Univariate Calculus

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Economics 300
Outline

• Rules of differentiation
  – Sum rule
  – Scale rule
  – Product rule
  – Power rule
  – Exponential rule
  – Chain rule
  – Quotient rule
  – Logarithmic rule
Outline

• Second derivatives and convexity
• Economic applications
  – Risk aversion and utility functions
  – Elasticities
  – Total revenue
  – Average, marginal, and total cost
Rules of differentiation

• Used to evaluate derivative of any function
• Much easier than working with basic definition

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]
Sum rule

\[ y = h(x) = f(x) + g(x) \]

\[ \frac{dy}{dx} = h'(x) = f'(x) + g'(x) \]
Scale rule

\[ y = h(x) = kf(x) \]

\[ \frac{dy}{dx} = h'(x) = kf'(x) \]
Product rule

\[ y = h(x) = f(x)g(x) \]

\[ \frac{dy}{dx} = h'(x) = f'(x)g(x) + f(x)g'(x) \]
Power rule

\[ y = h(x) = kx^n \]
\[ \frac{dy}{dx} = h'(x) = nkx^{n-1} \]
Exponential rule

\[ y = h(x) = e^{kx} \]

\[ \frac{dy}{dx} = h'(x) = ke^{kx} \]
Chain rule

\[ y = h(x) = g(f(x)) \]
\[ \frac{dy}{dx} = h'(x) = g'(f(x)) f'(x) \]

\[ y = g(u) \]
\[ u = f(x) \]
\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]
Quotient rule

\[ y = h(x) = \frac{f(x)}{g(x)} \]

\[ \frac{dy}{dx} = h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \]
Logarithmic rule

\[ y = f(x) = \ln x \]
\[ \frac{dy}{dx} = f'(x) = \frac{1}{x} \]

\[ y = f(x) = \log_b x \]
\[ \frac{dy}{dx} = f'(x) = \frac{1}{x} \frac{1}{\ln b} \]
Exercise

\[ y = \sqrt{x^3} \]  \[ \frac{dy}{dx} = \frac{3}{2} \sqrt{x} \]

\[ y = 25 \]

\[ y = \frac{7}{4x^4} \]  \[ \frac{dy}{dx} = -7x^{-5} \]

\[ y = 8x^2 + 3\sqrt{x} - 14 \]  \[ \frac{dy}{dx} = 16x + \frac{3}{2} x^{-1/2} \]

\[ y = \frac{1}{3} x^{-2} - 2x \]  \[ \frac{dy}{dx} = -\frac{2}{3} x^{-3} - 2 \]
Exercise

\[ y = 4 + 20x - 4x^2 \]
\[ y = e^{2x} \]
\[ y = 3 \ln x \]
\[ y = 2(x + 1)^2 \]

\[ \frac{dy}{dx} = 20 - 8x \]
\[ \frac{dy}{dx} = 2e^{2x} \]
\[ \frac{dy}{dx} = 3 \frac{1}{x} \]
\[ \frac{dy}{dx} = 4(x + 1) \]
\[
f(x) = (x + 1)^3 + (x^2 - 2x)^2 - 5
\]
\[
f'(x) = 3(x + 1)^2 + 2(x^2 - 2x)(2x - 2)
\]
\[
f(x) = (2x + 4)^{99} \quad f'(x) = 99(2x + 4)^{98} \cdot 2
\]
\[
f(x) = (e^x)^{ab} \quad f'(x) = ab(e^x)^{ab-1} \cdot e^x = abe^{abx}
\]
\[
f(x) = (e^{xa})^b \quad f'(x) = b(e^{xa})^{b-1} \cdot e^{xa} \cdot ax^{a-1} = ab(e^{xa})^b \cdot x^{a-1}
\]
\[
f(x) = (e^{a+bx+cx^2})^{10} \quad f'(x) = 10(e^{a+bx+cx^2})^{10} \cdot (b + 2cx)
\]
Exercise

\[ y = \log_2(2x + 3) \]
\[ y' = \frac{2}{2x + 3} \cdot \frac{1}{\ln 2} \]

\[ y = \log_4(8x^2) \]
\[ y' = \frac{16x}{8x^2} \cdot \frac{1}{\ln 4} = \frac{2}{x} \cdot \frac{1}{\ln 4} \]

\[ y = x^3 \log_2 x \]
\[ y' = 3x^2 \log_2 x + x^3 \cdot \frac{1}{x} \cdot \frac{1}{\ln 2} \]

\[ y = \frac{\log_3 x}{2x} \]
\[ y' = -\frac{1}{2}x^{-2} \log_3 x + \frac{1}{2} \cdot \frac{1}{x} \cdot \frac{1}{\ln 3} \]
Second derivative

