# Basics of Differential Calculus

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Economics 300

# Why differential calculus?

- Models explain economic behavior with system of equations
- What happens if a variable changes?
  - Comparative statics determines marginal change in economic behavior
    - How does change in tax rate alter consumption?
    - How does change in NBA collective bargaining agreement impact
      - share of NBA revenues going to players?
      - parity of teams across league?

# Why differential calculus?

- Economic models assume rational optimizers
  - Consumers maximize utility
  - Producers maximize profits
  - NBA owners maximize combination of wins and profits
- Optimization uses calculus to evaluate tradeoffs
  - How much to consume?
    - Consume until marginal utility = price
  - How much to produce?
    - Produce until marginal revenue = marginal cost
  - Which free agents to go for?

# Average rate of change over $[x_0,x_1]$

$$\frac{\Delta y}{\Delta x} \equiv \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \text{ where}$$

$$\Delta x \equiv x_1 - x_0 \text{ and } \Delta y \equiv y_1 - y_0$$

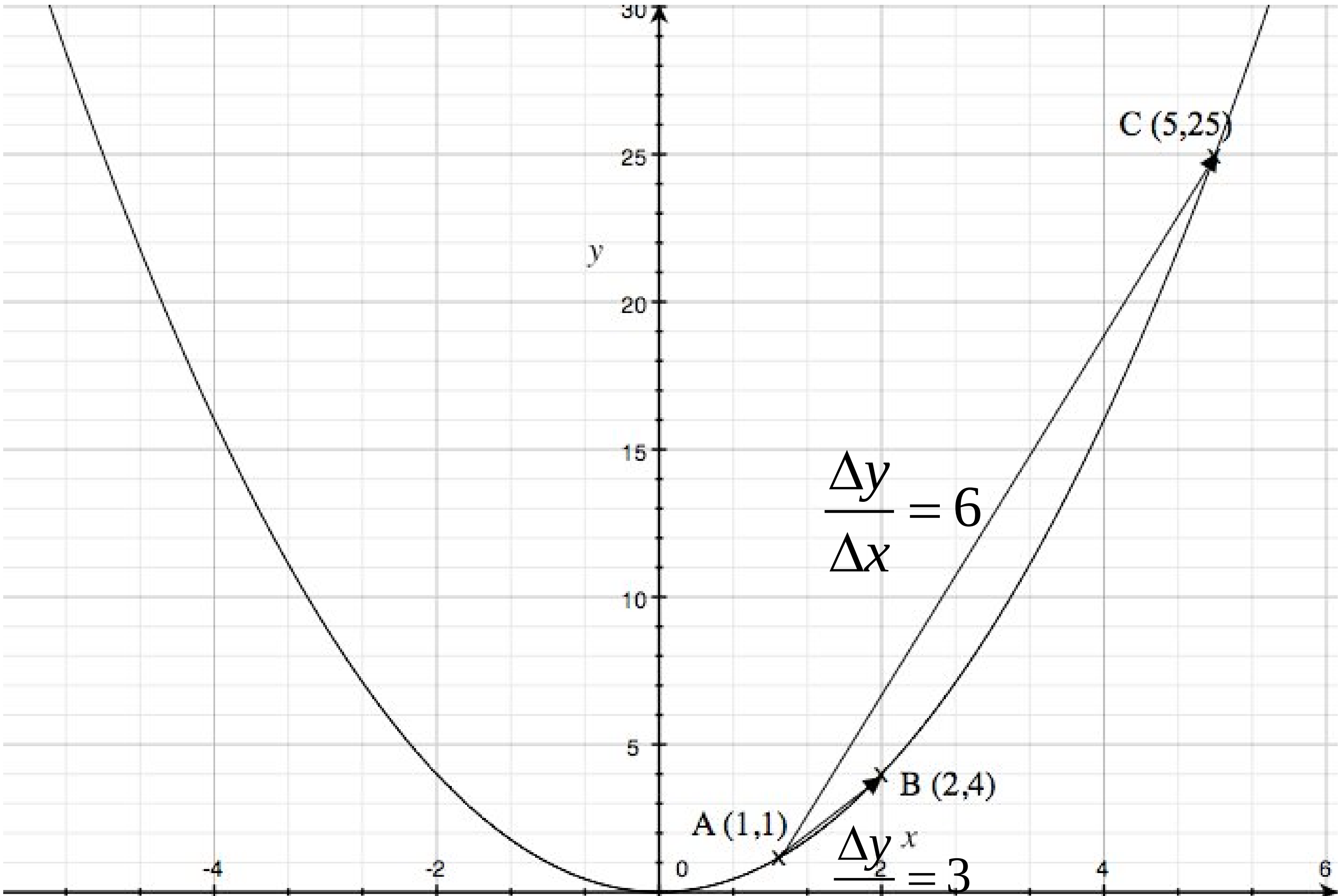
# Average rate of change examples

$$\frac{\Delta y}{\Delta x} \equiv \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$y = a + bx: \quad \frac{\Delta y}{\Delta x} = \frac{bx_1 - bx_0}{x_1 - x_0} = b$$

$$y = x^2: \quad \frac{\Delta y}{\Delta x} = \frac{x_1{}^2 - x_0{}^2}{x_1 - x_0} = \frac{(x_1 - x_0)(x_1 + x_0)}{x_1 - x_0} = x_1 + x_0$$

$$y = x^2$$




$$\frac{\Delta y}{\Delta x} = 6$$

$$\frac{\Delta y}{\Delta x} = 3$$

# Average rate of change and difference quotient

$$\frac{\Delta y}{\Delta x} \equiv \frac{f(x_1) - f(x)}{x_1 - x} \equiv \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$y = a + bx: \quad \frac{\Delta y}{\Delta x} = \frac{b(x + \Delta x) - bx}{\Delta x} = b$$

$$y = x^2: \quad \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

# Some properties

- Rate of change of sum = sum of rates of change
  - y, w, z are functions of x and y = w + z
  - Then $$\frac{\Delta y}{\Delta x} = \frac{\Delta(w+z)}{\Delta x} = \frac{\Delta w}{\Delta x} + \frac{\Delta z}{\Delta x}$$
- Scaling: $$\frac{\Delta(ay)}{\Delta x} = a\frac{\Delta y}{\Delta x}$$

