Univariate Calculus

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Economics 300

Outline

- Rules of differentiation
  - Sum rule
  - Scale rule
  - Product rule
  - Power rule
  - Exponential rule
  - Chain rule
  - Quotient rule
  - Logarithmic rule

Outline

- Second derivatives and convexity
- Economic applications
  - Risk aversion and utility functions
  - Elasticities
  - Total revenue
  - Average, marginal, and total cost

Rules of differentiation

- Used to evaluate derivative of any function
- Much easier than working with basic definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Sum rule

$$y = h(x) = f(x) + g(x)$$

$$\frac{dy}{dx} = h'(x) = f'(x) + g'(x)$$

Scale rule

$$y = h(x) = kf(x)$$

$$\frac{dy}{dx} = h'(x) = kf'(x)$$

Product rule

$$y = h(x) = f(x)g(x)$$

$$\frac{dy}{dx} = h'(x) = f'(x)g(x) + f(x)g'(x)$$

Power rule

$$y = h(x) = kx^n$$

$$\frac{dy}{dx} = h'(x) = nkx^{n-1}$$

Exponential rule

$$y = h(x) = e^{kx}$$

$$\frac{dy}{dx} = h'(x) = ke^{kx}$$

Chain rule

$$y = h(x) = g(f(x))$$

$$\frac{dy}{dx} = h'(x) = g'(f(x))\,f'(x)$$

$$y = g(u)$$

$$u = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Quotient rule

$$y = h(x) = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Logarithmic rule

$$y = f(x) = \ln x$$

$$\frac{dy}{dx} = f'(x) = \frac{1}{x}$$

$$y = f(x) = \log_b x$$

$$\frac{dy}{dx} = f'(x) = \frac{1}{x}\frac{1}{\ln b}$$

Exercise

$$y=\sqrt{x^3}$$

$$\frac{dy}{dx}=\frac{3}{2}\sqrt{x}$$

$$y=25$$

$$y=\frac{7}{4x^4}$$

$$\frac{dy}{dx}=-7x^{-5}$$

$$y=8x^2+3\sqrt{x}-14$$

$$\frac{dy}{dx}=16x+\frac{3}{2}x^{-1/2}$$

$$y=\frac{1}{3}x^{-2}-2x$$

$$\frac{dy}{dx}=-\frac{2}{3}x^{-3}-2$$

Exercise

$$y = 4 + 20x - 4x^2 \qquad \frac{dy}{dx} = 20 - 8x$$

$$y = e^{2x} \qquad \frac{dy}{dx} = 2e^{2x}$$

$$y = 3\ln x \qquad \frac{dy}{dx} = \frac{3}{x}$$

$$y = 2(x+1)^2 \qquad \frac{dy}{dx} = 4(x+1)$$

$$f(x) = (x+1)^3 + (x^2 - 2x)^2 - 5$$

$$f'(x) = 3(x+1)^2 + 2(x^2 - 2x)(2x - 2)$$

$$f(x) = (2x+4)^{99} \qquad f'(x) = 99(2x+4)^{98} 2$$

$$f(x) = (e^x)^{ab} \qquad f'(x) = ab(e^x)^{ab-1} e^x = abe^{abx}$$

$$f(x) = (e^{x^a})^b$$

$$f'(x) = b(e^{x^a})^{b-1} e^{x^a} ax^{a-1} = ab(e^{x^a})^b x^{a-1}$$

$$f(x) = (e^{a+bx+cx^2})^{10} \qquad f'(x) = 10(e^{a+bx+cx^2})^{10}(b+2cx)$$

Exercise

$$y = \log_2(2x+3) \qquad y' = \frac{2}{2x+3}\frac{1}{\ln 2}$$

$$y = \log_4(8x^2) \qquad y' = \frac{16x}{8x^2}\frac{1}{\ln 4} = \frac{2}{x}\frac{1}{\ln 4}$$

$$y = x^3 \log_2 x \qquad y' = 3x^2 \log_2 x + x^3 \frac{1}{x}\frac{1}{\ln 2}$$

$$y = \frac{\log_3 x}{2x} \qquad y' = -\frac{1}{2}x^{-2}\log_3 x + \frac{1}{2x}\frac{1}{x}\frac{1}{\ln 3}$$

Second derivative

- First derivative
  - Rate of change
  - Slope of the function
  - Velocity

$$\frac{dy}{dx} = f'(x)$$

- Second derivative
  - Rate of change of the rate of change
  - Rate of change of the slope of the function
  - Acceleration

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

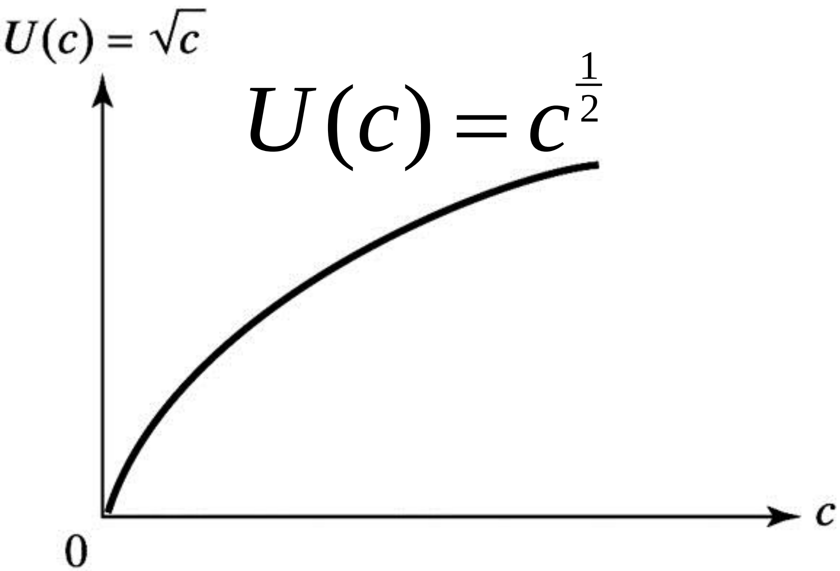
Example: square-root utility

$$U(c) = ac^{\frac{1}{2}}$$

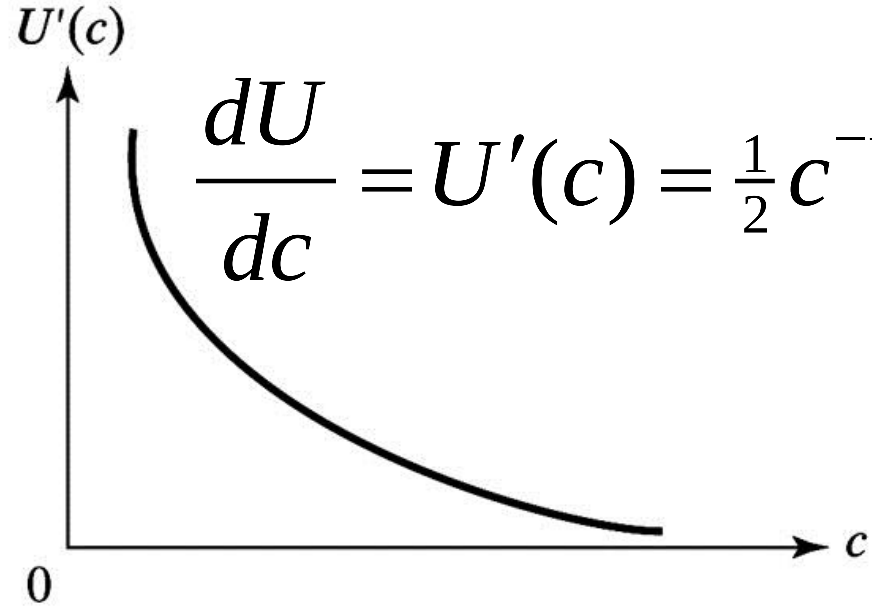
$$\frac{dU}{dc} = U'(c) = \tfrac{1}{2}ac^{-\frac{1}{2}}$$