• First derivative
  – Rate of change
  – Slope of the function
  – Velocity

  \[ \frac{dy}{dx} = f'(x) \]

• Second derivative
  – Rate of change of the rate of change
  – Rate of change of the slope of the function
  – Acceleration

  \[ \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} = f''(x) \]
Example: square-root utility

\[ U(c) = ac^{\frac{1}{2}} \]

\[ \frac{dU}{dc} = U'(c) = \frac{1}{2} ac^{-\frac{1}{2}} \]

\[ \frac{d^2U}{dc^2} = U''(c) = -\frac{1}{4} ac^{-\frac{3}{2}} \]
Square-root utility function

\[
U(c) = \sqrt{c}
\]

\[
U(c) = c^{\frac{1}{2}}
\]

\[
\frac{dU}{dc} = U'(c) = \frac{1}{2} c^{-\frac{1}{2}}
\]

\[
\frac{d^2U}{dc^2} = U''(c) = -\frac{1}{4} c^{-\frac{3}{2}}
\]
Example: logarithmic utility

\[ U(x) = \beta \ln x \]

\[
\frac{dU}{dx} = U'(x) = \frac{\beta}{x}
\]

\[
\frac{d^2U}{dx^2} = U''(x) = -\frac{\beta}{x^2}
\]
Logarithmic utility function

\[ U(x) = \beta \ln(x) \]

\[ U'(x) = \frac{\beta}{x} \]

\[ U''(x) = -\frac{\beta}{x^2} \]
Second derivative and convexity

• Strictly convex
  – Slope strictly increasing
  – Second derivative positive: \( f''(x) > 0 \)

• Strictly concave
  – Slope strictly decreasing
  – Second derivative negative: \( f''(x) < 0 \)
Second derivative and convexity

• Convex
  – Slope increasing
  – Second derivative $f''(x) \geq 0$

• Concave
  – Slope decreasing
  – Second derivative $f''(x) \leq 0$
Concave and convex functions

- **Strictly concave**
  - $f''(x) < 0$

- **Concave**
  - $f''(x) < 0$

- **Strictly convex**
  - $f''(x) > 0$

- **Convex**
  - $f''(x) > 0$
Which do you prefer?

- $12 for sure
- 50-50 chance of $6 or $18
- Expected value of both is $12
  \[0.5\times 6 + 0.5\times 18 = 12\]
- Risk averse
  - Prefer sure thing
  - \(U(12) > 0.5U(6) + 0.5U(18)\)
  - Concave utility
- Risk seeking
  - Prefer gamble
  - \(U(12) < 0.5U(6) + 0.5U(18)\)
  - Convex utility
Risk-averse and risk-loving utility

(a) Risk-Averse Individual

(b) Risk-Loving Individual
Exercise: Concave or convex?

\[ y = 18 + 12x - 6x^2 + x^3 \]
\[ y' = 12 - 12x + 3x^2 \]
\[ y'' = -12 + 6x \]

Concave \( y'' < 0 \)

Convex \( y'' > 0 \)
Elasticities

• How sensitive is demand to a change in price?
• Does revenue increase or decrease when price is raised?
• Depends on price elasticity of demand
  - % change in quantity / % change in price

\[ \varepsilon = \frac{\Delta Q / Q}{\Delta P / P} \]
Elasticities

\( \Delta Q \to 0 \), percentage change in \( Q \) is \( \frac{dQ}{Q} \)

\( \Delta P \to 0 \), percentage change in \( P \) is \( \frac{dP}{P} \)

Price elasticity of demand

\[ \varepsilon = \frac{dQ / Q}{dP / P} = \frac{dQ}{dP} \cdot \frac{P}{Q} \]
Elasticities

- \( \varepsilon < -1 \) demand is price elastic
  - quantity falls by more than 1% if price increases by 1%
- \( \varepsilon > -1 \) demand is price inelastic
  - quantity falls by less than 1% if price increases by 1%
- Elasticities in different markets are comparable, since based on % change
Elasticities with linear demand

\[ q = \alpha - \beta p \]
\[ \varepsilon = \frac{dq}{dp} \frac{p}{q} = -\beta \frac{p}{q} \]
\[ \varepsilon = -1 \Rightarrow q = \beta p \]
\[ \Rightarrow \beta p = \alpha - \beta p \]
\[ p = \frac{\alpha}{2\beta}, q = \frac{\alpha}{2} \]
Arc elasticity for discrete changes

\[ \varepsilon_{arc} = \frac{Q_B - Q_A}{\frac{(Q_A + Q_B)}{2}} = \frac{P_B - P_A}{\frac{(P_A + P_B)}{2}} \]
Alternative form for elasticity