# Application: quadratic

$$\frac{\Delta(x^2)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

$$y = ax^2 + bx + c$$

$$\frac{\Delta y}{\Delta x} = a\frac{\Delta(x^2)}{\Delta x} + b\frac{\Delta x}{\Delta x} + c\frac{\Delta 1}{\Delta x}$$

$$= a(2x + \Delta x) + b$$

# Application: cubic

$$\frac{\Delta(x^3)}{\Delta x} = \frac{(x+\Delta x)(x^2+2x\Delta x+\Delta x^2)-x^3}{\Delta x}$$

$$= \frac{(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3)-x^3}{\Delta x}$$

$$= 3x^2+3x\Delta x+\Delta x^2$$

$$y = gx^3+ax^2+bx+c$$

$$\frac{\Delta y}{\Delta x} = g\frac{\Delta(x^3)}{\Delta x}+a\frac{\Delta(x^2)}{\Delta x}+b\frac{\Delta x}{\Delta x}+c\frac{\Delta 1}{\Delta x}$$

$$= g(3x^2+3x\Delta x+\Delta x^2)+a(2x+\Delta x)+b$$

# Exercise

- Find difference quotient for each function

$y = 5x$ $$\frac{\Delta y}{\Delta x} = 5$$

$y = 30 - 15x$ $$\frac{\Delta y}{\Delta x} = -15$$

$y = 6x^2 + 2x + 9$ $$\frac{\Delta y}{\Delta x} = 6(2x + \Delta x) + 2$$

$y = 1 - x^2$ $$\frac{\Delta y}{\Delta x} = -(2x + \Delta x)$$

# Exercise

- Total revenue: TR = P Q
- Price: P = 10 – .5Q
- Difference quotient?

$$TR = (10 - .5Q)Q = -.5Q^2 + 10Q$$

$$\frac{\Delta TR}{\Delta Q} = -.5(2Q + \Delta Q) + 10$$

- If Q = 5, what is impact of 1 unit increase in Q?

$$\frac{\Delta TR}{\Delta Q} = -.5(2(5) + 1) + 10 = 4.5$$

# Derivative is difference quotient as Δx→0

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$y = a + bx: \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{b(x+\Delta x) - bx}{\Delta x} = b$$

$$y = x^2: \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

$$y = x^3: \frac{dy}{dx} = \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2$$

# Some properties

- Derivative of sum = sum of derivatives
  - y, w, z are functions of x and y = w + z
  - Then $$\frac{dy}{dx} = \frac{d(w+z)}{dx} = \frac{dw}{dx} + \frac{dz}{dx}$$
- Scaling: $$\frac{d(ay)}{dx} = a\frac{dy}{dx}$$
- Application $$y = gx^3 + ax^2 + bx + c$$

$$\frac{dy}{dx} = g\frac{d(x^3)}{dx} + a\frac{d(x^2)}{dx} + b\frac{dx}{dx} + c\frac{d1}{dx}$$

$$= 3gx^2 + 2ax + b$$

# Derivative is difference quotient as $\Delta x \to 0$

average rate of change ≡ difference quotient → derivative

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x)}{x_1 - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

