Exponential and Logarithmic Functions

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Economics 300

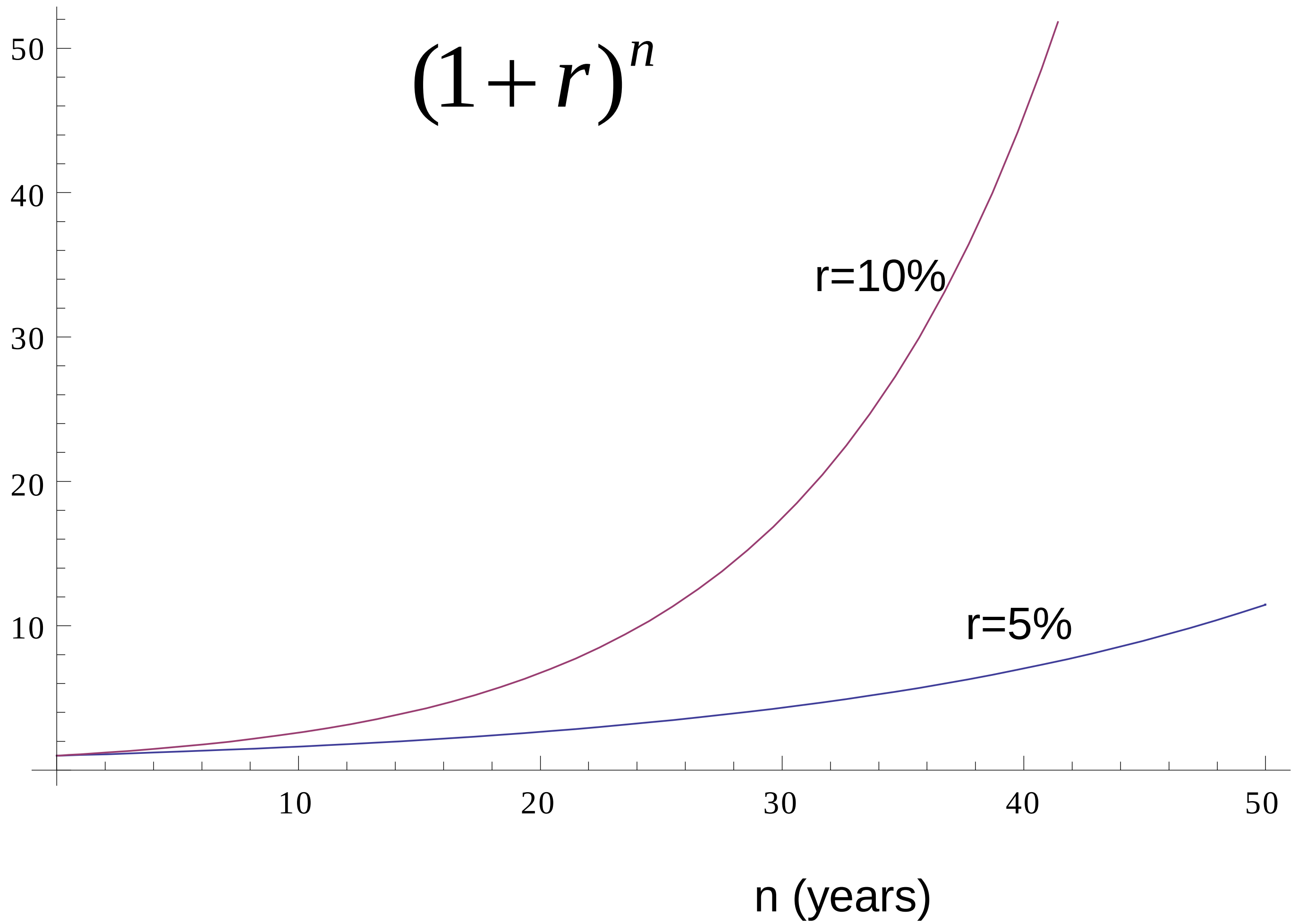
Modeling growth

- Exponential functions
  - Constant percentage growth per unit time
- Logarithmic functions
  - Inverse of exponential functions

Growth of money

- Interest rate r
- Value of X_t after 1 time period: $X_{t+1} = (1 + r)X_t$
  - r = 10%; $10 today is worth (1.1)10 = $11 next year
- Value of X_t after 2 time periods:
  $X_{t+2} = (1 + r)X_{t+1} = (1 + r)(1 + r)X_t = (1 + r)^2X_t$
- Value of X_t after n time periods
  $X_{t+n} = (1 + r)X_{t+n-1} = (1 + r)^nX_t$
- $1 earning 5% for 50 years = $11.47
- $1 earning 10% for 50 years = $117.39
  - Doubling interest rate has a huge impact

