Multivariate Calculus

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Economics 300

Multivariate calculus

- Calculus with single variable (univariate)

$$y = f(x)$$

- Calculus with many variables (multivariate)

$$y = f(x_1, x_2, ..., x_n)$$

Partial derivatives

- With single variable, derivative is change in y in response to an infinitesimal change in x
- With many variables, partial derivative is change in y in response to an infinitesimal change in a single variable x_i (hold all else fixed)
- Total derivative is change in all variables at once

Cobb-Douglas production function

$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$

- How does production change in L?

$$\frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}}$$

- Marginal product of labor (MPL)
- Partial derivatives use ∂ instead of d

Cobb-Douglas production function

$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$

- How does production change in K?

$$\frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}}$$

- Marginal product of capital (MPK)
- Partial derivatives use ∂ instead of d

Cobb-Douglas production function

$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$

- Note $$MPL = \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} > 0$$

$$MPK = \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}} > 0$$

- Produce more with more labor, holding capital fixed
- Produce more with more capital, holding labor fixed

Second-order partial derivatives

- Differentiate first-order partial derivatives

$$\frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \qquad \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}}$$

$$\frac{\partial}{\partial L}\left(\frac{\partial Q}{\partial L}\right) = \frac{\partial^2 Q}{\partial L^2} = -5K^{\frac{1}{2}}L^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial K}\left(\frac{\partial Q}{\partial K}\right) = \frac{\partial^2 Q}{\partial K^2} = -5K^{-\frac{3}{2}}L^{\frac{1}{2}}$$

Second-order partial derivatives

- Note

$$\frac{\partial}{\partial L}\left(\frac{\partial Q}{\partial L}\right) = \frac{\partial^2 Q}{\partial L^2} = -5K^{\frac{1}{2}}L^{-\frac{3}{2}} < 0$$

$$\frac{\partial}{\partial K}\left(\frac{\partial Q}{\partial K}\right) = \frac{\partial^2 Q}{\partial K^2} = -5K^{-\frac{3}{2}}L^{\frac{1}{2}} < 0$$

- MPL declines with more labor, holding capital fixed
- MPK declines with more capital, holding labor fixed
- Diminishing marginal product of labor and capital

Second-order cross partial derivatives

$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}} \qquad \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \qquad \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}}$$

- What happens to MPL when capital increases?

$$\frac{\partial}{\partial K}\left(\frac{\partial Q}{\partial L}\right) = \frac{\partial^2 Q}{\partial K \partial L} = 5K^{-\frac{1}{2}}L^{-\frac{1}{2}} > 0$$