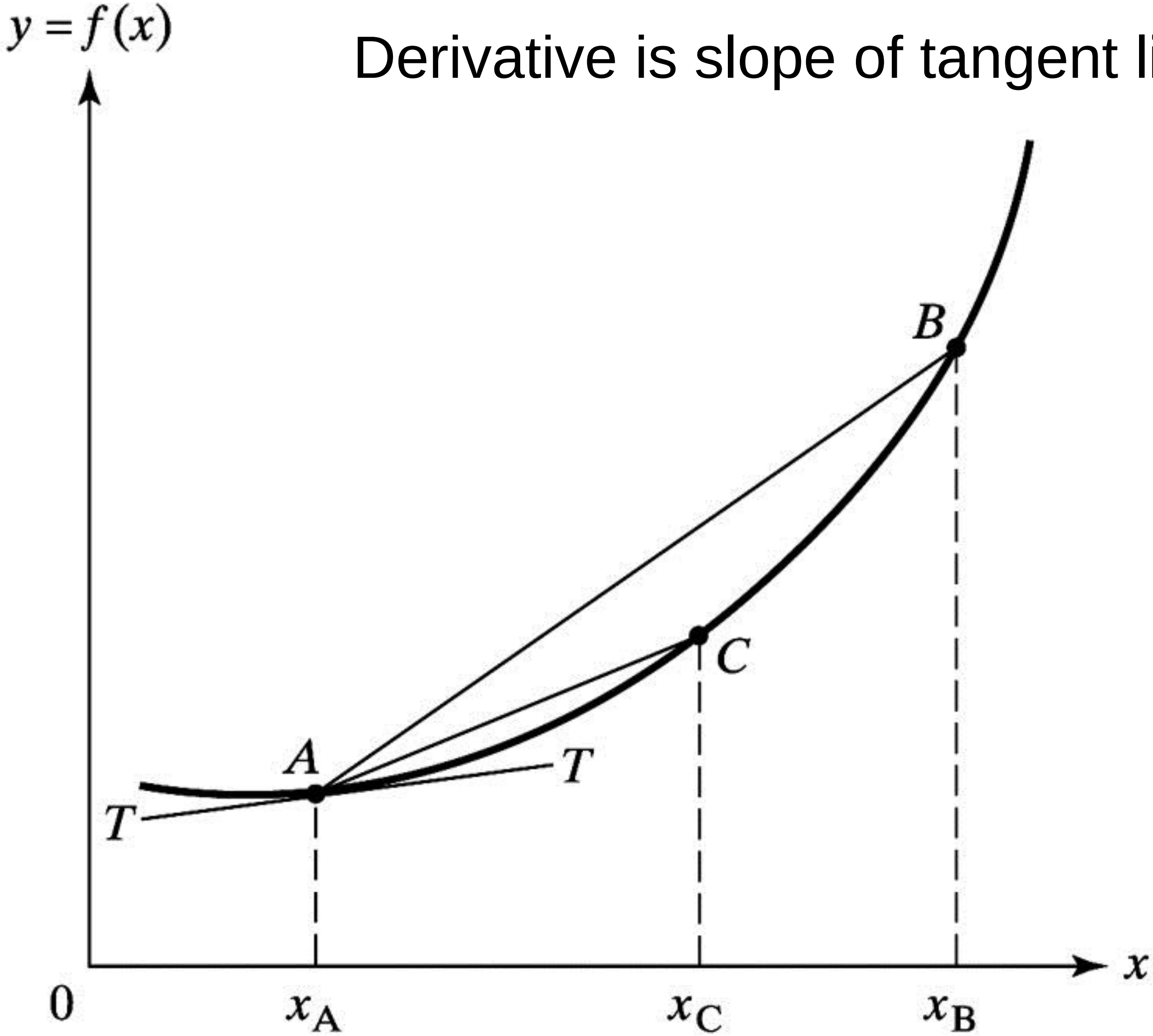
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Derivative is rate of change as $\Delta x \to 0$

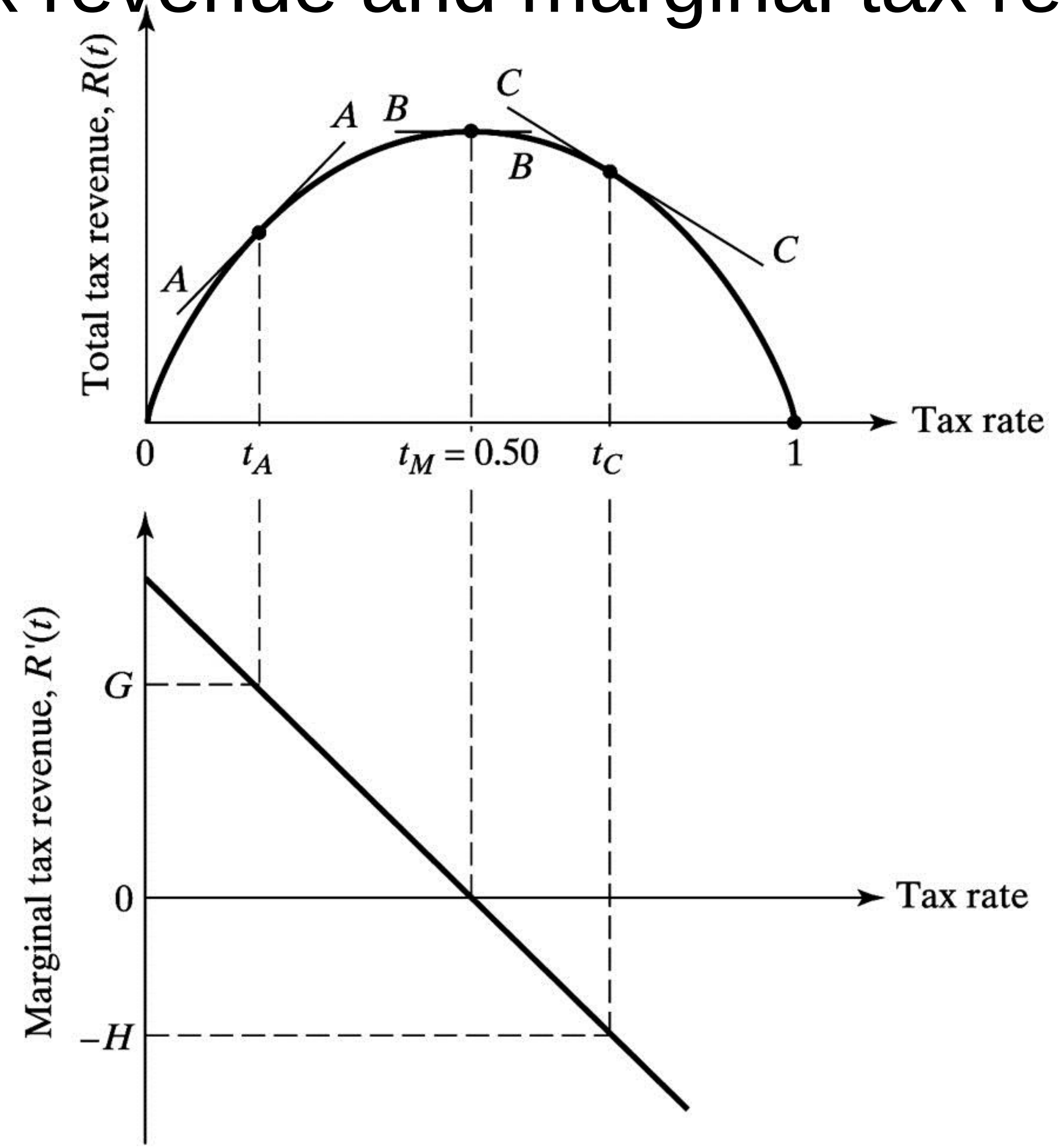
Derivative is instantaneous rate of change