Exponential growth



More frequent compounding

- Once per year: $(1+r)$
- Twice per year: $(1+\frac{r}{2})^2$
- k times per year: $(1+\frac{r}{k})^k = (1+\frac{1}{k/r})^{\frac{k}{r}r} = (1+\frac{1}{m})^{mr}$
- ∞ times per year: $\left(\lim_{m\to\infty}(1+\frac{1}{m})^m\right)^r = e^r$

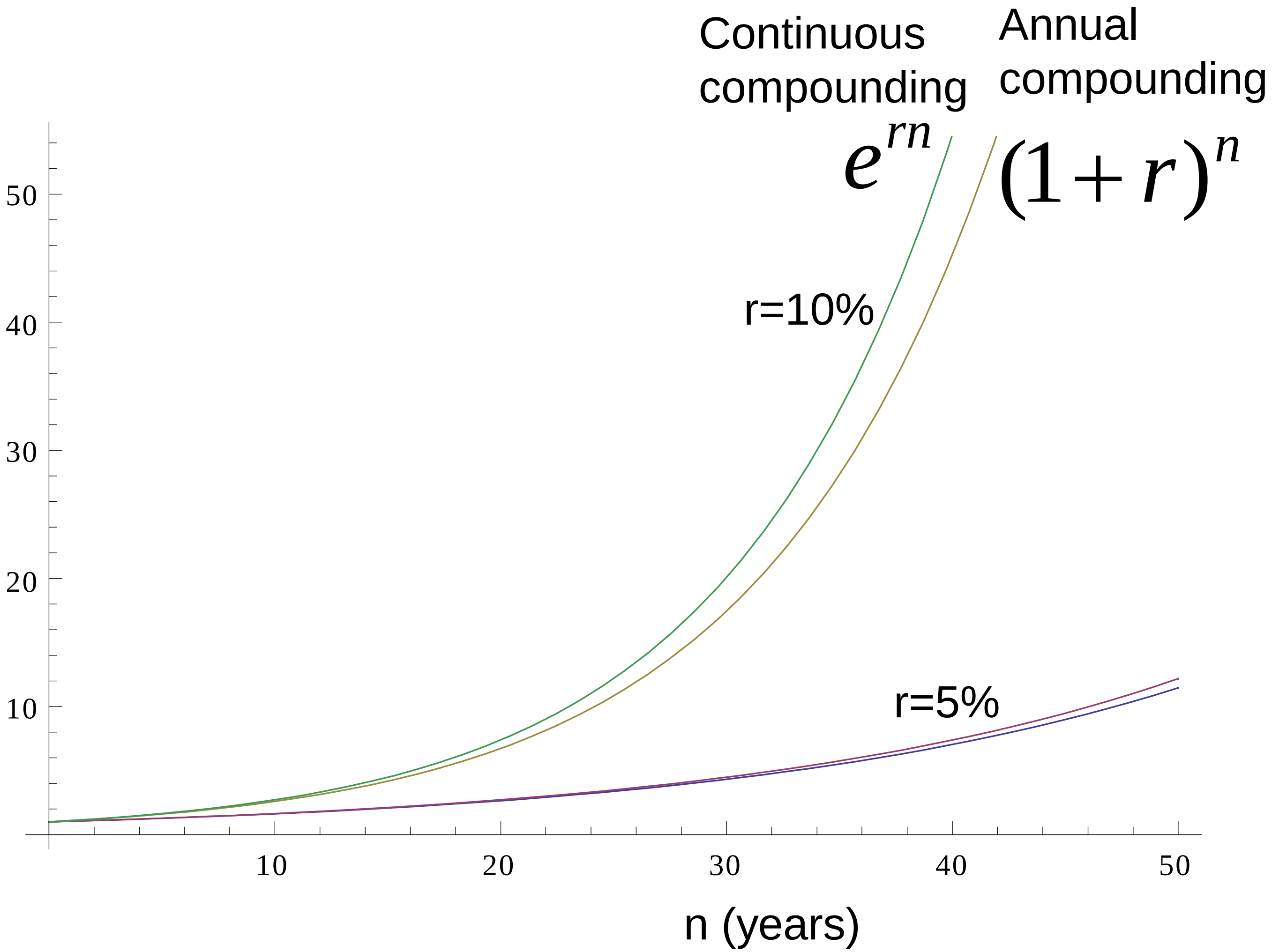
k	$(1+\frac{1}{k})^k$
1	2
10	2.593742
100	2.704814
10000	2.718146
100000000	2.718282