$$\frac{d^2U}{dc^2} = U''(c) = -\tfrac{1}{4}ac^{-\frac{3}{2}}$$

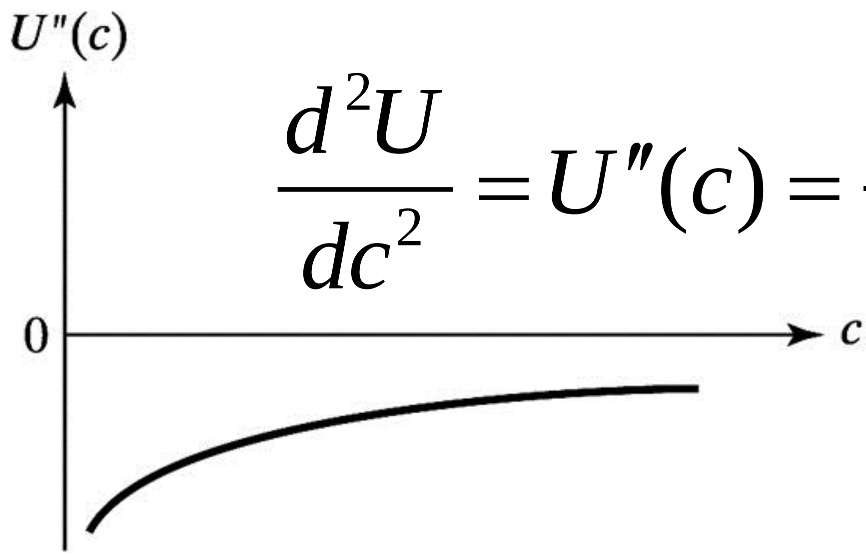
Square-root utility function



$$U(c) = c^{\frac{1}{2}}$$



$$\frac{dU}{dc} = U'(c) = \tfrac{1}{2} c^{-\frac{1}{2}}$$



$$\frac{d^2U}{dc^2} = U''(c) = -\tfrac{1}{4} c^{-\frac{3}{2}}$$

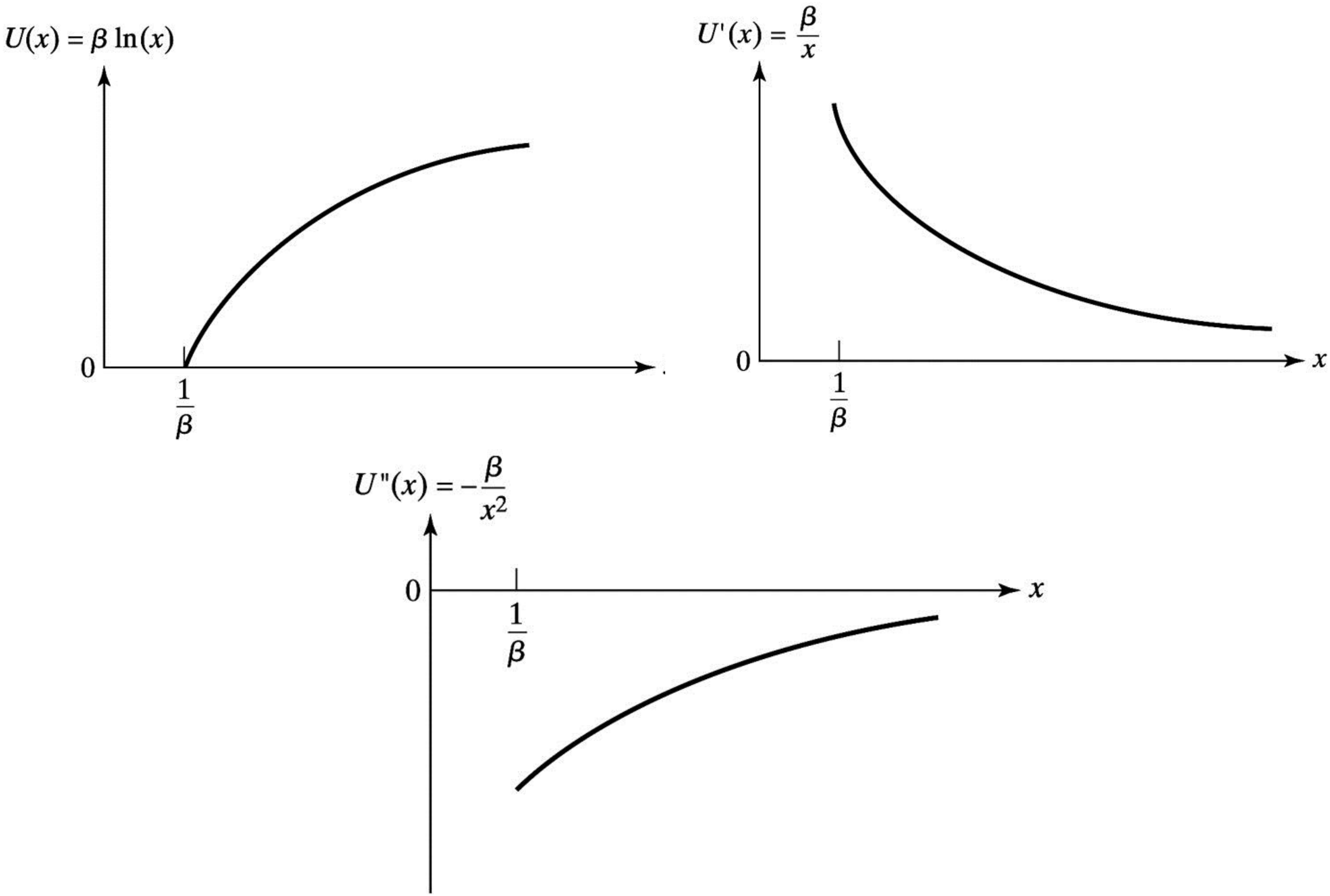
Example: logarithmic utility

$$U(x) = \beta \ln x$$

$$\frac{dU}{dx} = U'(x) = \frac{\beta}{x}$$

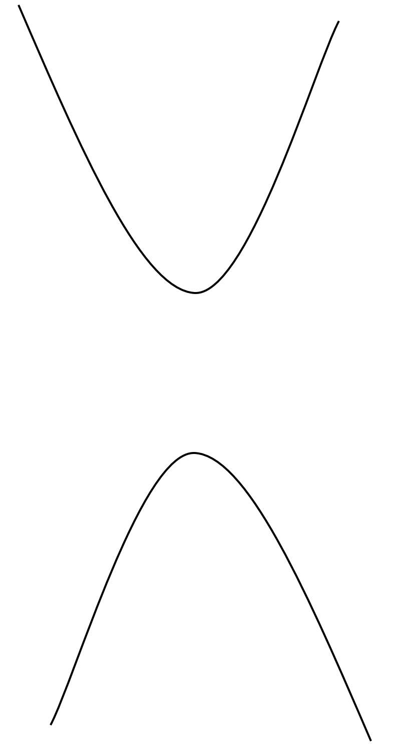
$$\frac{d^2U}{dx^2} = U''(x) = -\frac{\beta}{x^2}$$