# Tangent line is limit of secant line

Derivative is slope of tangent line



# Total tax revenue and marginal tax revenue



# Exercise

- Cigarette tax yields revenue R(t) = 50 + 25t – 75t$^2$
- What is marginal revenue?

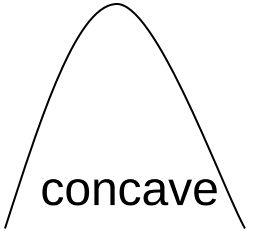
$$MR = \frac{dR}{dt} = 25 - 75(2t) = 25 - 150t$$

- What tax rate maximizes revenues?

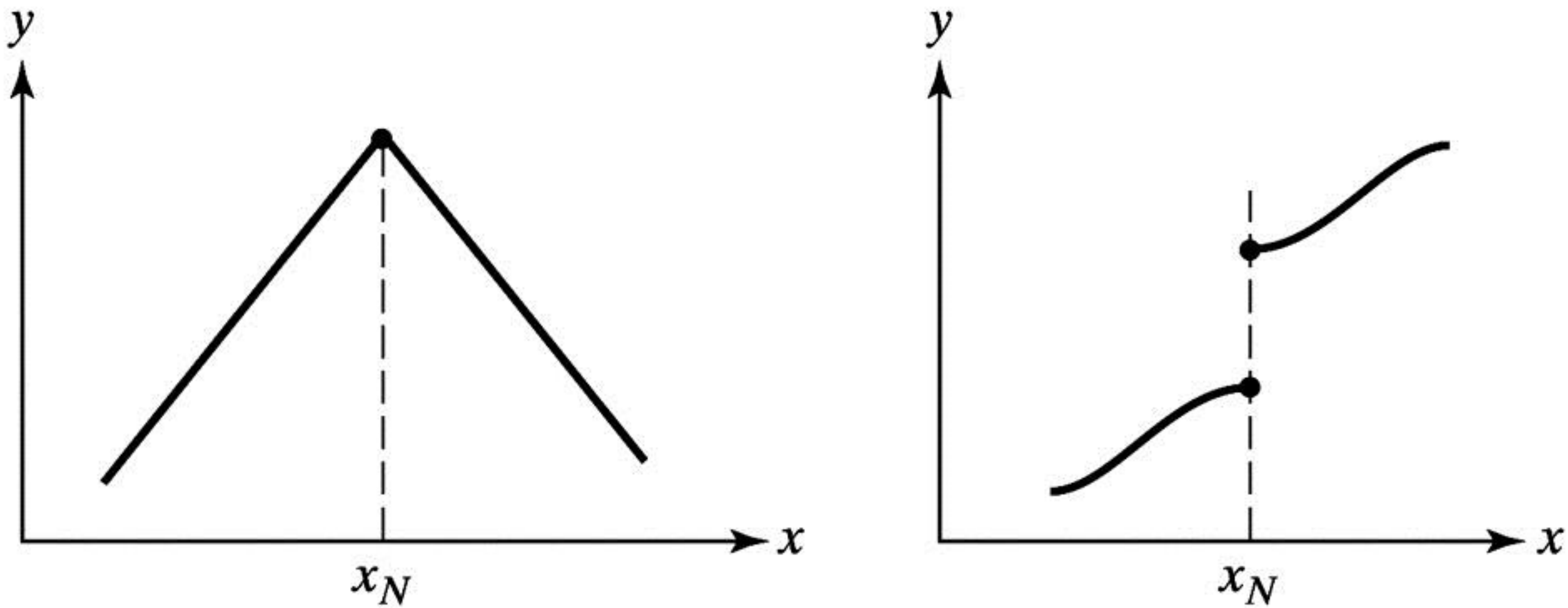
$$MR = 25 - 150t = 0$$

$$t^* = 25/150 = 1/6$$

- Why is this a maximum?