Price elasticity of demand

$$\varepsilon = \frac{dQ / Q}{dP / P} = \frac{dQ}{dP} \frac{P}{Q}$$

$$\frac{d \ln Q}{dQ} = \frac{1}{Q} \implies d \ln Q = \frac{dQ}{Q}$$

$$\frac{d \ln P}{dP} = \frac{1}{P} \implies d \ln P = \frac{dP}{P}$$

$$\varepsilon = \frac{dQ}{dP} \frac{P}{Q} = \frac{d \ln Q}{d \ln P}$$
Log-linear demand

\[
\ln(Q) = \alpha - \beta \ln(P)
\]

Graph (a)

\[
Q = \exp(\alpha) \cdot P^{-\beta} \quad \text{or} \quad \ln(Q) = \alpha - \beta \ln(P)
\]

Graph (b)
Infant mortality and income/capita

(a) Infant Mortality and GNP/Capita

(b) Logarithms of Infant Mortality and Income/Capita
Elasticity and other economic concepts

• Substitutes
  – Products with good substitutes are more elastic

• Complements
  – Products with many complements are less elastic

• Luxury goods tend to be more elastic

• Necessities tend to be less elastic
Do consumers or producers bear a tax?

- Depends on elasticity of demand and supply
  - Party least able to substitute away pays more of tax

\[
P_C = P_P + T
\]

\[
dP_C = dP_P + dT
\]

\[
Q_D(P_C) = Q_S(P_P)
\]

\[
\frac{dQ_D}{dP_C} dP_C = \frac{dQ_S}{dP_P} dP_P
\]

\[
\left( \frac{dQ_D}{dP_C} \frac{P}{Q} - \frac{dQ_S}{dP_P} \frac{P}{Q} \right) dP_P = -\frac{dQ_D}{dP_C} \frac{P}{Q} dT
\]

\[
\frac{dP_P}{dT} = \frac{\varepsilon_D}{\varepsilon_S - \varepsilon_D} \quad \text{and} \quad \frac{dP_C}{dT} = \frac{\varepsilon_S}{\varepsilon_S - \varepsilon_D}
\]
Impact of quantity change on revenue

• Depends on elasticity of demand
  – Producer wants to reduce quantity if it faces inelastic demand

\[ R = PQ \]

\[ \frac{dR}{dQ} = \frac{dP}{dQ} Q + P = P \left( 1 + \frac{dP}{dQ} \frac{Q}{P} \right) = P \left( 1 + \frac{1}{\varepsilon} \right) \]

• \( \varepsilon > -1 \) then revenue declines when produce more
• \( \varepsilon < -1 \) then revenue increases when produce more
• Monopolist allows produces in elastic portion of demand (\( \varepsilon < -1 \))
\[ P = 100 - Q; \ R = (100 - Q)Q \]
Exercise: Find elasticity

\[ y = 100 - 2x, \quad x = 4 \]
\[ \varepsilon = \frac{dy}{dx} \cdot \frac{x}{y} = -2 \cdot \frac{4}{92} = -\frac{8}{92} \]

\[ y = 16 - 8x + x^2, \quad x = 1.5 \]
\[ \varepsilon = (-8 + 2x) \cdot \frac{x}{y} = \frac{-2x}{4 - x} = -\frac{3}{2.5} \]

\[ \ln y = 6 - 0.5\ln x, \quad x = 3 \]
\[ \varepsilon = -0.5 \]

\[ y = 50x^{-2}, \quad x = 2 \]
\[ \varepsilon = -2 \cdot \frac{50x^{-3}x}{50x^{-2}} = -2 \]
Total, average, and marginal

- **Total product** \( Q = 100L^{1/2} \)
- **Average product** \( Q / L = 100L^{-1/2} \)
- **Marginal product** \( dQ / dL = 50L^{-1/2} \)

Average product > marginal product since concave
Total, average, and marginal

- **Total cost**
  \[ C = Q^3 - 15Q^2 + 93Q + 100 \]

- **Average cost**
  \[ \frac{C}{Q} \]

- **Marginal cost**
  \[ \frac{dC}{dQ} = 3Q^2 - 30Q + 93 \]
Average cost = marginal cost at min