Logarithmic utility function



Second derivative and convexity

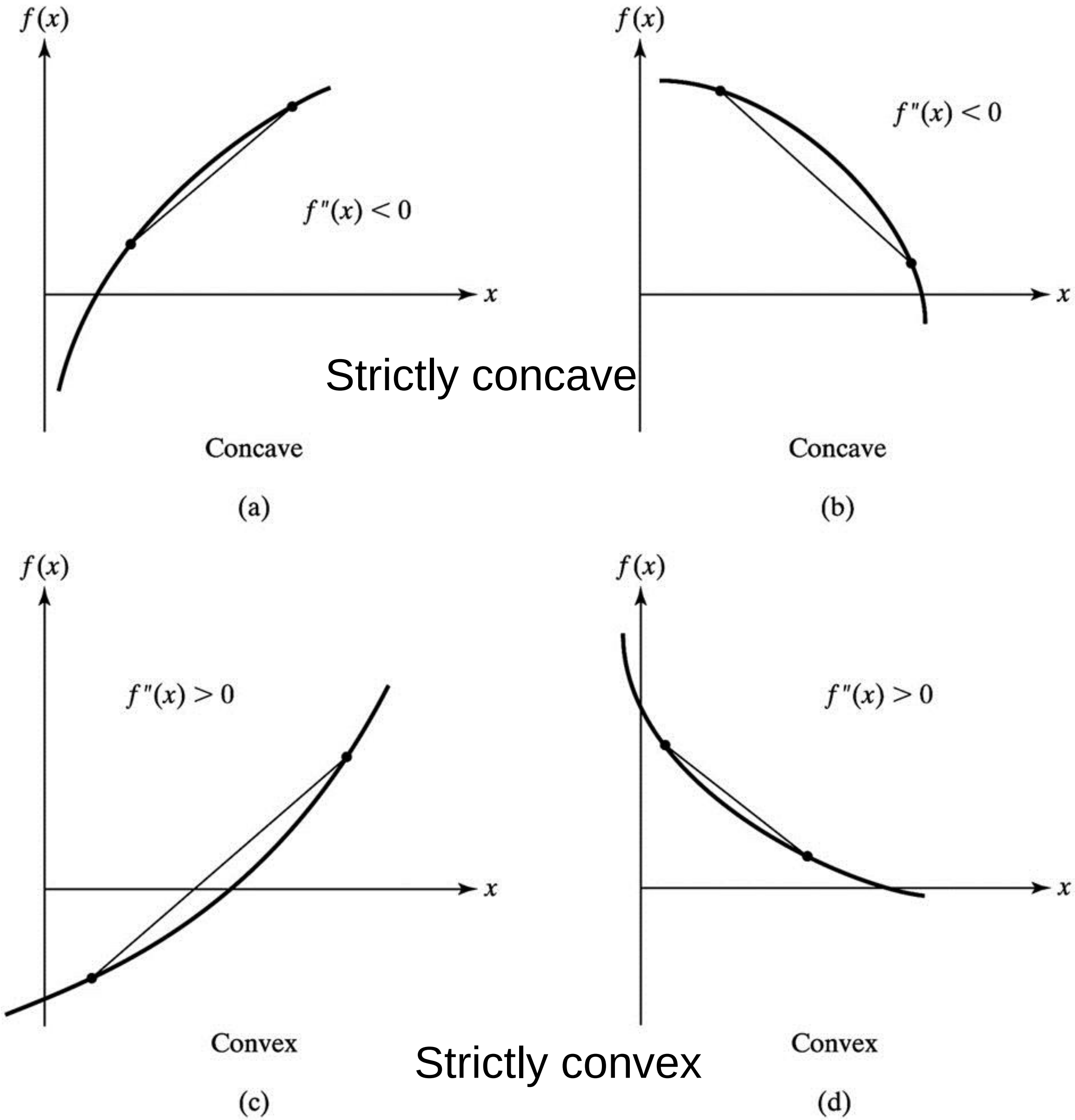
- Strictly convex
  - Slope strictly increasing
  - Second derivative positive: f”(x) > 0
- Strictly concave
  - Slope strictly decreasing
  - Second derivative negative: f”(x) < 0



Second derivative and convexity

- Convex
  - Slope increasing
  - Second derivative $f''(x) \geq 0$
- Concave
  - Slope decreasing
  - Second derivative $f''(x) \leq 0$

Concave and convex functions



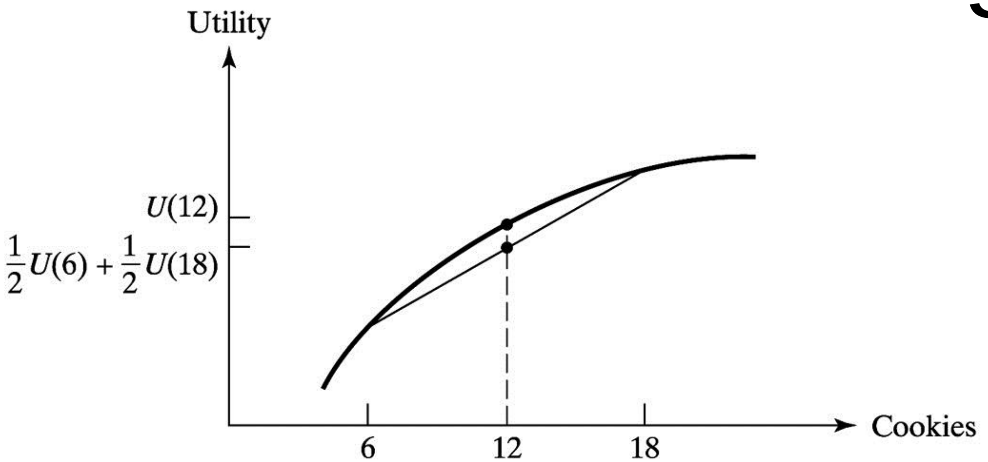
Strictly concave

Strictly convex

Which do you prefer?