# Functions not everywhere differentiable




A Function That Is Not Smooth

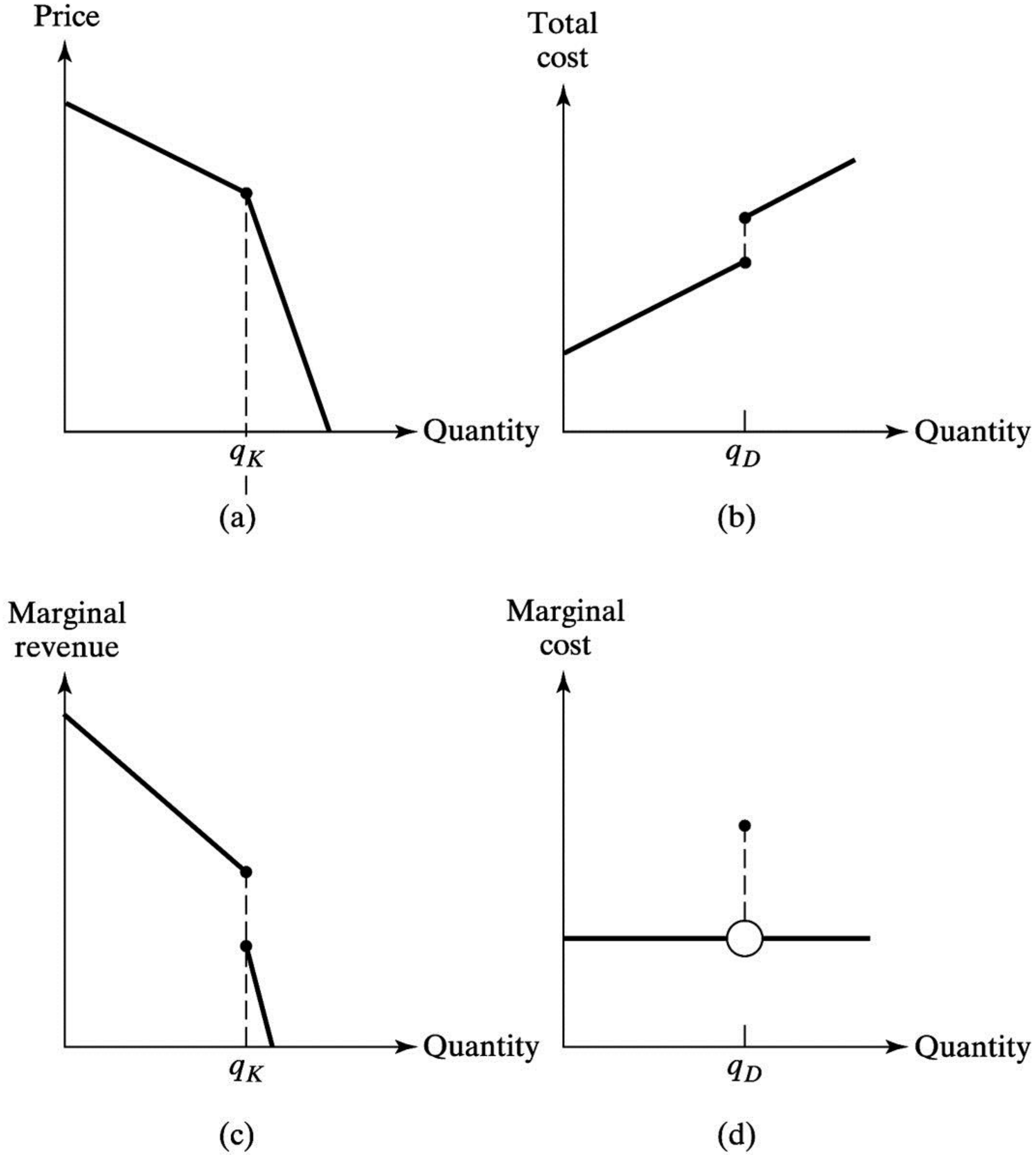
(a)

A Function That Is Not Continuous

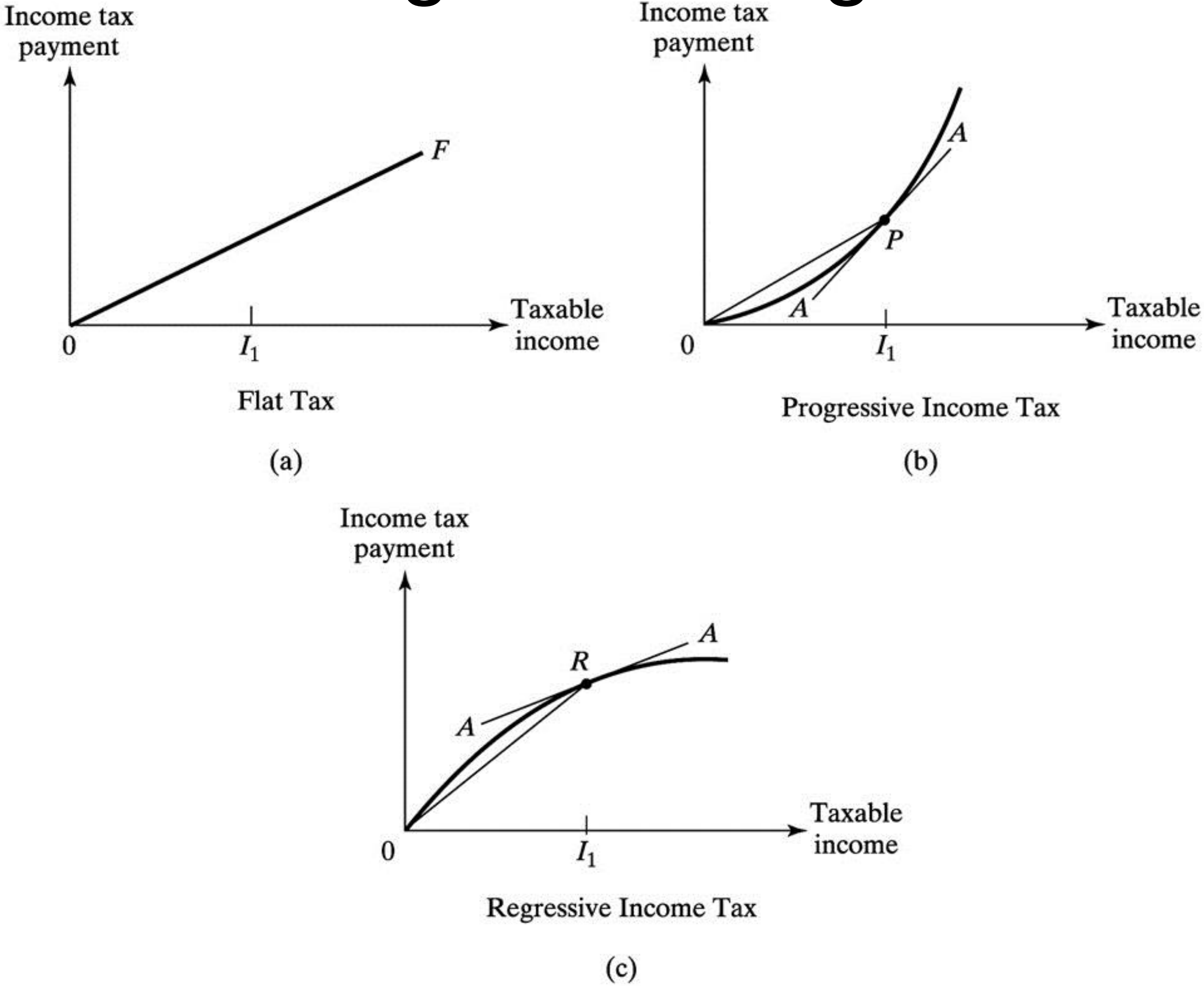
(b)