Most important constant in economics

$$e = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k =$$

2.71828182845904523536028747135266249775724709369995957496696762772407663035354759457138217852516642742746639193200305992181741359662904357290033429526059563073813232862794349076323382988075319525101901157383418793070215408914993488416750924476146066808226480016847741185374234544243710753907774499206955170276183860626133138458300075204493382656029760673711320070932870912744374704723069697720931014169283681902551510865746377211125238978442505695369677078544996996794686445490598793163688923009879312

Exponential growth



Effective rate vs. annual rate

- Annual rate of r_A = 10% with continuous compounding
- What is effective rate r_E over the year?

$$1 + r_E = e^{r_A}$$

$$r_E = e^{r_A} - 1$$

- r_A = 10% then r_E = 10.52% (continuous compounding)
- r_A = 10% then r_E = 10.50% (daily compounding)

$$r_E = (1 + \tfrac{1}{365})^{365 r_A} - 1$$

The Mating Game

A surprising application of e

[See mating-game.nb]

Present value (discrete)

- What is $1 next year worth today
- With r = 10%, than $1 today is worth $1.10 next year

$$X_{t+1} = (1+r)X_t$$

$$X_t = \frac{X_{t+1}}{1+r}$$

$$X_t = \frac{X_{t+n}}{(1+r)^n} = X_{t+n}(1+r)^{-n}$$

Present value (continuous)

- What is $1 next year worth today
- With r = 10%, than $1 today is worth $1.10 next year

$$X_{t+1} = e^{r} X_{t}$$

$$X_{t} = \frac{X_{t+1}}{e^{r}} = X_{t+1} e^{-r}$$

$$X_{t} = \frac{X_{t+n}}{e^{rn}} = X_{t+n} e^{-rn}$$

Net present value

- Investments generate costs and revenues over time
- What is the value today of the sequence of cash flows from an investment?

$$CF_t = \text{Revenue}_t - \text{Cost}_t$$

$$NPV = CF_0 + \frac{CF_1}{1+r} + \ldots + \frac{CF_n}{(1+r)^n} = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t}$$

$$NPV = \sum_{t=0}^{n} \delta^t CF_t, \quad \text{where} \quad \delta = \frac{1}{1+r} < 1$$

Examples

Discount rate $r > 0$

Discount factor $\delta = \frac{1}{1+r} < 1$

- Perpetuity: value of $1 each period forever

$$d_\infty = \frac{1}{1-\delta}$$

- Annuity: value of $1 each period for n periods

$$d_n = \frac{1-\delta^n}{1-\delta}$$

Example

$$\text{Discount rate } \; r > 0$$

$$\text{Discount factor } \; \delta = \frac{1}{1+r} < 1$$

- Perpetuity: value of $1 each period forever

$$d_{\infty} = 1 + \delta + \delta^2 + \ldots$$

$$\delta d_{\infty} = \delta + \delta^2 + \ldots \text{ , subtracting yields}$$

$$(1-\delta) d_{\infty} = 1$$

$$d_{\infty} = \frac{1}{1-\delta}$$

- r = 10%; δ = .909; perpetuity = $11.00

Example

$$\text{Discount rate } \; r > 0$$

$$\text{Discount factor } \; \delta = \frac{1}{1+r} < 1$$

- Annuity: value of $1 each period for n periods

$$d_n = 1 + \delta + \delta^2 + \ldots + \delta^{n-1}$$

$$\delta d_n = \delta + \delta^2 + \ldots + \delta^n \; \text{, subtracting yields}$$

