{du}{dV} = \frac{-1}{(24-2V)} + \frac{1}{2V}$$

$$\frac{d^2u}{dV^2} = \frac{-2}{(24-2V)^2} - \frac{1}{2V^2} < 0$$

- Hence, maximum

Constrained optimization with a budget line and indifference curves



$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

Slope of indifference curve

$$\frac{dS}{dV} = -\frac{\partial U / \partial V}{\partial U / \partial S} = -\frac{MU_V}{MU_S} = -\frac{1/2V}{1/2S} = -\frac{S}{V}$$

Slope of budget line

$$\frac{dS}{dV} = -\frac{P_V}{P_S} = -\frac{.5}{.25} = -2$$

At optimum: $\dfrac{MU_V}{MU_S} = \dfrac{P_V}{P_S}$

Exercise

$$y = x^2 + 2xz + 4z^2$$

subject to $\quad x + z = 8$

$$y = x^2 + 2xz + 4z^2$$

subject to $\quad x + z = 8$

$$x + z = 8$$

$$z = 8 - x$$

Then,

$$y = x^2 + 2x(8 - x) + 4(8 - x)^2$$

$$\mathrm{y} = x^2 + 16x - 2x^2 + 4(8 - x)^2$$

$$y = x^2 + 16x - 2x^2 + 4(8 - x)^2$$

$$\frac{dy}{dx} = 2x + 16 - 4x - 8(8 - x) = 0$$

$$16 - 2x - 64 + 8x = 0$$

$$\boxed{x^* = 8}$$

$$z = 8 - x \rightarrow \boxed{z^* = 0}$$

$$\frac{dy}{dx} = 2x + 16 - 4x - 8(8 - x)$$

$$\frac{d^2y}{dx^2} = 2 - 4 + 8 = 6 > 0$$

$\underline{|Min.}$

Exercise

$$y = 10x + 40z$$

subject to $x^{1/2} z^{1/2} = 2$

$$y = 10x + 40z$$

subject to $\mathrm{x}^{1/2} z^{1/2} = 2$

$$\mathrm{x}^{1/2} z^{1/2} = 2$$

$$z^{1/2} = \frac{2}{\mathrm{x}^{1/2}}$$

$$\boxed{z = \frac{4}{x}}$$

Then,

$$y = 10x + 40z$$

$$y = 10x + \frac{160}{x}$$

$$y = 10x + \frac{160}{x}$$

$$\frac{dy}{dx} = 10 - \frac{160}{x^2} = 0$$

$$10 = \frac{160}{x^2} \rightarrow x^2 = 16$$

$$\boxed{x^* = 4} \text{ or } \boxed{x^* = -4}$$

$$z = \frac{4}{x} \rightarrow \boxed{(4,1)} \text{ or } \boxed{(-4,-1)}$$

$$\frac{dy}{dx} = 10 - \frac{160}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{320}{x^3}$$

$$\text{At } x^* = 4, \ \frac{d^2y}{dx^2}(4) = \frac{320}{4^3} > 0 \rightarrow (4,1) \text{ minimum}$$

$$\text{At } x^* = -4, \ \frac{d^2y}{dx^2}(-4) = \frac{320}{(-4)^3} < 0 \rightarrow (-4,-1) \text{ maximum}$$

Constrained optimization

Choose (x_1, x_2) to maximize (or to minimize)

$$f(x_1, x_2)$$

subject to

$$g(x_1, x_2) = c$$

Substitution Method

1. Start with constraint (solve for one of the variables in terms of the other)
2. Substitute it into the objective function
3. Now the problem is a univariate optimization
4. FOC & SOC

- What if the constraint is very complicated or we have more than one constraint?