Differentiable $\Rightarrow$ Continuous

# Demand and cost functions



# Average vs. marginal




Flat Tax (a)

Progressive Income Tax (b)

Regressive Income Tax (c)

# Difference quotient of a polynomial

$$\frac{\Delta(1)}{\Delta x} = 0 \qquad \frac{\Delta(x)}{\Delta x} = 1$$

$$\frac{\Delta(x^2)}{\Delta x} = 2x + \Delta x$$

$$\frac{\Delta(x^3)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$$

$$\frac{\Delta(x^4)}{\Delta x} = 4x^3 + 6x^2\Delta x + 4x(\Delta x)^2 + (\Delta x)^3$$

# Exercise: y = $4x^2 - 8x + 3$

- Find roots (x such that y = 0).

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{8^2 - 4(4)(3)}}{2(4)}$$

$$= \frac{2 \pm \sqrt{4-3}}{2} = 1 \pm .5$$

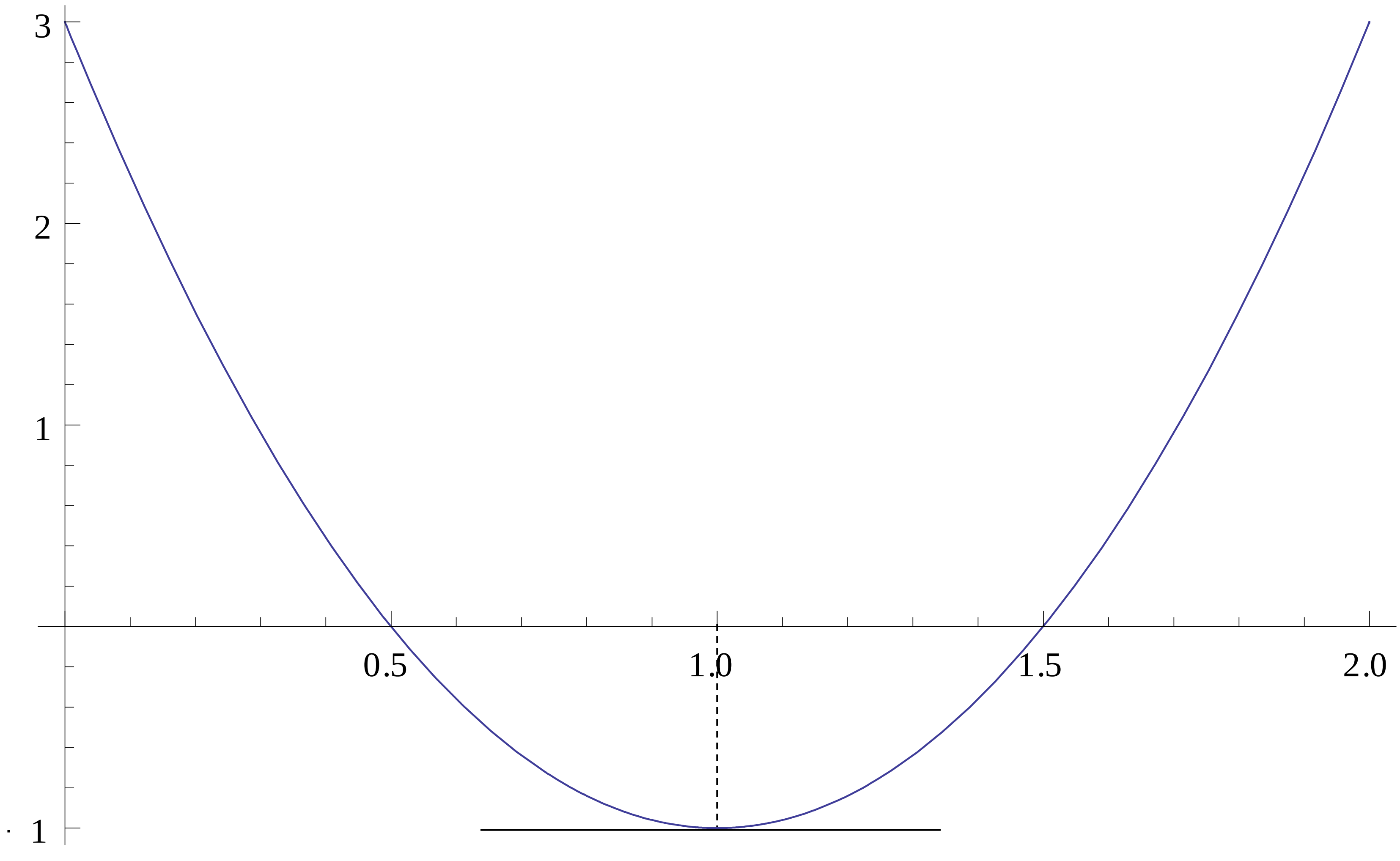
- Derivative

$$\frac{dy}{dx} = 8x - 8$$

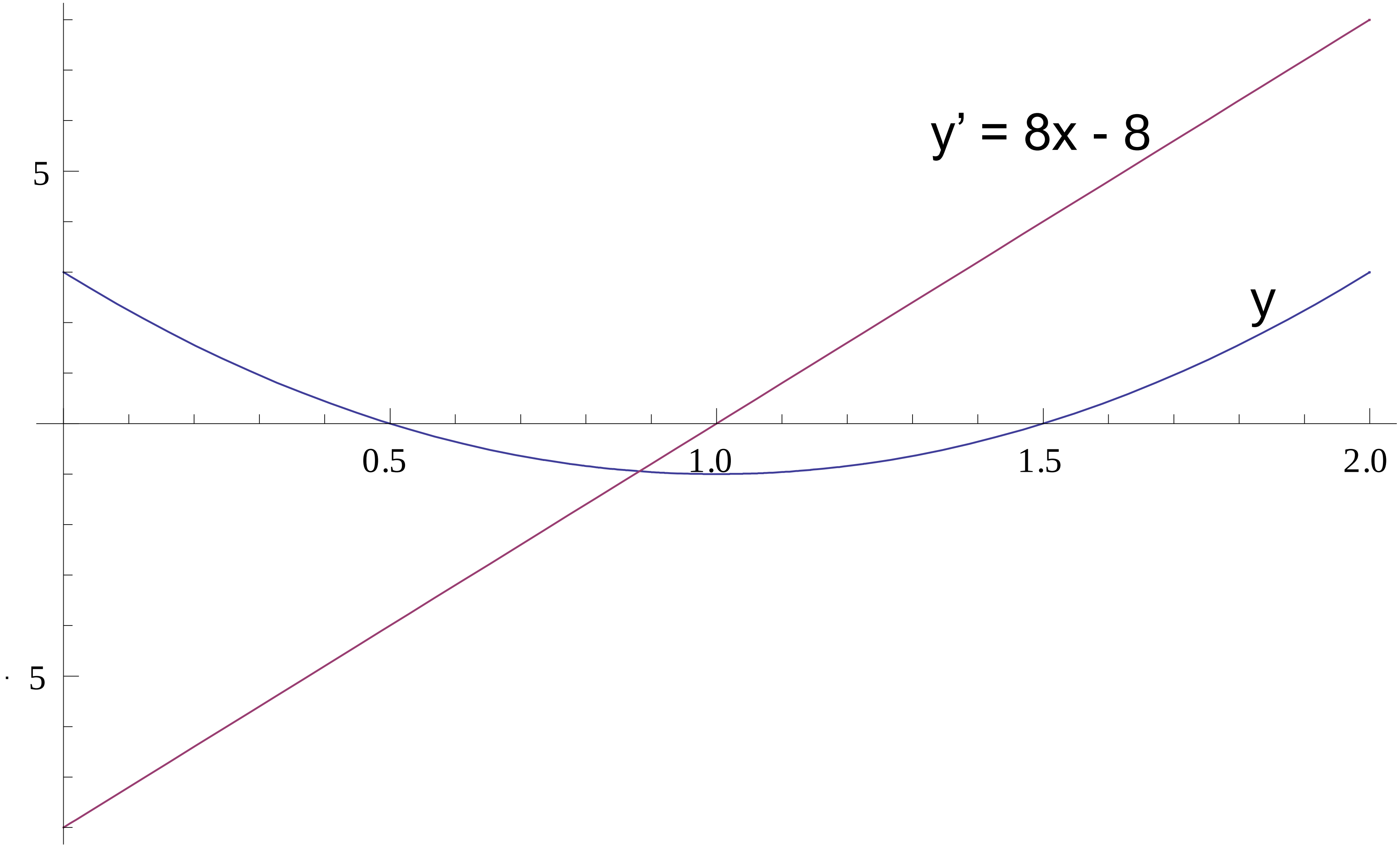