$$(1-\delta) d_n = 1 - \delta^n$$

$$d_n = \frac{1-\delta^n}{1-\delta}$$

- r = 10%; δ = .909; 20-year annuity = $9.36

Logarithms

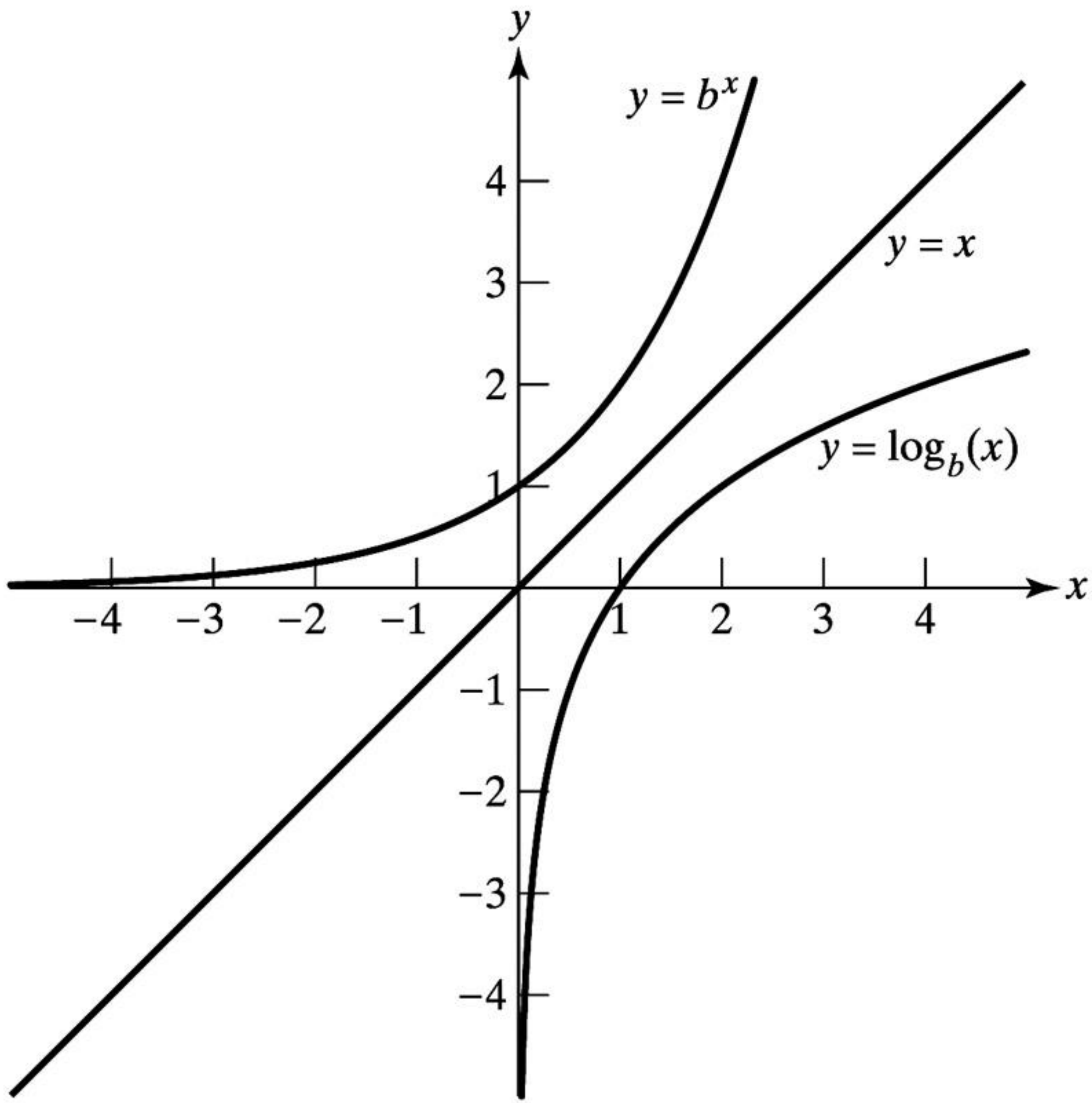
- Inverse of exponential function

$$y = \log_b(x) \text{ finds exponent y such that } b^y = x$$

- b is the base
- Most commonly b = 10 or b = *e*
- log base e is called natural logarithm: $y = \ln(x)$

