Extreme Values of Multivariate Functions

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Economics 300
Extreme values of multivariate functions

• In economics many problems reflect a need to choose among multiple alternatives
  – Consumers decide on consumption bundles
  – Producers choose a set of inputs
  – Policy-makers may choose several instruments to motivate behavior

• We now generalize the univariate techniques
Stationary points and tangent planes of bivariate functions

\[ g(x_1, x_2) = 6x_1 - x_1^2 + 16x_2 - 4x_2^2 \]

\[ h(x_1, x_2) = x_1^2 + 4x_2^2 - 2x_1 - 16x_2 + x_1x_2 \]
Slices of a bivariate function

\[ g = 6x_1 - x_1^2 + 16x_2 - 4x_2^2 \]

\[ g_1 = 6 - 2x_1 = 0 \]

\[ g_2 = 16 - 8x_2 = 0 \]
Multivariate first-order condition

- If $f(x_1, x_2, ..., x_n)$ is differentiable with respect to each of its arguments and reaches a maximum or a minimum at the stationary point, $(x_1^*, ..., x_n^*)$, then each of the partial derivatives evaluated at that point equals zero, i.e.

$$f_1(x_1^*, ..., x_n^*) = 0$$

$$...$$

$$...$$

$$...$$

$$f_n(x_1^*, ..., x_n^*) = 0$$
Second-order condition in the bivariate case $f(x_1, x_2)$

**First total differential**

$$y = f(x_1, x_2)$$

$$dy = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$$

i.e.

$$dy = f_1dx_1 + f_2dx_2$$
Second-order condition in the bivariate case \( f(x_1, x_2) \)

**Second total differential**

\[
d^2 y = \frac{\partial [dy]}{\partial x_1} dx_1 + \frac{\partial [dy]}{\partial x_2} dx_2
\]

\[
= \frac{\partial [f_1 dx_1 + f_2 dx_2]}{\partial x_1} dx_1 + \frac{\partial [f_1 dx_1 + f_2 dx_2]}{\partial x_2} dx_2
\]

\[
= f_{11} \cdot (dx_1)^2 + 2 f_{12} \cdot (dx_1 \cdot dx_2) + f_{22} \cdot (dx_2)^2
\]
Extreme values and multivariate functions

Sufficient condition for a local maximum (minimum)

- If the second total derivative evaluated at a stationary point of a function $f(x_1,x_2)$ is negative (positive) for any $dx_1$ and $dx_2$, then that stationary point represents a local maximum (minimum) of the function.
Extreme values and multivariate functions

Sufficient Condition for a Local Minimum:

\[ d^2 y > 0 \text{ if } f_{11} > 0 \text{ and } f_{22} - \frac{(f_{12})^2}{f_{11}} > 0 \]

Sufficient Condition for a Local Minimum:

\[ d^2 y > 0 \text{ if } f_{11} > 0 \text{ and } f_{11}f_{22} > f_{12}^2 \]
Extreme values and multivariate functions

Sufficient Condition for a Local Maximum:

\[ d^2y < 0 \text{ if } f_{11} < 0 \text{ and } f_{22} - \frac{(f_{12})^2}{f_{11}} < 0 \]

Sufficient Condition for a Local Maximum:

\[ d^2y < 0 \text{ if } f_{11} < 0 \text{ and } f_{11}f_{22} > f_{12}^2 \]
Extreme values of multivariate functions – bivariate case

- Choose \((x_1,x_2)\) to maximize (or to minimize) \(f(x_1,x_2)\)
First Order Conditions:

\[ f_1(x_1, x_2) = 0 \quad \text{and} \quad f_2(x_1, x_2) = 0 \]

stationary points

\( (x_1^*, x_2^*) \)
Second Order Conditions

**Local Minimum if**
\[ f_{11}(x_1^*, x_2^*) > 0 \]

**and**
\[ f_{11}(x_1^*, x_2^*) f_{22}(x_1^*, x_2^*) > \left( f_{12}(x_1^*, x_2^*) \right)^2 \]

**Local Maximum if**
\[ f_{11}(x_1^*, x_2^*) < 0 \]

**and**
\[ f_{11}(x_1^*, x_2^*) f_{22}(x_1^*, x_2^*) > \left( f_{12}(x_1^*, x_2^*) \right)^2 \]
Exercises

• Choose \((x_1,x_2)\) to minimize

\[ f(x_1, x_2) = 4x_1 + 2x_2^2 + x_1^2 + x_2 \]
\[ f(x_1, x_2) = 4x_1 + 2x_2^2 + x_1^2 + x_2 \]