- Lagrange method

Lagrange method – bivariate case

The Lagrangian function for a constraint maximization (or minimization) problem with the objective function $f(x_1, x_2)$ subject to the equality constraints that

$$g(x_1, x_2) = c \text{ is}$$

$$\boxed{\mathfrak{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(g(x_1, x_2) - c\right)}$$

Lagrange method – bivariate case

$$\mathfrak{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(g(x_1, x_2) - c\right)$$

$$\boxed{\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1} = 0}$$

$$\boxed{\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2} = 0}$$

$$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial \lambda} = 0 \rightarrow \boxed{g(x_1, x_2) = c}$$

Example

- Recall the soup and veggies problem
- Choose (S,V) to maximize

$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

Subject to

$$6 = 0.25S + 0.5V$$

$$\mathfrak{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(g(x_1, x_2) - c\right)$$

$$\mathfrak{L}(S, V, \lambda) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V) - \lambda\left(0.25S + 0.5V - 6\right)$$

$$\mathfrak{L}(S,V,\lambda)=\frac{1}{2}\ln(S)+\frac{1}{2}\ln(V)-\lambda\left(0.25S+0.5V-6\right)$$

$$\frac{\partial\mathfrak{L}(S,V,\lambda)}{\partial S}=0$$

$$\frac{\partial\mathfrak{L}(S,V,\lambda)}{\partial V}=0$$

$$0.25S+0.5V=6$$

$$\mathfrak{L}(S,V,\lambda) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V) - \lambda\left(0.25S + 0.5V - 6\right)$$

$$\frac{\partial\mathfrak{L}(S,V,\lambda)}{\partial S} = 0 \rightarrow \frac{1}{2S} - 0.25\lambda = 0$$

$$\frac{\partial\mathfrak{L}(S,V,\lambda)}{\partial V} = 0 \rightarrow \frac{1}{2V} - 0.5\lambda = 0$$

$$0.25S + 0.5V = 6$$

Three equations, three unknowns

$$\frac{\partial \mathfrak{L}(S, V, \lambda)}{\partial S} = 0 \rightarrow \frac{1}{2S} - 0.25\lambda = 0 \rightarrow \boxed{S = \frac{2}{\lambda}}$$

$$\frac{\partial \mathfrak{L}(S, V, \lambda)}{\partial V} = 0 \rightarrow \frac{1}{2V} - 0.5\lambda = 0 \rightarrow \boxed{V = \frac{1}{\lambda}}$$

$$0.25S + 0.5V = 6$$

$$S = \frac{2}{\lambda}$$

$$V = \frac{1}{\lambda}$$

$$0.25S + 0.5V = 6 \rightarrow 0.25\frac{2}{\lambda} + 0.5\frac{1}{\lambda} = 6$$

$$S = \frac{2}{\lambda}$$

$$V = \frac{1}{\lambda}$$

$$0.25S + 0.5V = 6 \rightarrow 0.25\frac{2}{\lambda} + 0.5\frac{1}{\lambda} = 6$$

$$\rightarrow \boxed{\lambda^* = \frac{1}{6}}$$

$$\boxed{S^* = 12} \qquad \boxed{V^* = 6}$$

Lagrangian function with multiple constraints

Recall the previous example:

Budget = 6

Price of soup (P_S)= 1/4

Price of veggies (P_V)= 1/2

Price of Juice (P_J)= 1/12

Budget constraint:

$$(1/4)S + (1/2)V + (1/12)J = 6$$

Also, you want to consume 24 ounces of liquid, i.e.

S + J = 24

And the utility function is given as

$$U(S,V,J) = \frac{1}{3}\ln(S) + \frac{1}{3}\ln(V) + \frac{1}{3}\ln(J)$$

Choose (S,V,J) to maximize

$$U(S,V,J) = \frac{1}{3}\ln(S) + \frac{1}{3}\ln(V) + \frac{1}{3}\ln(J)$$

subject to

(budget constraint) (1/4)S + (1/2)V + (1/12)J = 6 (λ)

(liquidity constraint) S + J = 24 (μ)

$$\mathfrak{L}(S,V,J,\lambda,\mu) =$$

$$\frac{1}{3}\ln(S)+\frac{1}{3}\ln(V)+\frac{1}{3}\ln(J)-\lambda\left(\frac{1}{4}S+\frac{1}{2}V+\frac{1}{12}J-6\right)-\mu(S+J-24)$$

$$\mathfrak{L}(S, V, J, \lambda, \mu) =$$

$$\frac{1}{3}\ln(S) + \frac{1}{3}\ln(V) + \frac{1}{3}\ln(J) - \lambda\left(\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6\right) - \mu(S + J - 24)$$