- Extreme value

$$\frac{dy}{dx} = 8x - 8 = 0 \Rightarrow x^* = 1$$

# Exercise: $y = 4x^2 - 8x + 3$



# Exercise: $y = 4x^2 - 8x + 3$



# Differential vs. derivative

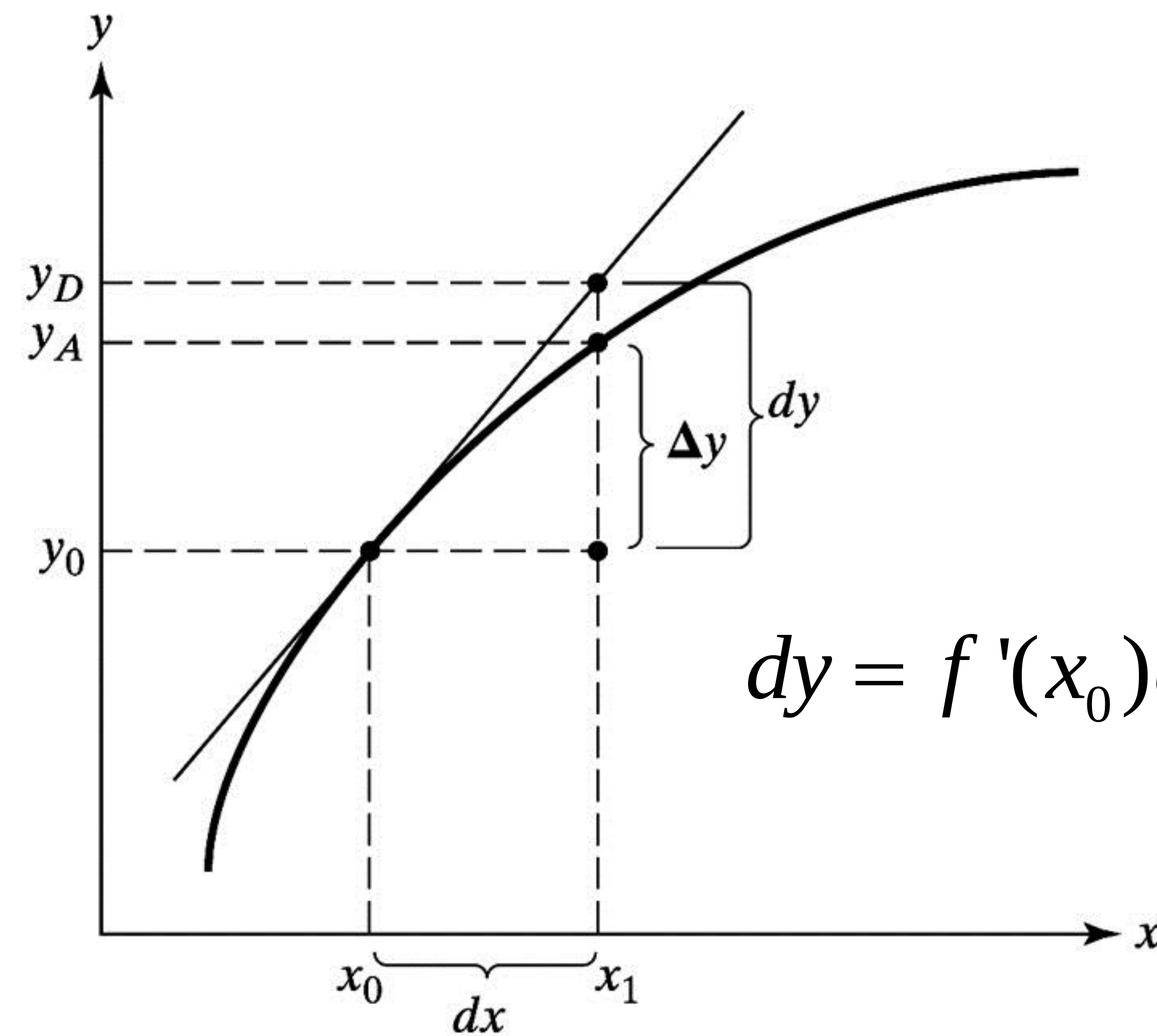
- Derivative is rate of change as $\Delta x \to 0$

$$\frac{\Delta y}{\Delta x} \to \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

- Differential is change in y along tangent line

$$dy = f'(x)dx$$

# Differential approximation and actual change




$$dy = f'(x_0)dx$$

# Exercise: $y = 16 - 4x + x^3$

- What is rate of change? $$\frac{\Delta y}{\Delta x} = -4 + 3x^2 + 3x\Delta x + \Delta x^2$$
- What is derivative? $$\frac{dy}{dx} = -4 + 3x^2$$
- What is differential? $$dy = (3x^2 - 4)dx$$