$$y = \log(x)$$

Log is inverse of exponential function



Log base 10

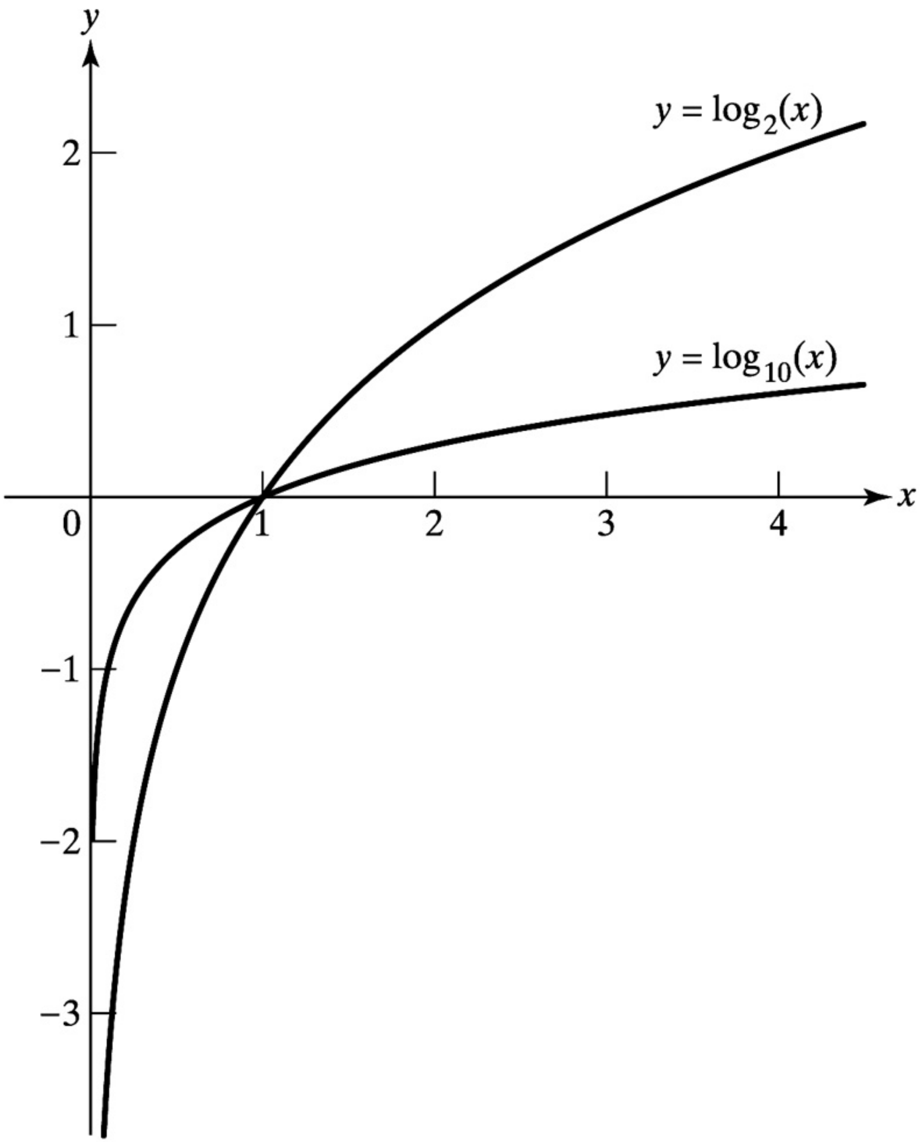
- Example of log base 10
- x 1 10 100 1000 10000
- y 0 1 2 3 4

