Multivariate Calculus

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Economics 300
Multivariate calculus

• Calculus with single variable (univariate)
  \[ y = f(x) \]

• Calculus with many variables (multivariate)
  \[ y = f(x_1, x_2, \ldots, x_n) \]
Partial derivatives

• With single variable, derivative is change in $y$ in response to an infinitesimal change in $x$
• With many variables, partial derivative is change in $y$ in response to an infinitesimal change in a single variable $x_i$ (hold all else fixed)
• Total derivative is change in all variables at once
Cobb-Douglas production function

\[ Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}} \]

- How does production change in \( L \)?

\[ \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \]

- Marginal product of labor (MPL)
- Partial derivatives use \( \partial \) instead of \( d \)
Cobb-Douglas production function

\[ Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}} \]

- How does production change in K?

\[ \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}} \]

- Marginal product of capital (MPK)
- Partial derivatives use \( \partial \) instead of \( d \)
Cobb-Douglas production function

\[ Q = 20K^{\frac{1}{2}} L^{\frac{1}{2}} \]

• Note

\[ MPL = \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}} L^{-\frac{1}{2}} > 0 \]

\[ MPK = \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}} L^{\frac{1}{2}} > 0 \]

• Produce more with more labor, holding capital fixed

• Produce more with more capital, holding labor fixed
Second-order partial derivatives

• Differentiate first-order partial derivatives

\[ \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \quad \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}} \]

\[ \frac{\partial}{\partial L} \left( \frac{\partial Q}{\partial L} \right) = \frac{\partial^2 Q}{\partial L^2} = -5K^{\frac{1}{2}}L^{-\frac{3}{2}} \]

\[ \frac{\partial}{\partial K} \left( \frac{\partial Q}{\partial K} \right) = \frac{\partial^2 Q}{\partial K^2} = -5K^{-\frac{3}{2}}L^{\frac{1}{2}} \]
Second-order partial derivatives

- Note
  \[
  \frac{\partial}{\partial L} \left( \frac{\partial Q}{\partial L} \right) = \frac{\partial^2 Q}{\partial L^2} = -5K^{\frac{1}{2}}L^{-\frac{3}{2}} < 0
  \]
  \[
  \frac{\partial}{\partial K} \left( \frac{\partial Q}{\partial K} \right) = \frac{\partial^2 Q}{\partial K^2} = -5K^{-\frac{3}{2}}L^{\frac{1}{2}} < 0
  \]

- MPL declines with more labor, holding capital fixed
- MPK declines with more capital, holding labor fixed
- Diminishing marginal product of labor and capital
Second-order cross partial derivatives

\[ Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}} \]

\[ \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \]

\[ \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}} \]

- What happens to MPL when capital increases?

\[ \frac{\partial}{\partial K} \left( \frac{\partial Q}{\partial L} \right) = \frac{\partial^2 Q}{\partial K \partial L} = 5K^{-\frac{1}{2}}L^{-\frac{1}{2}} > 0 \]

- What happens to MPK when labor increases?

\[ \frac{\partial}{\partial L} \left( \frac{\partial Q}{\partial K} \right) = \frac{\partial^2 Q}{\partial L \partial K} = 5K^{-\frac{1}{2}}L^{-\frac{1}{2}} > 0 \]

- By Young’s Theorem cross-partial derivatives are equal

\[ \frac{\partial^2 Q}{\partial L \partial K} = \frac{\partial^2 Q}{\partial K \partial L} = 5K^{-\frac{1}{2}}L^{-\frac{1}{2}} > 0 \]
Labor demand at different levels of capital

\[ MPL = \frac{\partial Q}{\partial L} \]

\[ \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} > 0 \]

\[ \frac{\partial^2 Q}{\partial L^2} = -5K^{\frac{1}{2}}L^{-\frac{3}{2}} < 0 \]
$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$
Contour plot

• Isoquant is 2D projection in K-L space of Cobb-Douglas production with output fixed at $Q^*$

$$Q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$
Implicit function theorem

• Each isoquant is implicit function
  \[ Q^* = AK^{\frac{1}{2}}L^{\frac{1}{2}} \]

• Implicit since K and L vary as dependent variable
  \( Q^* \) is a fixed parameter

• Solving for K as function of L results in explicit function
  \[ K(L) = \left( \frac{Q^*}{A} \right)^{\frac{2}{L}} \]

• Use implicit function theorem when can’t solve \( K(L) \)
Implicit function theorem

For an implicit function $F(y, x_1, ..., x_n) = k$ defined at point $(y^0, x_1^0, ..., x_n^0)$ with continuous partial derivatives $F_y(y^0, x_1^0, ..., x_n^0) \neq 0$, there is a function $y = f(x_1, ..., x_n)$ defined in neighborhood of $(x_1^0, ..., x_n^0)$ corresponding to $F(y, x_1, ..., x_n) = k$ such that

$$y^0 = f(x_1^0, ..., x_n^0)$$

$$F(y^0, x_1^0, ..., x_n^0) = k$$

$$f_i(x_1^0, ..., x_n^0) = -\frac{F_{x_i}(y^0, x_1^0, ..., x_n^0)}{F_y(y^0, x_1^0, ..., x_n^0)}$$
Verifying implicit function theorem

- Cobb-Douglas production
  \[ Q^* = AK^{\frac{1}{2}}L^{\frac{1}{2}} \]

- Explicit function for isoquant
  \[ K(L) = \left(\frac{Q^*}{A}\right)^2 \]