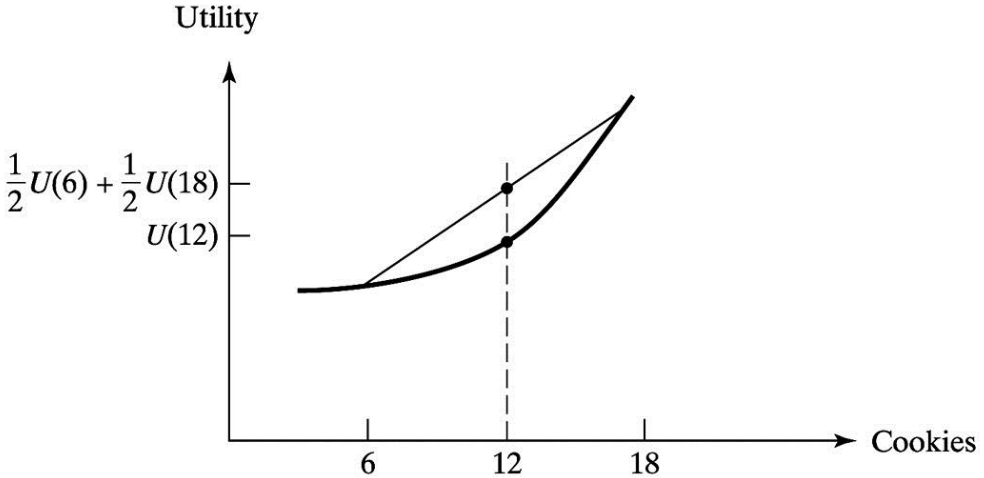
- \$12 for sure
- 50-50 chance of \$6 or \$18
- Expected value of both is \$12
  $.5(\$6) + .5(\$18) = \$12$
- Risk averse
  - Prefer sure thing
  - $U(12) > .5U(6) + .5U(18)$
  - Concave utility
- Risk seeking
  - Prefer gamble
  - $U(12) < .5U(6) + .5U(18)$
  - Convex utility

Risk-averse and risk-loving utility



Risk-Averse Individual

(a)



Risk-Loving Individual

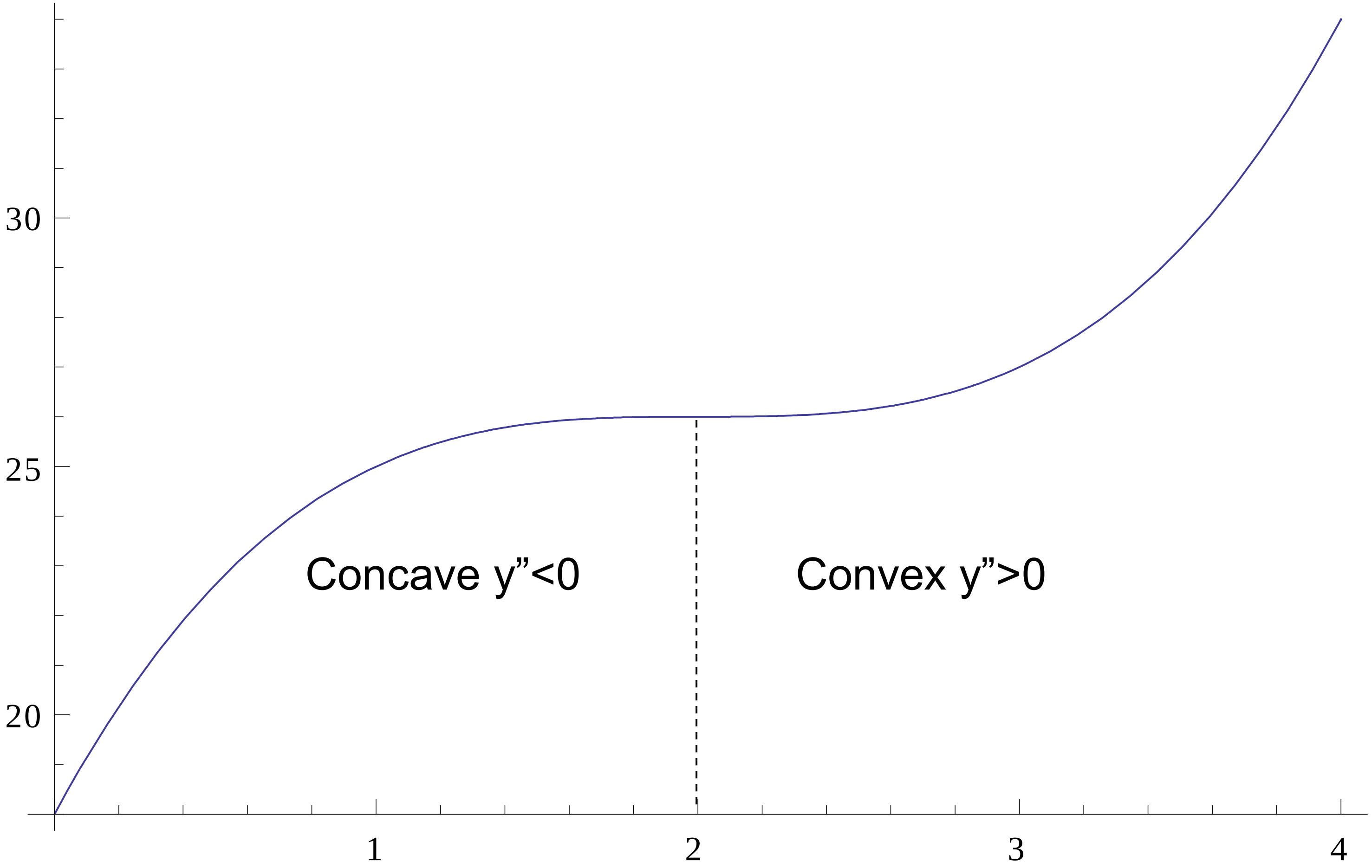
(b)

Exercise: Concave or convex?

$$y = 18 + 12x - 6x^2 + x^3$$

$$y' = 12 - 12x + 3x^2$$

$$y'' = -12 + 6x$$



Elasticities

- How sensitive is demand to a change in price?
- Does revenue increase or decrease when price is raised?
- Depends on price elasticity of demand
    - % change in quantity / % change in price