- Let $x_0 = 2$; $\Delta x = 8$ $$\Delta y \approx (3(2^2) - 4)(8) = 64$$ $$\Delta y = (3(2^2) - 4 + 3(2)(8) + 8^2)(8) = 960$$
- Let $x_0 = 2$; $\Delta x = .2$ $$\Delta y \approx (3(2^2) - 4)(.2) = 1.6$$ $$\Delta y = (3(2^2) - 4 + 3(2)(.2) + .2^2)(.2) = 1.848$$

# Exercise
# investment (I) and cost of borrowing (r)

$$I = f(r) = 600 - 150r + 400r^2$$

$$\text{Compute: } \Delta I = f'(r) \cdot \Delta r$$

$$r_0 = 2\%;\ \Delta r_1 = .5\%;\ \Delta r_2 = 1\%$$

$$\Delta I_1 \approx (800r_0 - 150)\Delta r_1 = (800(.02) - 150)(.005) = -.67$$

$$\Delta I_2 \approx (800r_0 - 150)\Delta r_2 = (800(.02) - 150)(.01) = -1.34$$

$$I_0 = 600 - 150(.02) + 400(.02^2) = 597.16$$

$$I_1 = 600 - 150(.025) + 400(.025^2) = 596.5$$

$$I_2 = 600 - 150(.03) + 400(.03^2) = 595.86$$

$$\Delta I_1 = 596.5 - 597.16 = -.66$$

$$\Delta I_2 = 595.86 - 597.16 = -1.3$$