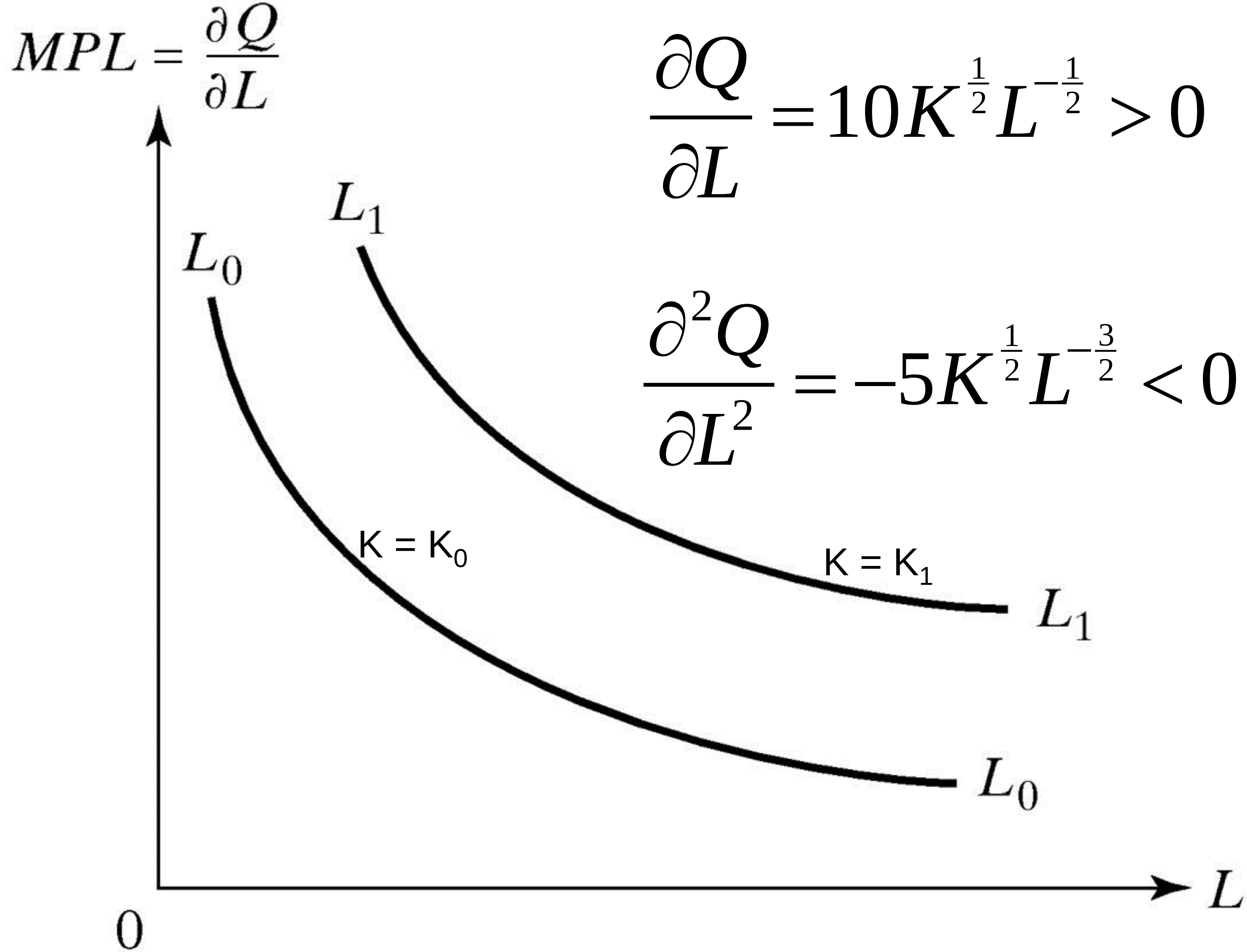
- What happens to MPK when labor increases?

$$\frac{\partial}{\partial L}\left(\frac{\partial Q}{\partial K}\right) = \frac{\partial^2 Q}{\partial L \partial K} = 5K^{-\frac{1}{2}}L^{-\frac{1}{2}} > 0$$

- By Young's Theorem cross-partials are equal

$$\frac{\partial^2 Q}{\partial L \partial K} = \frac{\partial^2 Q}{\partial K \partial L} = 5K^{-\frac{1}{2}}L^{-\frac{1}{2}} > 0$$