$$x = 10^y$$

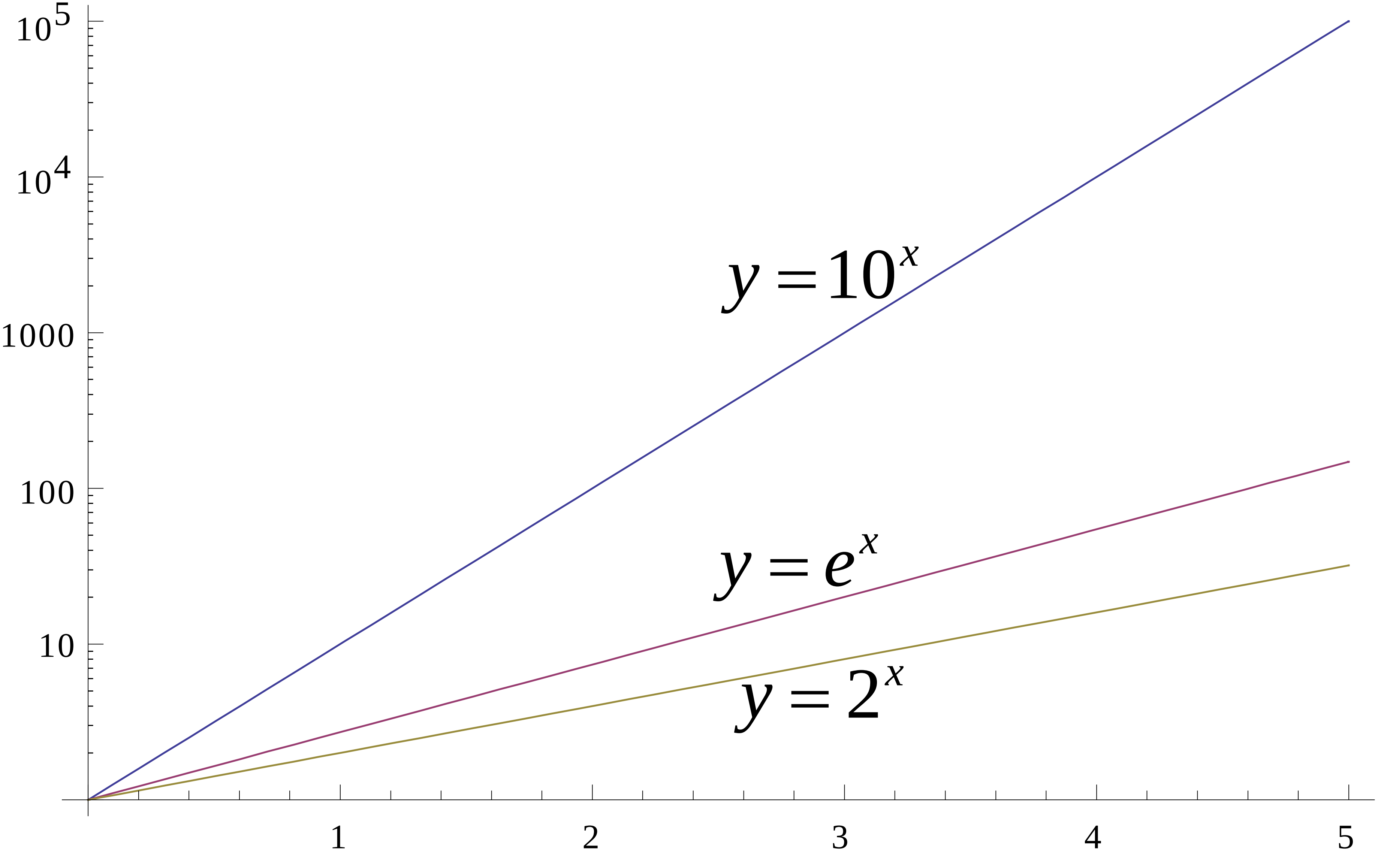
$$y = \log_{10}(x)$$

- Examples of log scales
  - Shock waves (Richter scale for earthquakes) (2011: Virginia 5.8, Japan 9.0; 1585 times larger)
  - Sound waves (decibels for sound)
  - Radio waves (Hz, kHz, MHz, GHz)

Log base 10 and base 2



Log plot of exponential growth



Properties of logarithmic functions

$$\log_b b = 1$$

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^y = y \log_b x$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Natural logs (base e)