$$\varepsilon = \frac{\Delta Q / Q}{\Delta P / P}$$

Elasticities

$$\Delta Q \to 0, \text{ percentage change in Q is } \frac{dQ}{Q}$$

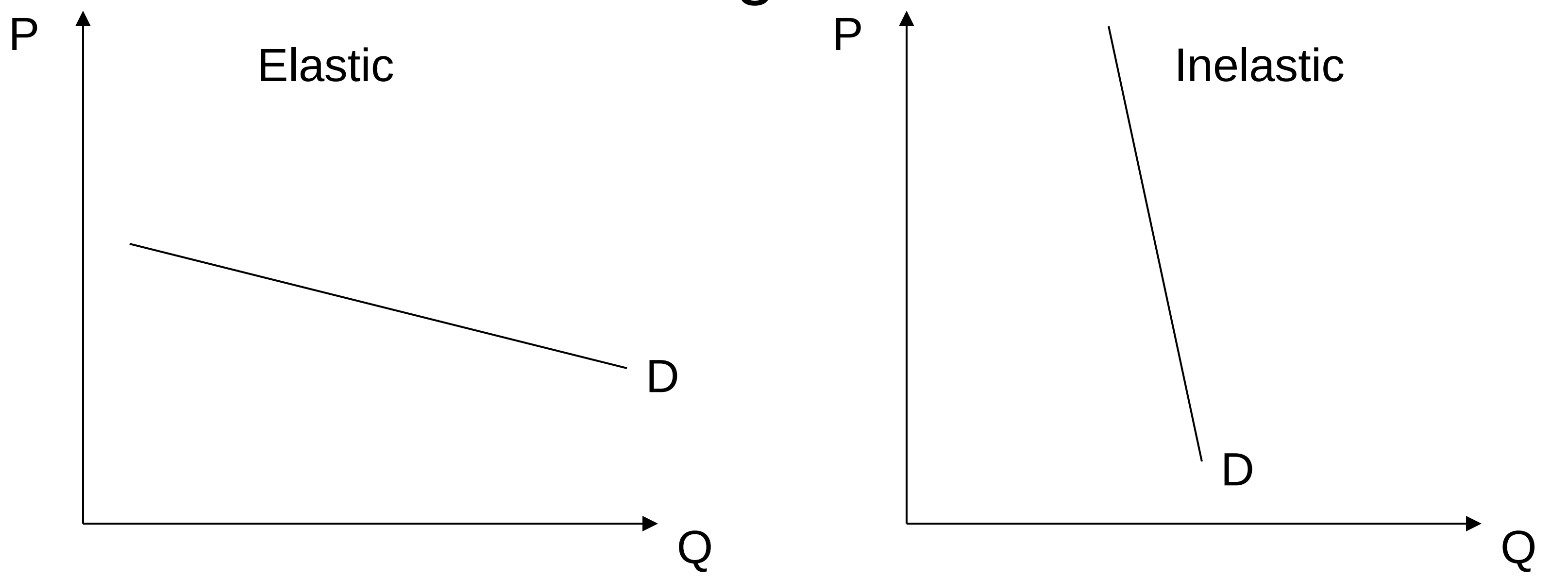
$$\Delta P \to 0, \text{ percentage change in P is } \frac{dP}{P}$$

Price elasticity of demand

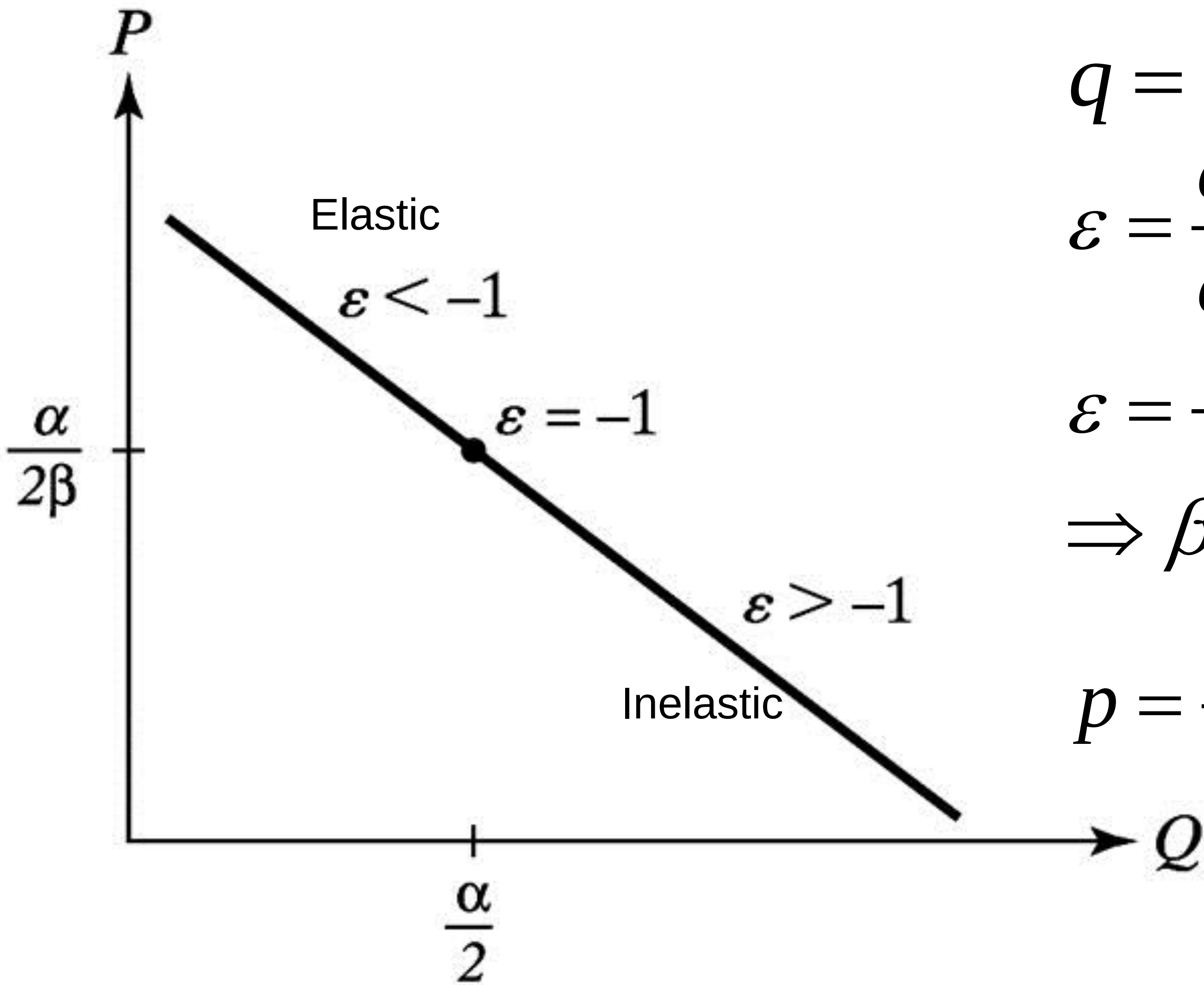
$$\varepsilon = \frac{dQ/Q}{dP/P} = \frac{dQ}{dP}\frac{P}{Q}$$

Elasticities

- $\varepsilon < -1$ demand is price elastic
  - quantity falls by more than 1% if price increases by 1%
- $\varepsilon > -1$ demand is price inelastic
  - quantity falls by less than 1% if price increases by 1%
- Elasticities in different markets are comparable, since based on % change



Elasticities with linear demand



$$q = \alpha - \beta p$$

$$\varepsilon = \frac{dq}{dp}\frac{p}{q} = -\beta\frac{p}{q}$$

$$\varepsilon = -1 \Rightarrow q = \beta p$$

$$\Rightarrow \beta p = \alpha - \beta p$$

$$p = \frac{\alpha}{2\beta}, q = \frac{\alpha}{2}$$

Arc elasticity for discrete changes

$$\varepsilon_{arc} = \frac{\dfrac{Q_B - Q_A}{(Q_A + Q_B)/2}}{\dfrac{P_B - P_A}{(P_A + P_B)/2}}$$