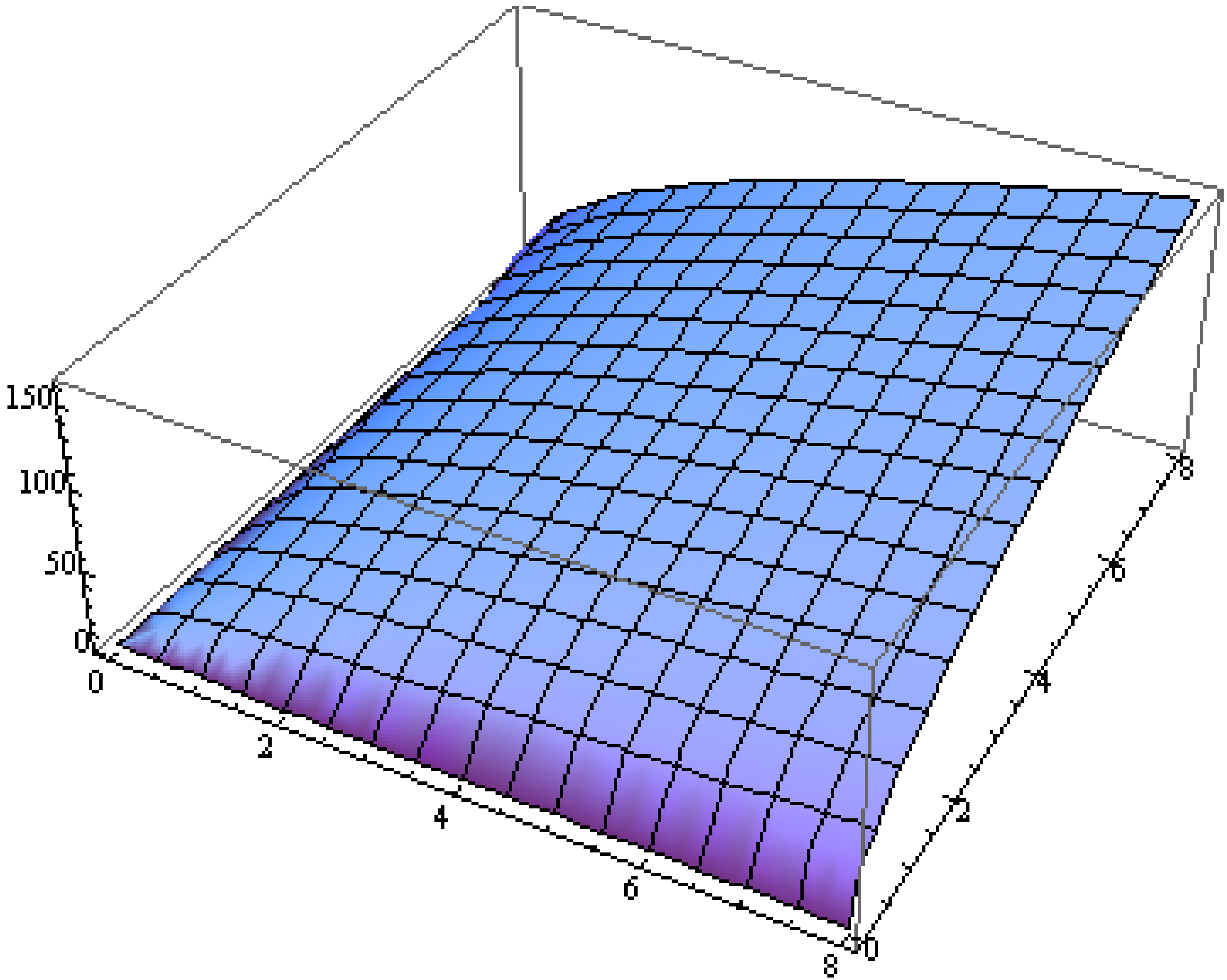
Labor demand at different levels of capital