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial J} = \frac{1}{3J} - \frac{\lambda}{12} - \mu = 0$$

$$\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J = 6$$

$$S + J = 24$$

5 equations, 5 unknowns (S,V,J,λ,μ)

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0 \rightarrow \lambda = 4\left(\frac{1}{3S} - \mu\right)$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0 \rightarrow \boxed{\lambda = \frac{2}{3V}}$$

So,

$$\frac{2}{3V} = 4\left(\frac{1}{3S} - \mu\right) \rightarrow \boxed{\mu = \frac{1}{3S} - \frac{1}{6V}}$$

$$\frac{\partial \mathfrak{L}}{\partial J} = \frac{1}{3J} - \frac{\lambda}{12} - \mu = 0 \rightarrow \frac{1}{3J} - \frac{1}{12}\frac{2}{3V} - \left(\frac{1}{3S} - \frac{1}{6V}\right) = 0$$

$$\frac{1}{3J} - \frac{1}{18V} - \frac{1}{3S} + \frac{1}{6V} = 0$$

$$\frac{-1+3}{18V} = \frac{1}{3S} - \frac{1}{3J}$$

$$\frac{1}{9V} = \frac{J-S}{3SJ} \rightarrow \boxed{V = \frac{SJ}{3(J-S)}}$$

$$S + J = 24 \rightarrow \boxed{S = 24 - J}$$

$$\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J = 6 \rightarrow$$

$$\frac{1}{4}(24 - J) + \frac{1}{2}\left(\frac{SJ}{3(J - S)}\right) + \frac{1}{12}J = 6$$

$$\boxed{J^* = 16}$$

$$S = 24 - J \rightarrow \boxed{S^* = 8}$$

$$V = \frac{SJ}{3(S - J)} \rightarrow \boxed{V^* = \frac{16}{3}}$$

Constrained optimization, part 3

- Substitution method
- Lagrange method

Envelope theorem

The change in the value function when a parameter changes is equal to the derivative of the Lagrangian function with respect to the parameter, evaluated at the optimum choices.

Example 1
Lagrange multiplier as shadow price

Recall Lagrange Method – bivariate case

The Lagrangian function for constrained maximization (or minimization) problem with the objective function $f(x_1, x_2)$ subject to the equality constraints that $g(x_1, x_2) = c$ is

$$\boxed{\mathfrak{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(g(x_1, x_2) - c\right)}$$

$$\boxed{\mathfrak{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(g(x_1, x_2) - c\right)}$$

$$\frac{\partial f}{\partial c} = \frac{\partial \mathfrak{L}}{\partial c} = \lambda^*$$

- Recall the Soup & veggies example

Budget (c) = 6

Price of soup (P_S)= 0.25

Price of veggies (P_V)= 0.50

$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

What is the marginal effect of budget changes in the utility?

$$\frac{\partial U}{\partial c} = ?$$

- Utility from the consumption of soup and veggies is

$$U(S,V) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V)$$

$$\mathfrak{L}(S,V,\lambda) = \frac{1}{2}\ln(S) + \frac{1}{2}\ln(V) - \lambda\left(P_S S + P_V V - c\right)$$

$$\boxed{\lambda^* = \frac{1}{6}}$$

$$\boxed{S^* = 12} \qquad \boxed{V^* = 6}$$

$$\frac{\partial U}{\partial c} = \frac{\partial \mathfrak{L}}{\partial c} = \lambda^* = \frac{1}{6}$$

Example 2
Hotelling's lemma

- A firm uses two inputs x_1 and x_2, that cost w_1 and w_2, to produce good y, which sells for price p
- Production function is $g(x_1, x_2)$
- How does profit change as prices change?