Alternative form for elasticity

Price elasticity of demand

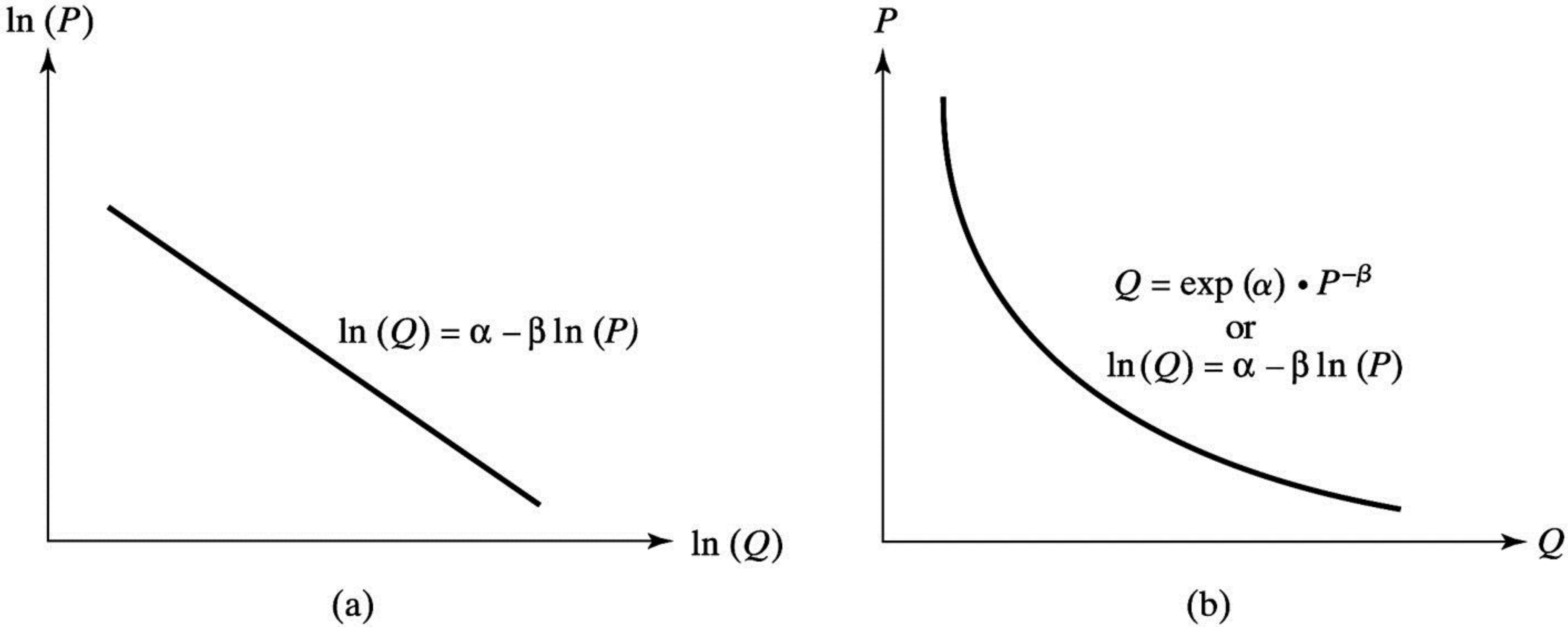
$$\varepsilon = \frac{dQ\,/\,Q}{dP\,/\,P} = \frac{dQ}{dP}\frac{P}{Q}$$

$$\frac{d\ln Q}{dQ} = \frac{1}{Q} \Rightarrow d\ln Q = \frac{dQ}{Q}$$

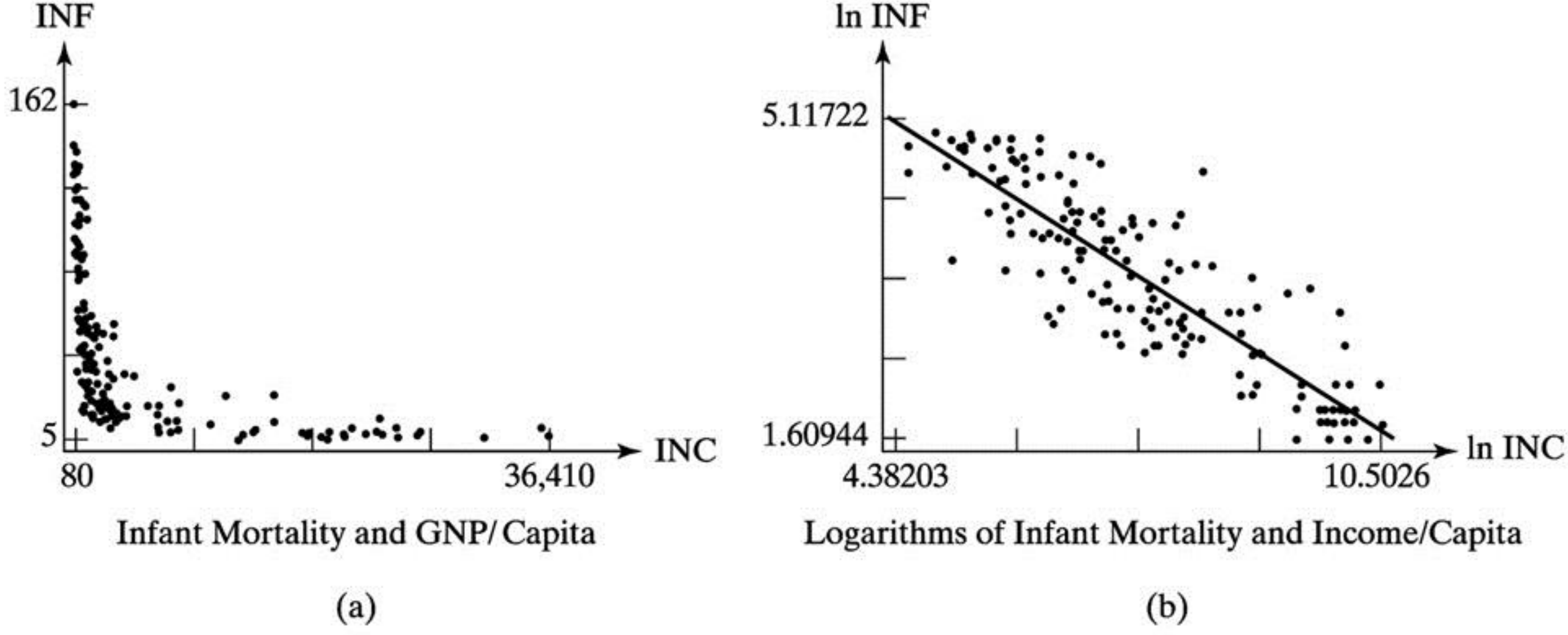
$$\frac{d\ln P}{dP} = \frac{1}{P} \Rightarrow d\ln P = \frac{dP}{P}$$

$$\varepsilon = \frac{dQ}{dP}\frac{P}{Q} = \frac{d\ln Q}{d\ln P}$$

Log-linear demand



Infant mortality and income/capita



Elasticity and other economic concepts

- Substitutes
  - Products with good substitutes are more elastic
- Complements
  - Products with many complements are less elastic
- Luxury goods tend to be more elastic
- Necessities tend to be less elastic

Do consumers or producers bear a tax?