$$\frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} > 0$$

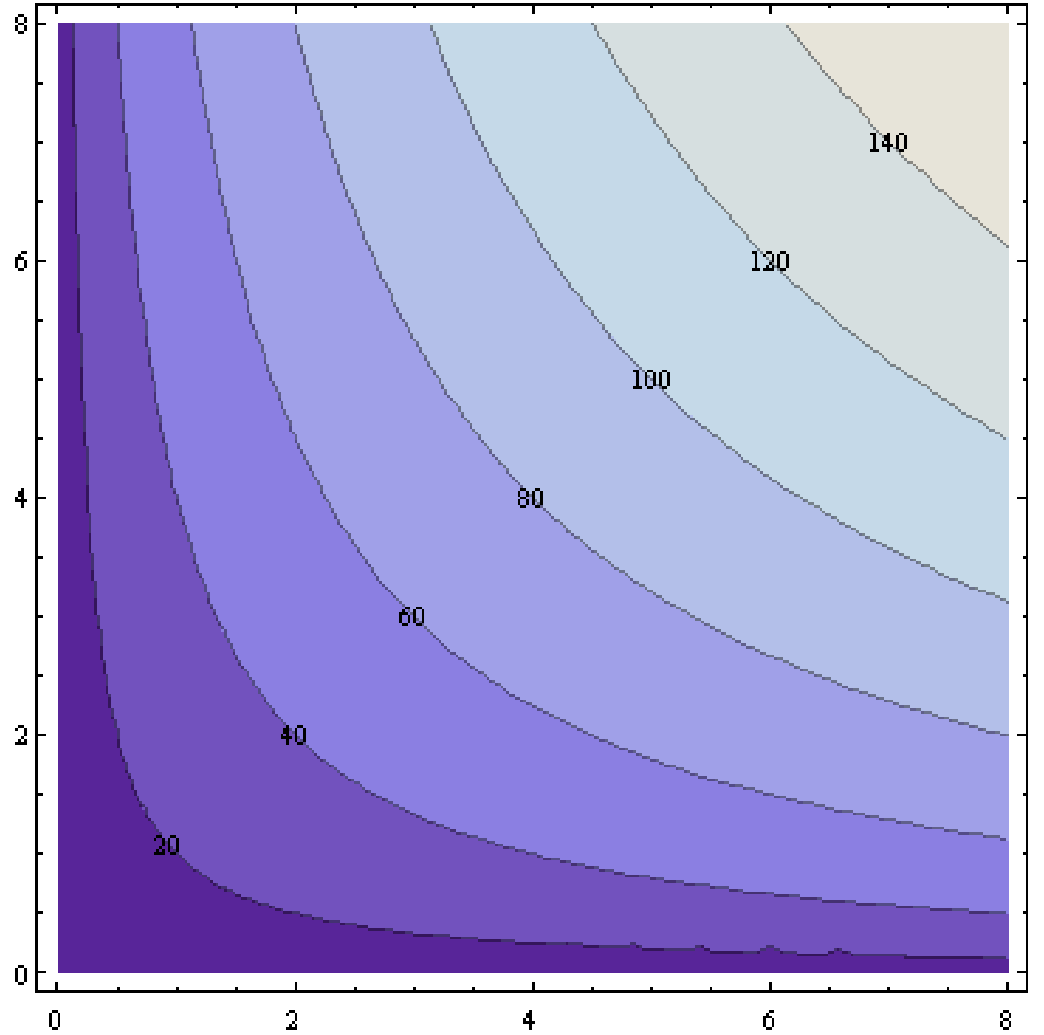
$$\frac{\partial^2 Q}{\partial L^2} = -5K^{\frac{1}{2}}L^{-\frac{3}{2}} < 0$$

$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$



Contour plot

- Isoquant is 2D projection in K-L space of Cobb-Douglas production with output fixed at Q*



$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$

Implicit function theorem

- Each isoquant is implicit function

$$Q^* = AK^{\frac{1}{2}}L^{\frac{1}{2}}$$

- Implicit since K and L vary as dependent variable Q^* is a fixed parameter
- Solving for K as function of L results in explicit function

$$K(L) = \frac{\left(\frac{Q^*}{A}\right)^2}{L}$$

- Use implicit function theorem when can't solve K(L)

Implicit function theorem

For an implicit function $F(y, x_1, \ldots, x_n) = k$ defined at point $(y^0, x_1^0, \ldots, x_n^0)$ with continuous partial derivatives $F_y(y^0, x_1^0, \ldots, x_n^0) \neq 0$, there is a function $y = f(x_1, \ldots, x_n)$ defined in neighborhood of $(x_1^0, \ldots, x_n^0)$ corresponding to $F(y, x_1, \ldots, x_n) = k$ such that

$$y^0 = f(x_1^0, \ldots, x_n^0)$$

$$F(y^0, x_1^0, \ldots, x_n^0) = k$$

$$f_i(x_1^0, \ldots, x_n^0) = -\frac{F_{x_i}(y^0, x_1^0, \ldots, x_n^0)}{F_y(y^0, x_1^0, \ldots, x_n^0)}$$

Verifying implicit function theorem

- Cobb-Douglas production $$Q^* = AK^{\frac{1}{2}}L^{\frac{1}{2}}$$
- Explicit function for isoquant $$K(L) = \frac{\left(\frac{Q^*}{A}\right)^2}{L}$$
- Derivative of isoquant $$\frac{dK(L)}{dL} = -\frac{\left(\frac{Q^*}{A}\right)^2}{L^2} = -\frac{KL}{L^2} = -\frac{K}{L}$$
- Slope of isoquant = marginal rate of technical substitution
  - Essential in determining optimal mix of production inputs