- Firm chooses x_1 and x_2 to maximize its profit

$$\pi(p, w_1, w_2) = py - w_1 x_1 - w_2 x_2$$

$$\text{subject to } g(x_1, x_2) = y$$

$$\boxed{\mathfrak{L}(p,w_1,w_2) = py - w_1x_1 - w_2x_2 - \lambda\big(g(x_1,x_2) - y\big)}$$

$$\frac{\partial\pi(p,w_1,w_2)}{\partial p} = \frac{\partial\mathfrak{L}(p,w_1,w_2)}{\partial p} = y^* > 0$$

$$\frac{\partial\pi(p,w_1,w_2)}{\partial w_1} = \frac{\partial\mathfrak{L}(p,w_1,w_2)}{\partial w_1} = -x_1^* < 0$$

$$\frac{\partial\pi(p,w_1,w_2)}{\partial w_2} = \frac{\partial\mathfrak{L}(p,w_1,w_2)}{\partial w_2} = -x_2^* < 0$$

Optimization with inequality constraints Kuhn-Tucker method

Choose x and y to maximize

$$f(x,y) = x - \frac{x^2}{2} + y^2$$

subject to

$$\frac{x^2}{2} + y^2 \leq \frac{9}{8} \qquad (\lambda)$$

$$-y \leq 0 \qquad (\mu)$$

$$\mathfrak{L} = x - \frac{x^2}{2} + y^2 - \lambda\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) - \mu(-y)$$

$$\frac{\partial \mathfrak{L}}{\partial x} = 1 - x - \lambda x = 0$$

$$\frac{\partial \mathfrak{L}}{\partial y} = 2y - 2\lambda y + \mu = 0$$

$$\frac{x^2}{2} + y^2 - \frac{9}{8} \leq 0 \text{ and } \lambda \geq 0$$

$$-y \leq 0 \text{ and } \mu \geq 0$$

$$\lambda\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) = 0$$

$$\mu(-y) = 0$$

Complementary slackness conditions:

1. If price > 0, then constraint is binding
2. If constraint slack, price = 0

Solution

- There are 4 cases:

$$\text{Case 1: } \lambda = 0 \text{ and } \mu = 0$$

$$\text{Case 2: } \lambda \neq 0 \text{ and } \mu = 0$$

$$\text{Case 3: } \lambda = 0 \text{ and } \mu \neq 0$$

$$\text{Case 4: } \lambda \neq 0 \text{ and } \mu \neq 0$$

Case 1: $\lambda = 0$ and $\mu = 0$

$$\frac{\partial \mathfrak{L}}{\partial x} = 1 - x - \lambda x = 0 \text{ becomes } 1 - x = 0 \rightarrow \boxed{x^* = 1}$$

$$\frac{\partial \mathfrak{L}}{\partial y} = 2y - 2\lambda y + \mu = 0 \text{ becomes } 2y = 0 \rightarrow \boxed{y^* = 0}$$

Check if (1,0) satisifes other conditions

$$\frac{x^2}{2} + y^2 - \frac{9}{8} = \frac{1}{2} - \frac{9}{8} \le 0$$

$$-y = 0 \le 0$$

So, (1,0) is a candidate for max.

Case 2: $\lambda \neq 0$ and $\mu = 0$

$$\frac{\partial \mathfrak{L}}{\partial y} = 2y - 2\lambda y + \mu = 0 \text{ becomes } 2y - 2\lambda y = 0 \rightarrow \boxed{\lambda^* = 1} \text{ provided that } y \neq 0$$

$$\frac{\partial \mathfrak{L}}{\partial x} = 1 - x - \lambda x = 0 \rightarrow 1 - x - x = 0 \rightarrow \boxed{x^* = \frac{1}{2}}$$

$$\lambda\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) = 0 \rightarrow \frac{x^2}{2} + y^2 - \frac{9}{8} = 0 \text{ since } \lambda \neq 0$$

$$\frac{(1/2)^2}{2} + y^2 - \frac{9}{8} = 0 \rightarrow \boxed{y^* = 1}$$

Check if $(\frac{1}{2},1)$ satisifes remaining condition

$$-y = -1 \leq 0$$

So, $(\frac{1}{2},1)$ is a candidate for max.