- Depends on elasticity of demand and supply
  - Party least able to substitute away pays more of tax

$$P_C = P_P + T$$

$$Q_D(P_C) = Q_S(P_P)$$

$$dP_C = dP_P + dT$$

$$\frac{dQ_D}{dP_C} dP_C = \frac{dQ_S}{dP_P} dP_P$$

$$\left(\frac{dQ_D}{dP_C} \frac{P}{Q} - \frac{dQ_S}{dP_P} \frac{P}{Q} \right) dP_P = -\frac{dQ_D}{dP_C} \frac{P}{Q} dT$$

$$\frac{dP_P}{dT} = \frac{\varepsilon_D}{\varepsilon_S - \varepsilon_D} \quad \text{and} \quad \frac{dP_C}{dT} = \frac{\varepsilon_S}{\varepsilon_S - \varepsilon_D}$$

Impact of quantity change on revenue

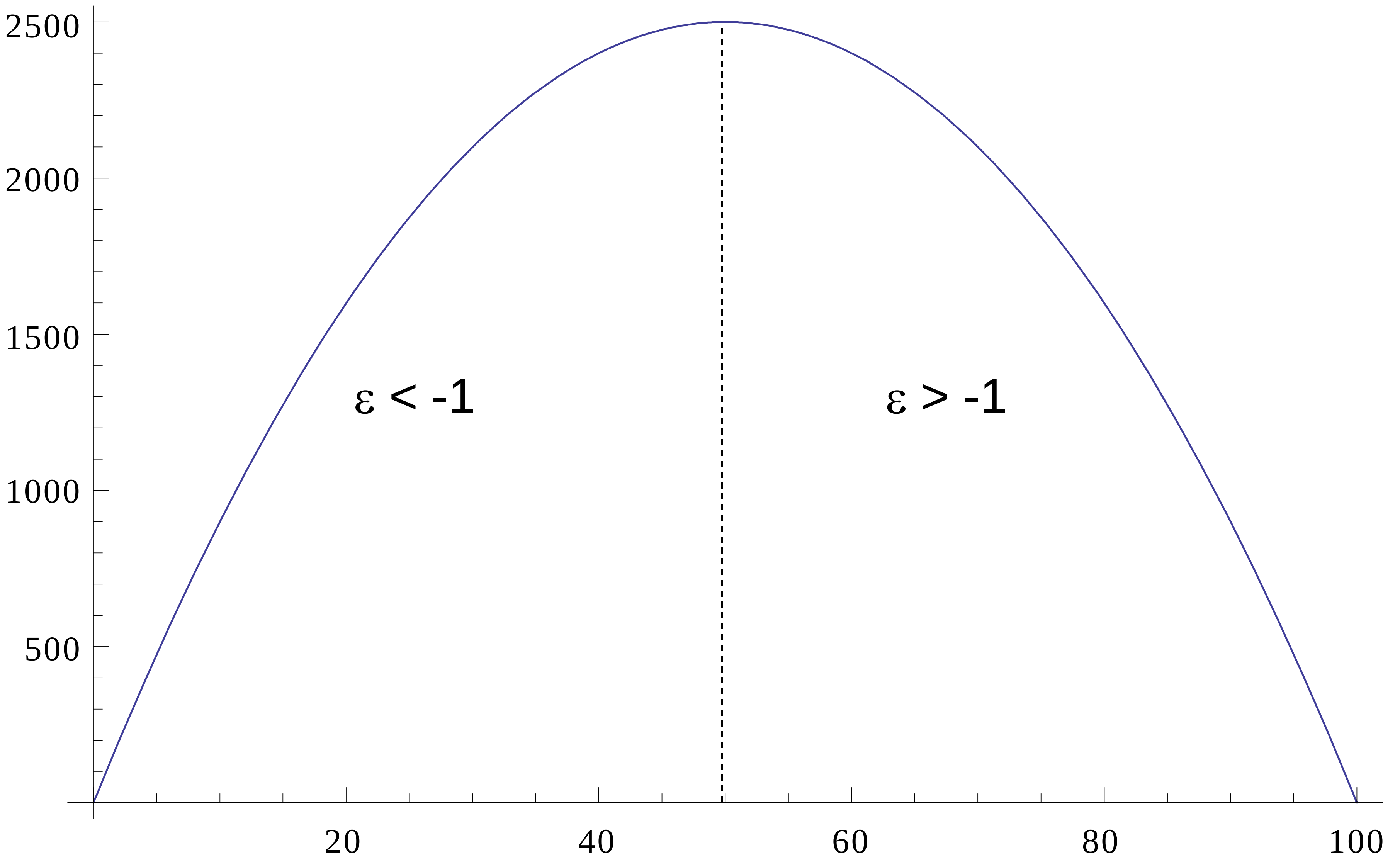
- Depends on elasticity of demand
  - Producer wants to reduce quantity if it faces inelastic demand

$$R = PQ$$

$$\frac{dR}{dQ} = \frac{dP}{dQ}Q + P = P\left(1 + \frac{dP}{dQ}\frac{Q}{P}\right) = P\left(1 + \frac{1}{\varepsilon}\right)$$

- $\varepsilon > -1$ then revenue declines when produce more
- $\varepsilon < -1$ then revenue increases when produce more
- Monopolist allows produces in elastic portion of demand ($\varepsilon < -1$)

P = 100 – Q; R = (100 – Q)Q



Exercise: Find elasticity

$$y = 100 - 2x, x = 4 \qquad \varepsilon = \frac{dy}{dx}\frac{x}{y} = -2\frac{4}{92} = -8/92$$