Implicit function theorem

$$f_i(x_1^0,...,x_n^0) = -\frac{F_{x_i}(y^0,x_1^0,...,x_n^0)}{F_y(y^0,x_1^0,...,x_n^0)}$$

$$Q = AK^{\frac{1}{2}}L^{\frac{1}{2}} \qquad \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \qquad \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}}$$

$$\frac{dK}{dL} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{10K^{\frac{1}{2}}L^{-\frac{1}{2}}}{10K^{-\frac{1}{2}}L^{\frac{1}{2}}} = -\frac{K}{L}$$

Implicit function theorem from differential

- For multivariate function $Q = F(K, L)$
- Differential is

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL$$

- Holding quantity fixed (along the isoquant)

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL = 0$$

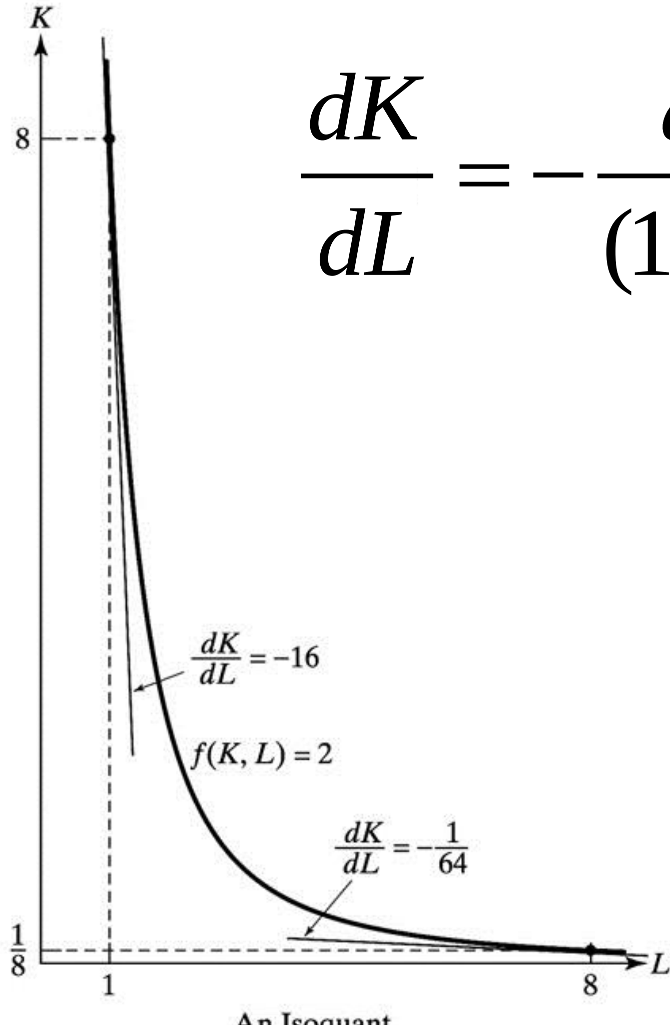
- Thus

$$\frac{dK}{dL} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{MPL}{MPK}$$

Isoquants for Cobb-Douglas

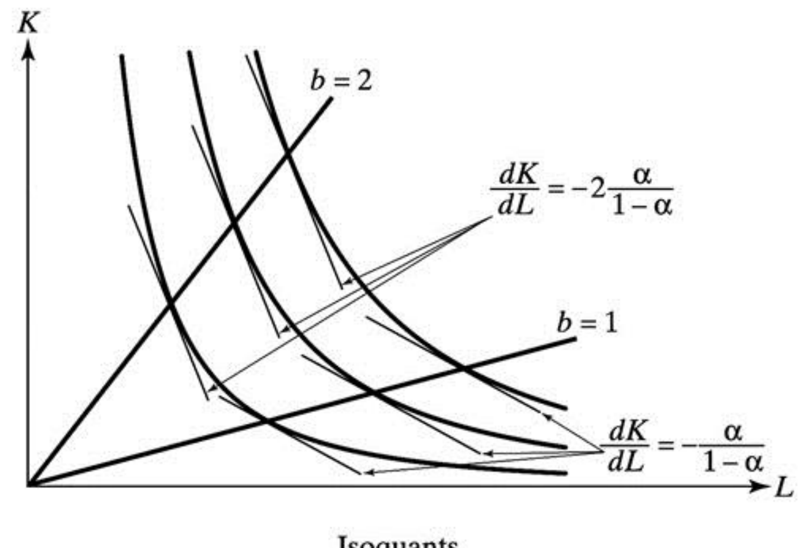
$$Q = AK^{1-\alpha}L^{\alpha}$$

$$\frac{dK}{dL} = -\frac{\alpha AK^{1-\alpha}L^{\alpha-1}}{(1-\alpha)AK^{-\alpha}L^{\alpha}} = -\frac{\alpha}{1-\alpha}\frac{K}{L}$$



An Isoquant
(b)

$$K = bL$$



Isoquants
(c)

Homogeneous functions

- When all independent variables increase by factor s, what happens to output?
  - When production function is homogeneous of degree one, output also changes by factor s