Case 3: $\lambda = 0$ and $\mu \neq 0$

$$\frac{\partial \mathfrak{L}}{\partial x} = 1 - x - \lambda x = 0 \text{ becomes } 1 - x = 0 \rightarrow \boxed{x^* = 1}$$

$$\mu(-y) = 0 \text{ implies that } \boxed{y^* = 0} \text{ since } \mu \neq 0$$

$$\frac{\partial \mathfrak{L}}{\partial y} = 2y - 2\lambda y + \mu = 0 \text{ becomes } \mu = 0 \text{ not possible!!}$$

So, no candidate from this case.

Case 4: $\lambda \neq 0$ and $\mu \neq 0$

$\mu(-y) = 0$ implies $\boxed{y^* = 0}$ since $\mu \neq 0$

$\frac{\partial \mathfrak{L}}{\partial y} = 2y - 2\lambda y + \mu = 0$ becomes $\mu = 0$ not possible!!!

No candidate from this case.

Two candidates (1,0) and $(\frac{1}{2},1)$

$$f(x,y) = x - \frac{x^2}{2} + y^2$$

$$f(1,0) = 1 - \frac{1}{2} + 0 = \frac{1}{2}$$

$$f(\frac{1}{2},1) = \frac{1}{2} - \frac{(1/2)^2}{2} + 1 = \frac{11}{8}$$

$$f(\frac{1}{2},1) = \frac{11}{8} > \frac{1}{2} = f(1,0)$$

Hence, the max. is $(\frac{1}{2},1)$.

- At the optimal solution (1/2,1):

$$\frac{(1/2)^2}{2} + 1^2 = \frac{9}{8} \qquad \text{(binding)}$$

$$-1 \le 0 \qquad \text{(nonbinding)}$$

Exercise

Choose (S,V,J) to maximize

$$U(S,V,J)=\frac{1}{3}\ln(S)+\frac{1}{3}\ln(V)+\frac{1}{3}\ln(J)$$

subject to

(budget constraint) $(1/4)S+(1/2)V+(1/12)J \le 6$ (λ)

(liquidity constraint) $S+J \le 40$ (μ)

Set up the Lagrangian function for this problem

$$\mathfrak{L}(S, V, J, \lambda, \mu) =$$

$$\frac{1}{3}\ln(S) + \frac{1}{3}\ln(V) + \frac{1}{3}\ln(J) - \lambda\left(\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6\right) - \mu(S + J - 40)$$

$$\mathfrak{L}(S, V, J, \lambda, \mu) =$$

$$\frac{1}{3}\ln(S) + \frac{1}{3}\ln(V) + \frac{1}{3}\ln(J) - \lambda\left(\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6\right) - \mu(S + J - 40)$$

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial J} = \frac{1}{3J} - \frac{\lambda}{12} - \mu = 0$$

$$\lambda\left(\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6\right) = 0$$

$$\mu(S + J - 40) = 0$$

$$\lambda \geq 0,\ \ \mu \geq 0, \quad \frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J \leq 6, \quad S + J \leq 40$$

Solution

- There are 4 cases:

$$\text{Case 1: } \lambda = 0 \text{ and } \mu = 0$$

$$\text{Case 2: } \lambda \neq 0 \text{ and } \mu = 0$$

$$\text{Case 3: } \lambda = 0 \text{ and } \mu \neq 0$$

$$\text{Case 4: } \lambda \neq 0 \text{ and } \mu \neq 0$$

Case 1: λ=0 and μ=0

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0 \text{ becomes } \frac{1}{3S} = 0 \text{ not possible!!!}$$