$$y = 16 - 8x + x^2, x = 1.5$$

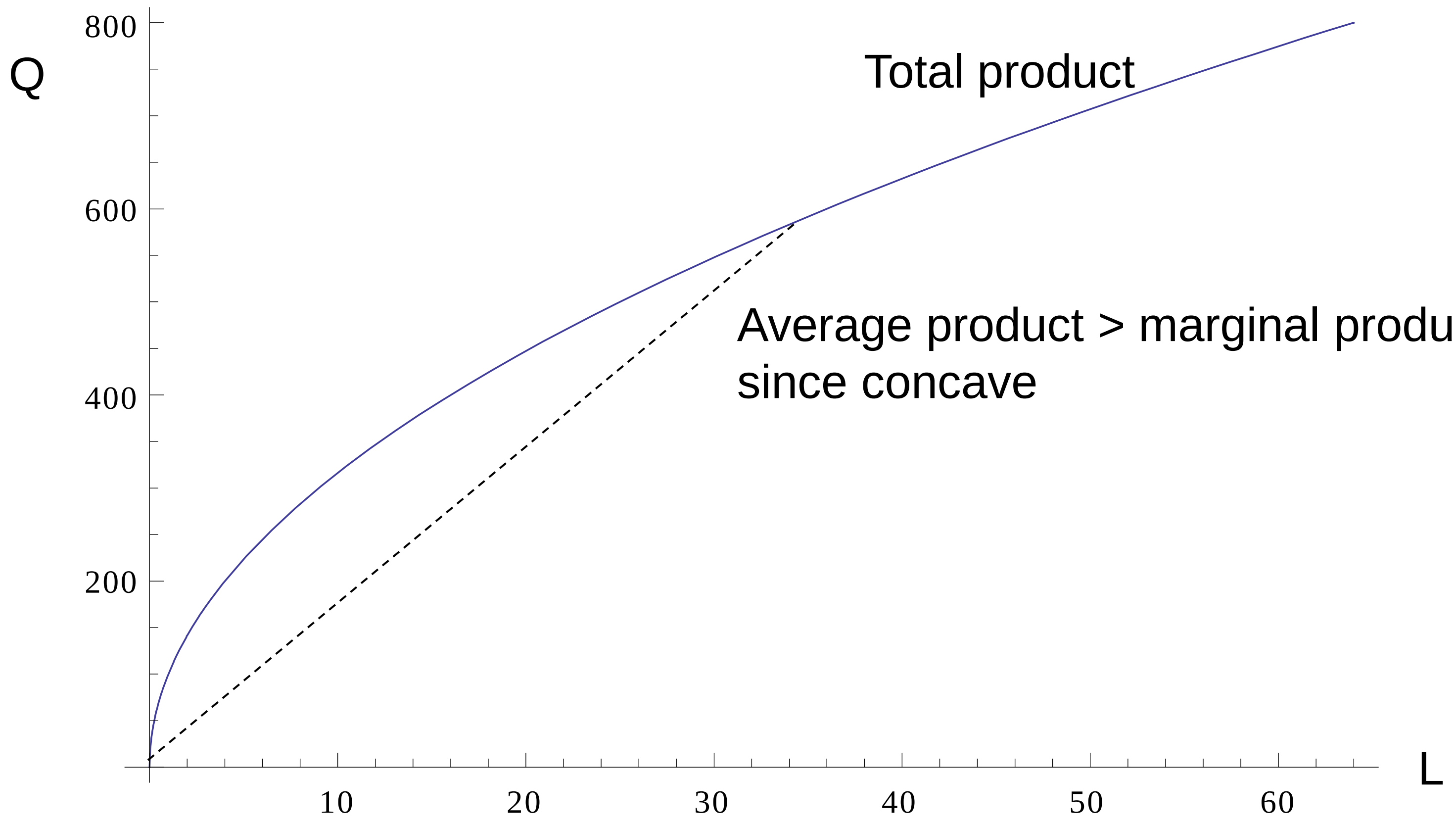
$$\varepsilon = (-8 + 2x)\frac{x}{y} = \frac{-2x}{4 - x} = -3/2.5$$

$$\ln y = 6 - .5\ln x, x = 3 \qquad \varepsilon = -.5$$

$$y = 50x^{-2}, x = 2 \qquad \varepsilon = -2\frac{50x^{-3}x}{50x^{-2}} = -2$$

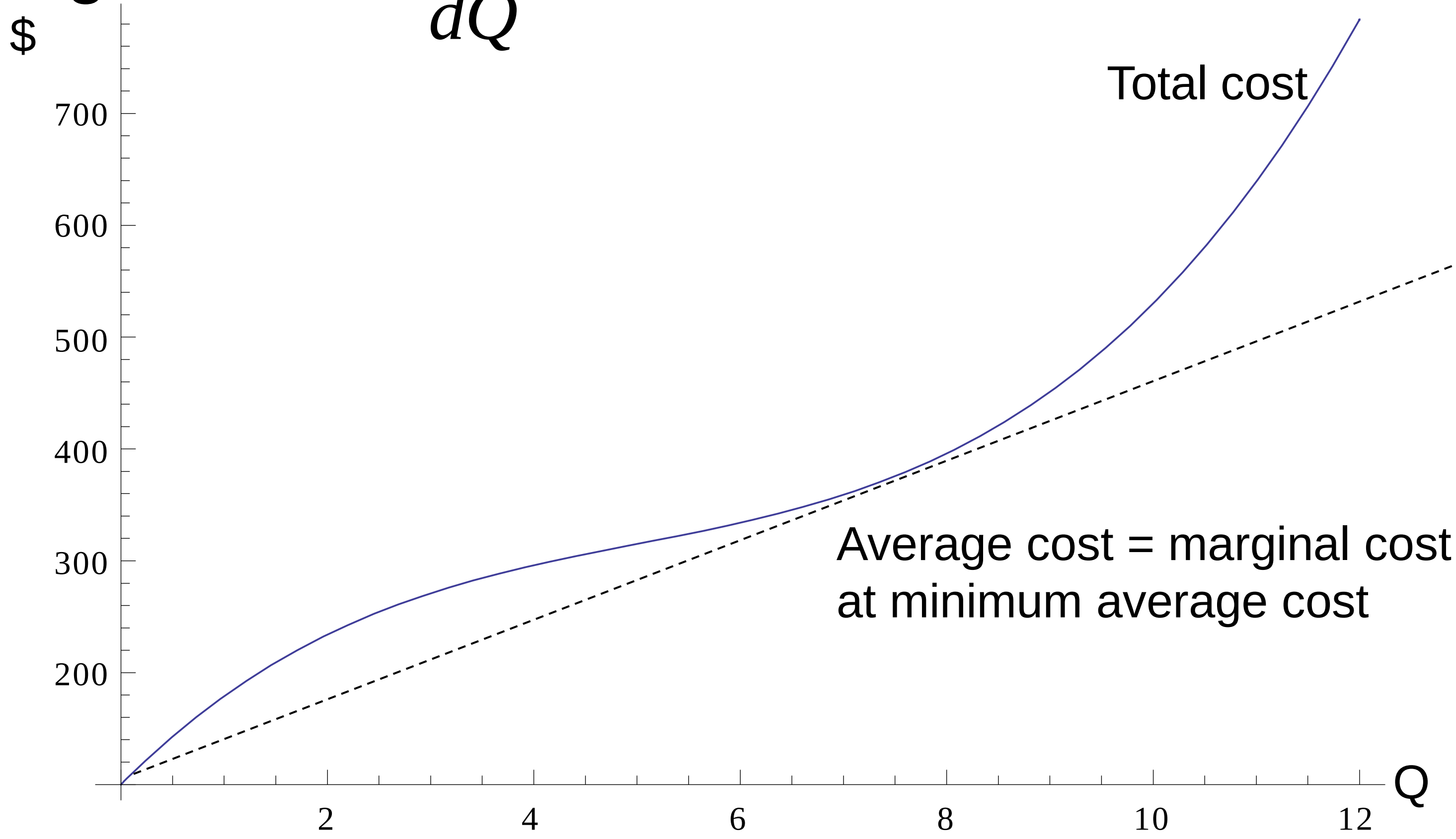
Total, average, and marginal

- Total product $Q = 100L^{1/2}$
- Average product $Q / L = 100L^{-1/2}$
- Marginal product $dQ / dL = 50L^{-1/2}$



Total, average, and marginal

- Total cost $$C = Q^3 - 15Q^2 + 93Q + 100$$
- Average cost $$C/Q$$
- Marginal cost $$\frac{dC}{dQ} = 3Q^2 - 30Q + 93$$



Average cost = marginal cost at min