- A function is homogeneous of degree k if

$$s^k Q = F(sK, sL)$$

Homogeneous functions

- Is Cobb-Douglas production function homogeneous?

$$Q = AK^{1-\alpha}L^{\alpha}$$

$$A(sK)^{1-\alpha}(sL)^{\alpha} = As^{1-\alpha}s^{\alpha}K^{1-\alpha}L^{\alpha} = sQ$$

- Yes, homogeneous of degree 1
- Production function has *constant returns to scale*
  - Doubling inputs, doubles output

Homogeneous functions

- Is production function $Q = AK^{\alpha}L^{\beta}$ homogeneous?

$$A(sK)^{\alpha}(sL)^{\beta} = As^{\alpha+\beta}K^{\alpha}L^{\beta} = s^{\alpha+\beta}Q$$

- Yes, homogeneous of degree $\alpha+\beta$
- For $\alpha+\beta = 1$, *constant returns to scale*
  - Doubling inputs, doubles output
- For $\alpha+\beta > 1$, *increasing returns to scale*
  - Doubling inputs, more than doubles output
- For $\alpha+\beta < 1$, *decreasing returns to scale*
  - Doubling inputs, less than doubles output

Properties of homogeneous functions

- Partial derivatives of a homogeneous of degree k function are homogeneous of degree k-1

$$Q = AK^{1-\alpha}L^{\alpha}$$

$$\frac{\partial Q}{\partial L} = \alpha AK^{1-\alpha}L^{\alpha-1}$$

$$\alpha A(sK)^{1-\alpha}(sL)^{\alpha-1} = \alpha As^{0}K^{1-\alpha}L^{\alpha-1} = \frac{\partial Q}{\partial L}$$