Case 2: $\lambda \neq 0$ and μ=0

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0 \text{ becomes } \frac{1}{3S} - \frac{\lambda}{4} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial J} = \frac{1}{3J} - \frac{\lambda}{12} - \mu = 0 \text{ becomes } \frac{1}{3J} - \frac{\lambda}{12} = 0$$

$$\lambda\left(\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6\right) = 0 \text{ becomes } \frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6 = 0$$

$$\frac{1}{3S} - \frac{\lambda}{4} = 0 \text{ implies that } S = \frac{4}{3\lambda}$$

$$\frac{1}{3V} - \frac{\lambda}{2} = 0 \text{ implies that } V = \frac{2}{3\lambda}$$

$$\frac{1}{3J} - \frac{\lambda}{12} = 0 \text{ implies that } J = \frac{4}{\lambda}$$

Then,

$$\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6 = 0 \text{ becomes } \frac{1}{4}\frac{4}{3\lambda} + \frac{1}{2}\frac{2}{3\lambda} + \frac{1}{12}\frac{4}{\lambda} - 6 = 0$$

$$\frac{1}{3\lambda} + \frac{1}{3\lambda} + \frac{1}{3\lambda} - 6 = 0, \quad \frac{1}{\lambda} = 6, \boxed{\lambda^* = \frac{1}{6}}$$

$$\boxed{\lambda^* = \frac{1}{6}}$$

$$S = \frac{4}{3\lambda}, \boxed{S^* = 8}$$

$$V = \frac{2}{3\lambda}, \boxed{V^* = 4}$$

$$J = \frac{4}{\lambda}, \boxed{J^* = 24}$$

Check that S+J $\leq$ 40

This is the optimum, since second constraint is slack!

Case 3: $\lambda = 0$ and $\mu \neq 0$

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0 \text{ becomes } \frac{1}{3S} - \mu = 0$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0 \text{ becomes } \frac{1}{3V} = 0 \text{ not possible!!!}$$

Case 4: $\lambda \neq 0$ and $\mu \neq 0$

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial J} = \frac{1}{3J} - \frac{\lambda}{12} - \mu = 0$$

$\lambda\left(\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6\right) = 0$ becomes $\frac{1}{4}S + \frac{1}{2}V + \frac{1}{12}J - 6 = 0$

$\mu(S + J - 40) = 0$ becomes S+J=40

5 equations, 5 unknowns (S,V,J,λ,μ)

$$\frac{\partial \mathfrak{L}}{\partial S} = \frac{1}{3S} - \frac{\lambda}{4} - \mu = 0 \rightarrow \lambda = 4\left(\frac{1}{3S} - \mu\right)$$

$$\frac{\partial \mathfrak{L}}{\partial V} = \frac{1}{3V} - \frac{\lambda}{2} = 0 \rightarrow \boxed{\lambda = \frac{2}{3V}}$$

So,

$$\frac{2}{3V} = 4\left(\frac{1}{3S} - \mu\right) \rightarrow \boxed{\mu = \frac{1}{3S} - \frac{1}{6V}}$$

Two messy solutions (neither possible)

$$\lambda = \frac{1+\sqrt{19}}{24}$$

$$\mu = \frac{3-\sqrt{19}}{160} < 0$$

$$S = \frac{8}{3}(7-\sqrt{19})$$

$$J = \frac{8}{3}(8+\sqrt{19})$$

$$V = \frac{8}{9}(-1+\sqrt{19})$$

$$\lambda = \frac{1-\sqrt{19}}{24} < 0$$

$$\mu = \frac{3+\sqrt{19}}{160}$$

$$S = \frac{8}{3}(7+\sqrt{19})$$

$$J = \frac{8}{3}(8+\sqrt{19})$$

$$V = \frac{8}{9}(-1-\sqrt{19})$$

Unique solution

$$\boxed{\lambda^* = \frac{1}{6}}$$

$$S = \frac{4}{3\lambda}, \boxed{S^* = 8}$$

$$V = \frac{2}{3\lambda}, \boxed{V^* = 4}$$

$$J = \frac{4}{\lambda}, \boxed{J^* = 24}$$

$$\boxed{\mu^* = 0}$$

Note: S+J $<$ 40