- Derivative of isoquant
  \[ \frac{dK(L)}{dL} = -\frac{Q^*}{A^2} = -\frac{KL}{L^2} = -\frac{K}{L} \]

- Slope of isoquant = marginal rate of technical substitution
  - Essential in determining optimal mix of production inputs
Implicit function theorem

\[ f_i(x_1^0, \ldots, x_n^0) = -\frac{F_{x_i}(y_0^0, x_1^0, \ldots, x_n^0)}{F_y(y_0^0, x_1^0, \ldots, x_n^0)} \]

\[ Q = AK^{\frac{1}{2}}L^{\frac{1}{2}} \quad \frac{\partial Q}{\partial L} = 10K^{\frac{1}{2}}L^{-\frac{1}{2}} \quad \frac{\partial Q}{\partial K} = 10K^{-\frac{1}{2}}L^{\frac{1}{2}} \]

\[ \frac{dK}{dL} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{10K^{\frac{1}{2}}L^{-\frac{1}{2}}}{10K^{-\frac{1}{2}}L^{\frac{1}{2}}} = -\frac{K}{L} \]
Implicit function theorem from differential

- For multivariate function $Q = F(K, L)$
- Differential is
  \[ dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL \]
- Holding quantity fixed (along the isoquant)
  \[ dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL = 0 \]
- Thus
  \[ \frac{dK}{dL} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{MPL}{MPK} \]
Isoquants for Cobb-Douglas

\[ Q = AK^{1-\alpha}L^\alpha \]

\[
\frac{dK}{dL} = -\frac{\alpha AK^{1-\alpha}L^{\alpha-1}}{(1-\alpha)AK^{-\alpha}L^\alpha} = -\frac{\alpha}{1-\alpha} \frac{K}{L}
\]

\[ K = bL \]
Homogeneous functions

- When all independent variables increase by factor $s$, what happens to output?
  - When production function is homogeneous of degree one, output also changes by factor $s$

- A function is homogeneous of degree $k$ if

$$s^k Q = F(sK, sL)$$
Homogeneous functions

- Is Cobb-Douglas production function homogeneous?
  \[ Q = AK^{1-\alpha}L^\alpha \]
  \[ A(sK)^{1-\alpha}(sL)^\alpha = As^{1-\alpha}s^\alpha K^{1-\alpha}L^\alpha = sQ \]

- Yes, homogeneous of degree 1
- Production function has \textit{constant returns to scale}
  - Doubling inputs, doubles output
Homogeneous functions

- Is production function $Q = AK^\alpha L^\beta$ homogeneous?

  \[ A(sK)^\alpha (sL)^\beta = As^{\alpha+\beta} K^\alpha L^\beta = s^{\alpha+\beta} Q \]

- Yes, homogeneous of degree $\alpha+\beta$

- For $\alpha+\beta = 1$, constant returns to scale
  - Doubling inputs, doubles output

- For $\alpha+\beta > 1$, increasing returns to scale
  - Doubling inputs, more than doubles output

- For $\alpha+\beta < 1$, decreasing returns to scale
  - Doubling inputs, less than doubles output
Properties of homogeneous functions

- Partial derivatives of a homogeneous of degree \( k \) function are homogeneous of degree \( k-1 \)

\[
Q = AK^{1-\alpha} L^\alpha
\]

\[
\frac{\partial Q}{\partial L} = \alpha AK^{1-\alpha} L^{\alpha-1}
\]

\[
\alpha A(sK)^{1-\alpha} (sL)^{\alpha-1} = \alpha As^0 K^{1-\alpha} L^{\alpha-1} = \frac{\partial Q}{\partial L}
\]

- Cobb-Douglas partial derivatives don’t change as you scale up production
Isoquants for Cobb-Douglas

\[ Q = AK^{1-\alpha} L^\alpha \]

\[
\frac{dK}{dL} = -\frac{\alpha AK^{1-\alpha} L^{\alpha-1}}{(1-\alpha)AK^{-\alpha} L^\alpha} = -\frac{\alpha}{1-\alpha} \frac{K}{L}
\]

\[ K = bL \]
Euler’s theorem

• Any function \( Q = F(K, L) \) has the differential

\[
dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL
\]

• Euler’s Theorem: For a function homogeneous of degree \( k \),

\[
kQ = \frac{\partial Q}{\partial K} K + \frac{\partial Q}{\partial L} L
\]

• Let \( Q = GDP \) and Cobb-Douglas

\[Q = AK^{1-\alpha} L^\alpha\]

• Marginal products are associated with factor prices

\[
\frac{\partial Q}{\partial L} L = \alpha AK^{1-\alpha} L^\alpha = \alpha Q
\]

\[
\frac{\partial Q}{\partial K} K = (1 - \alpha)Q
\]

In US, \( \alpha \approx .67 \)
Chain rule

\[ y = f(x_1, \ldots, x_n) \]

\[ x_i = g_i(t_1, \ldots, t_m) \]

\[ \frac{\partial y}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t_i} \]
Chain rule exercise

\[ y = 3x^2 - 2xw + w^2 \]

\[ x = 8z - 18 \]

\[ w = 4z \]

What is \( \frac{dy}{dz} \)?

\[
\frac{dy}{dz} = \frac{\partial y}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial y}{\partial w} \frac{\partial w}{\partial z}
\]

\[
= (6x - 2w)8 + (2w - 2x)4
\]