**FOC:**

\[ f_1 = 4 + 2x_1 = 0 \quad \Rightarrow \quad x_1^* = -2 \]

\[ f_2 = 4x_2 + 1 = 0 \quad \Rightarrow \quad x_2^* = \frac{-1}{4} \]
\[ f_1 = 4 + 2x_1 \]
\[ f_2 = 4x_2 + 1 \]

**SOC**: We need to find \( f_{11}, f_{12}, f_{22} \)

If \( f_{11} > 0 \) and \( f_{11} \cdot f_{22} > (f_{12})^2 \), then local min
\[ f_1 = 4 + 2x_1 \]
\[ f_2 = 4x_2 + 1 \]

**SOC:**

\[ f_{11} = 2 \]
\[ f_{12} = 0 \]
\[ f_{22} = 4 \]

Observe that \( f_{11} \)

\[ f_{11} \cdot f_{22} = 2(4) = 8 > 0 = (f_{12})^2 \]

Hence, \((−2, \frac{-1}{4})\) is local minimum.
Exercise 2

• Find the local max and local min of

\[ f(x_1, x_2) = 8x_1 - 7x_2^2 - x_1^2 + 14x_2 \]
$f(x_1, x_2) = 8x_1 - 7x_2^2 - x_1^2 + 14x_2$

$FOC:\n\begin{align*}
f_1 &= 8 - 2x_1 = 0 \rightarrow x_1^* = 4 \\
f_2 &= -14x_2 + 14 = 0 \rightarrow x_2^* = 1
\end{align*}$
\( f_1 = 8 - 2x_1 \)
\( f_2 = -14x_2 + 14 \)

**SOC:**
\( f_{11} = -2 \)
\( f_{12} = 0 \)
\( f_{22} = -14 \)

Observe that \( f_{11} = -2 < 0 \) and
\[ f_{11} \cdot f_{22} = (-2)(-14) = 28 > 0 = (f_{12})^2 \]
Hence, \((4,1)\) is local max.
Exercise 3

• Find the local max and local min of

\[ f(x_1, x_2) = -2x_1 + 4x_2^2 + x_1^2 - 16x_2 + x_1x_2 \]
\[ f(x_1, x_2) = -2x_1 + 4x_2^2 + x_1^2 - 16x_2 + x_1x_2 \]

**FOC:**

\[ f_1 = -2 + 2x_1 + x_2 = 0 \]

\[ f_2 = 8x_2 - 16 + x_1 = 0 \]
\[ f_1 = -2 + 2x_1 + x_2 = 0 \]
\[ f_2 = 8x_2 - 16 + x_1 = 0 \]

\[-2 + 2x_1 + x_2 = 0 \rightarrow x_2 = 2 - 2x_1\]

\[8(2 - 2x_1) - 16 + x_1 = 0 \rightarrow x_1^* = 0\]

\[x_2^* = 2\]
$f_1 = -2 + 2x_1 + x_2$

$f_2 = 8x_2 - 16 + x_1$

$SOC:$

$f_{11} = 2$

$f_{12} = 1$

$f_{22} = 8$

Observe that $f_{11} = 2 > 0$ and

$f_{11} \cdot f_{22} = (2)(8) = 16 > 1 = (f_{12})^2$

Hence, $(0, 2)$ is local min.
Exercise 4

• Find the local max and local min of

\[ f(x_1, x_2) = -x_1 - \frac{1}{8} x_2^2 - \frac{1}{2} x_1^2 + x_2 + x_1 x_2 \]
\[ f(x_1, x_2) = -x_1 - \frac{1}{8} x_2^2 - \frac{1}{2} x_1^2 + x_2 + x_1 x_2 \]