- Continuous growth models
- Same properties hold

$$\ln e = 1$$

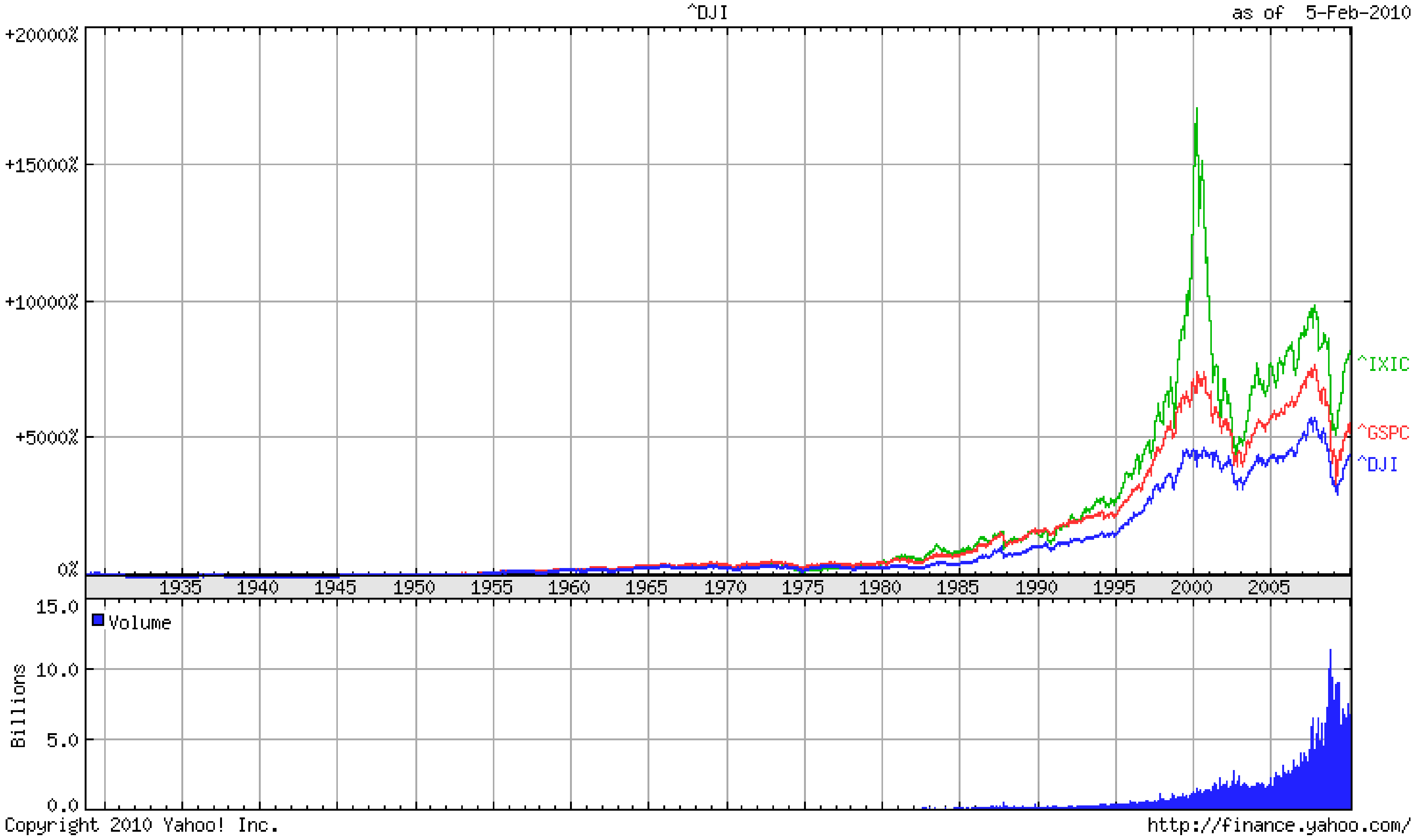
$$\ln xy = \ln x + \ln y$$

$$\ln \frac{x}{y} = \ln x - \ln y$$

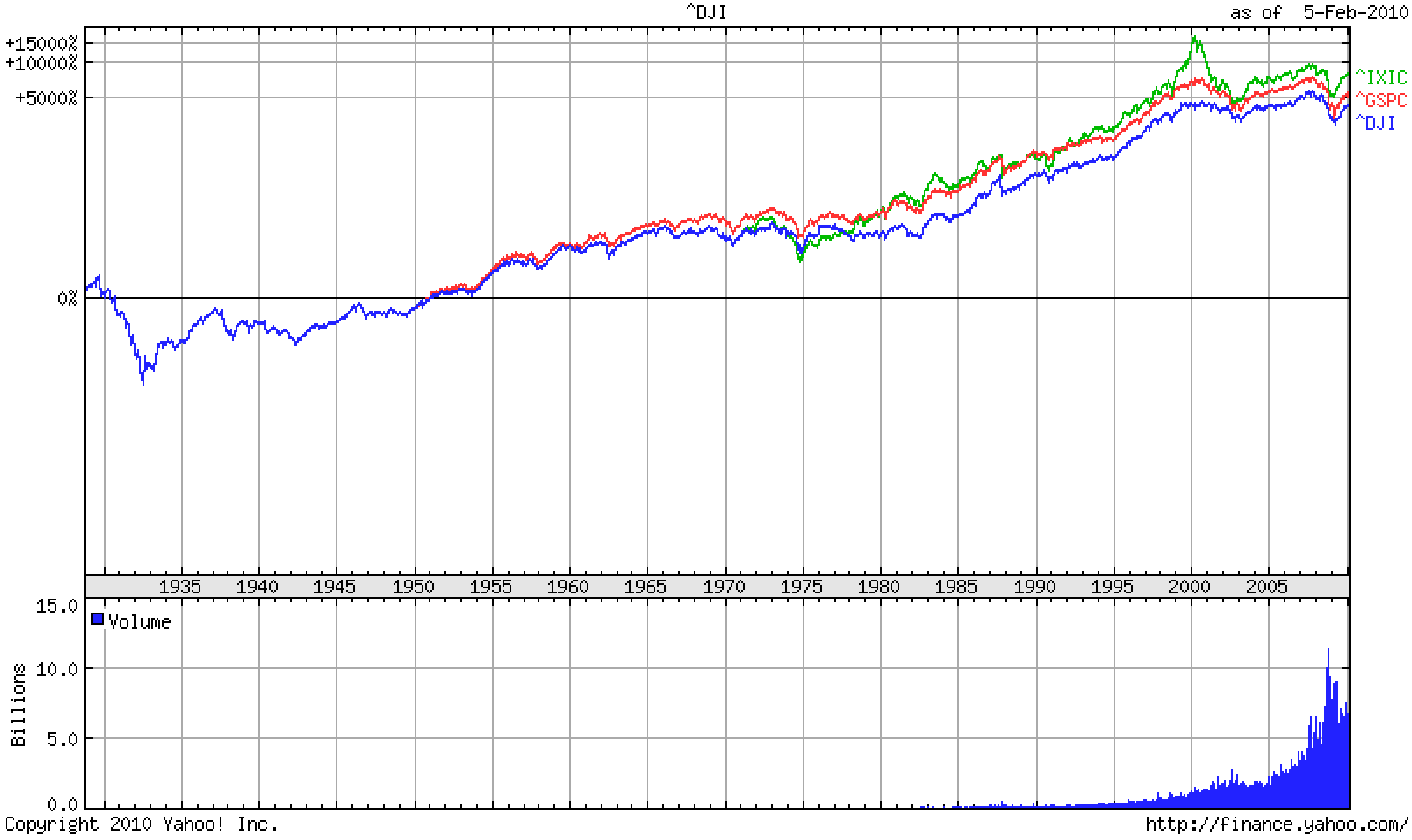
$$\ln x^y = y \ln x$$

- Example: Yahoo Finance (plotting stock history)