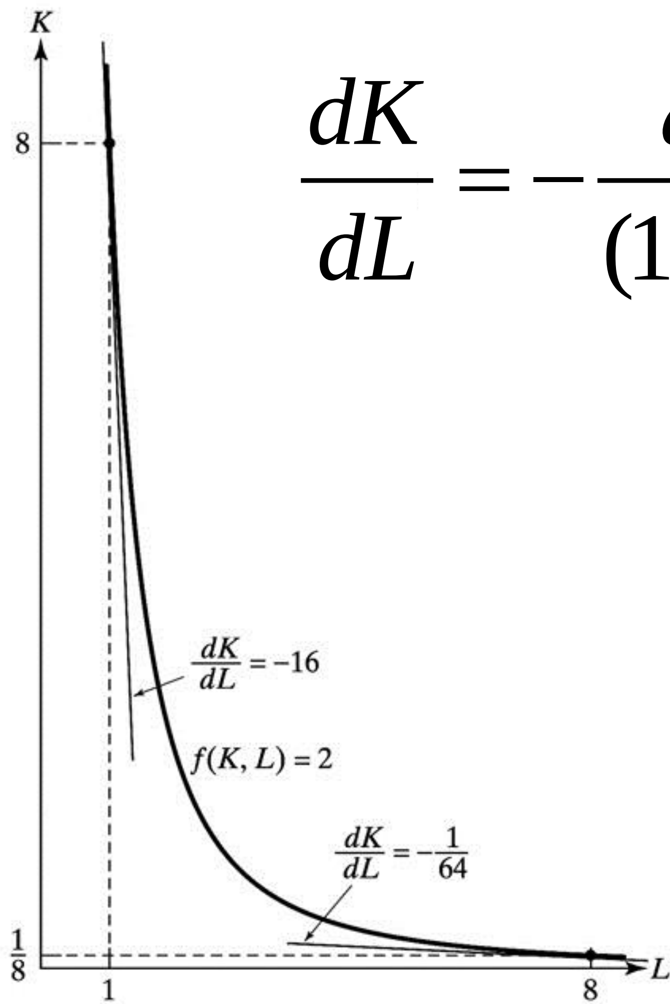
- Cobb-Douglas partial derivatives don’t change as you scale up production

Isoquants for Cobb-Douglas

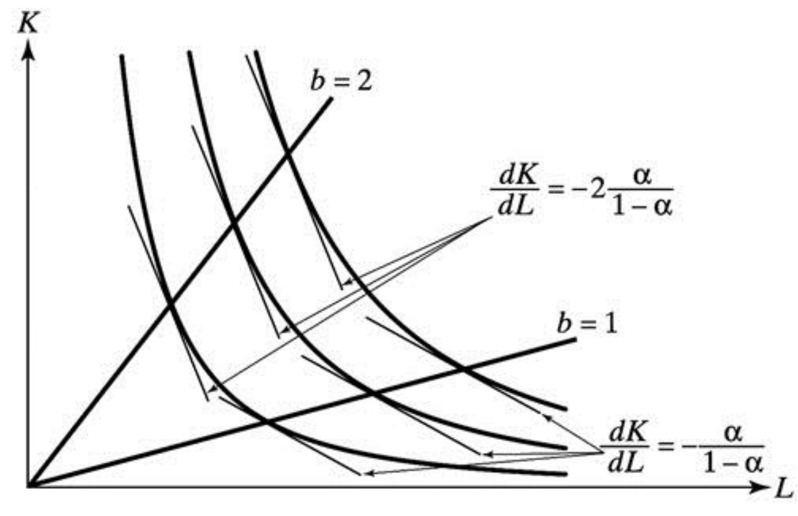
$$Q = AK^{1-\alpha}L^{\alpha}$$

$$\frac{dK}{dL} = -\frac{\alpha AK^{1-\alpha}L^{\alpha-1}}{(1-\alpha)AK^{-\alpha}L^{\alpha}} = -\frac{\alpha}{1-\alpha}\frac{K}{L}$$

$$K = bL$$



An Isoquant
(b)



Isoquants
(c)

Euler's theorem

- Any function Q = F(K,L) has the differential

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL$$

- Euler's Theorem: For a function homogeneous of degree k,

$$kQ = \frac{\partial Q}{\partial K} K + \frac{\partial Q}{\partial L} L$$

- Let Q = GDP and Cobb-Douglas $Q = AK^{1-\alpha} L^{\alpha}$
- Marginal products are associated with factor prices

$$\frac{\partial Q}{\partial L} L = \alpha A K^{1-\alpha} L^{\alpha} = \alpha Q \qquad \frac{\partial Q}{\partial K} K = (1-\alpha) Q$$

In US, $\alpha \approx .67$

Chain rule

$$y = f(x_1, ..., x_n)$$

$$x_i = g_i(t_1, ..., t_m)$$

$$\frac{\partial y}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + ... + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Chain rule exercise

$$y = 3x^2 - 2xw + w^2$$

$$x = 8z - 18$$

$$w = 4z$$

What is dy/dz?

$$\frac{dy}{dz} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial z}$$

$$= (6x - 2w)8 + (2w - 2x)4$$