**FOC:**

\[ f_1 = -1 - x_1 + x_2 = 0 \]

\[ f_2 = -\frac{1}{4} x_2 + 1 + x_1 = 0 \]
\[ f_1 = -1 - x_1 + x_2 = 0 \]
\[ f_2 = -\frac{1}{4} x_2 + 1 + x_1 = 0 \]

\[-1 - x_1 + x_2 = 0 \rightarrow x_2 = 1 + x_1 \]

\[-\frac{1}{4} (1 + x_1) + 1 + x_1 = 0 \rightarrow 1 + x_1 - 4 - 4x_1 = 0 \]

\[ x_1^* = -1 \]

\[ x_2^* = 0 \]
\[ f_1 = -1 - x_1 + x_2 \]
\[ f_2 = -\frac{1}{4} x_2 + 1 + x \]

**SOC:**
\[ f_{11} = -1 \]
\[ f_{12} = 1 \]
\[ f_{22} = -\frac{1}{4} \]

Observe that \( f_{11} = -1 < 0 \) and
\[ f_{11} \cdot f_{22} = (-1)(-\frac{1}{4}) = \frac{1}{4} < 1 = (f_{12})^2 \]

Hence, no concl.
Exercise 6

- Find the local max and local min of

\[ f(x_1, x_2) = -\frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 + x_2 \]
\[
f(x_1, x_2) = -\frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 + x_2
\]

**FOC:**

\[
f_1 = -x_1^2 = 0 \rightarrow x_1^* = 0
\]

\[
f_2 = -x_2 + 1 = 0 \rightarrow x_2^* = 1
\]
\[ f_1 = -x_1^2 \]
\[ f_2 = -x_2 + 1 \]

\text{SOC:}
\[ f_{11} = -2x_1 \]
\[ f_{12} = 0 \]
\[ f_{22} = -1 \]
At $(0,1)$

\[ \begin{align*}
SOC:\quad & f_{11} = -2x_1 = 0 \\
& f_{12} = 0 \\
& f_{22} = -1
\end{align*} \]

Observe that $f_{11} = 0$ and

\[ f_{11} \cdot f_{22} = (0)(-1) = 0 = 0 = (f_{12})^2 \]

Hence, no concl.
Exercise 7

• Find the local max and local min of

\[ f(x_1, x_2) = x_1 - \frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 + x_2 \]
\[ f(x_1, x_2) = x_1 - \frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 + x_2 \]

**FOC:** 

\[ f_1 = 1 - x_1^2 = 0 \quad \Rightarrow \quad x_1^* = -1 \quad \text{or} \quad x_1^* = 1 \]

\[ f_2 = -x_2 + 1 = 0 \quad \Rightarrow \quad x_2^* = 1 \]

Two stationary points \((-1,1)\) and \((1,1)\)
\[ f_1 = 1 - x_1^2 \]
\[ f_2 = -x_2 + 1 \]

**SOC:**
\[ f_{11} = -2x_1 \]
\[ f_{12} = 0 \]
\[ f_{22} = -1 \]
At \((1,1)\)

**SOC**: 

\[f_{11} = -2x_1 = -2\]  
\[f_{12} = 0\]  
\[f_{22} = -1\]

Observe that \(f_{11} = -2 < 0\) and  
\[f_{11} \cdot f_{22} = (-2)(-1) = 2 > 0 = (f_{12})^2\]

Hence, \((1,1)\) is local max.
At \((-1, 1)\)

**SOC**: 

\[ f_{11} = -2x_1 = -2(-1) = 2 \]
\[ f_{12} = 0 \]
\[ f_{22} = -1 \]

Observe that \(f_{11} = 2 > 0\) and 
\[ f_{11} \cdot f_{22} = (2)(-1) = -2 < 0 = (f_{12})^2 \]

Hence, at \((-1, 1)\) no concl.