DJ vs. S&P vs. Nasdaq (linear scale)



DJ vs. S&P vs. Nasdaq (log scale)



Average return from stocks

- return r
- Dow Jones $742.12 in February 1978
- Dow Jones $12650.36 in February 2008

$$742.12e^{30r} = 12650.36$$

$$\ln(742.12) + 30r = \ln(12650.36)$$

$$r = \frac{\ln(12650.36) - \ln(742.12)}{30} = 9.45\%$$

Average return from stocks

- return r
- Dow Jones $742.12 in February 1978
- Dow Jones $9908.39 in February 2010

$$742.12e^{32r} = 9908.39$$

$$\ln(742.12) + 32r = \ln(9908.39)$$

$$r = \frac{\ln(9908.39) - \ln(742.12)}{32} = 8.10\%$$

Average return from stocks

- return r
- Dow Jones $742.12 in February 1978
- Dow Jones $15,801.79 in February 2014

$$742.12e^{36r} = 15801.79$$

$$\ln(742.12) + 36r = \ln(15801.79)$$

$$r = \frac{\ln(15801.79) - \ln(742.12)}{36} = 8.50\%$$

Average return from stocks

- return r, accounting for inflation
- Dow Jones $742.12 in Feb 1978; CPI 62.5
- Dow Jones $15,801.79 in Feb 2014; CPI 234.1

$$(742.12/62.5)e^{36r} = 15801.79/234.1$$

$$\ln(742.12/62.5) + 36r = \ln(15801.79/234.1)$$

$$r = \frac{\ln(15801.79/234.1) - \ln(742.12/62.5)}{36} = 4.83\%$$

Average growth rate

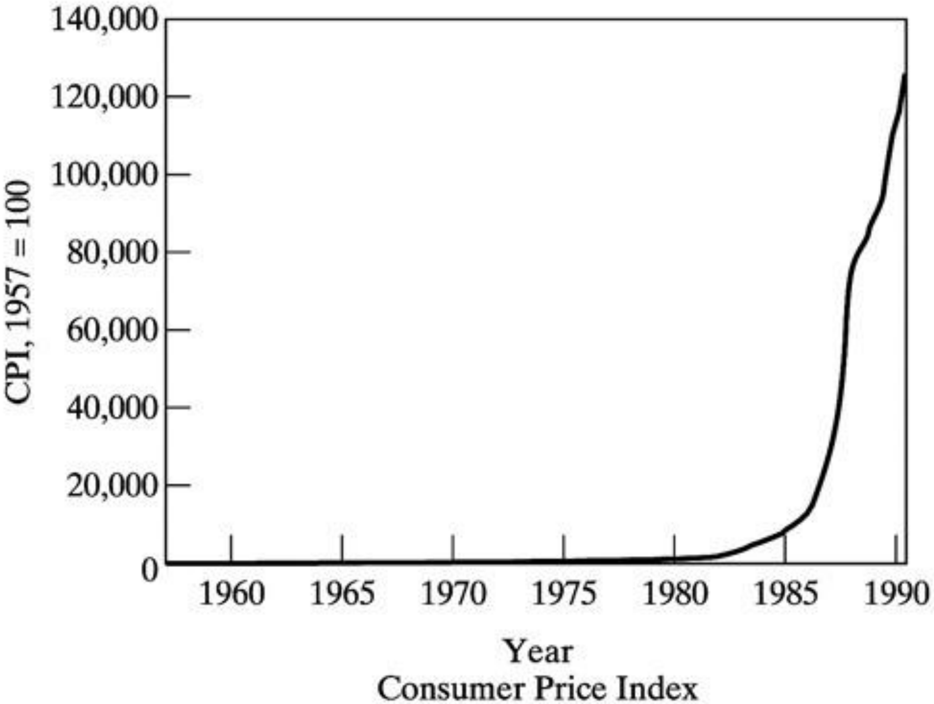
- Value at time 0: V_0
- Value at time T: V_T
- Assume constant percentage growth per unit time

$$V_0 e^{rT} = V_T$$

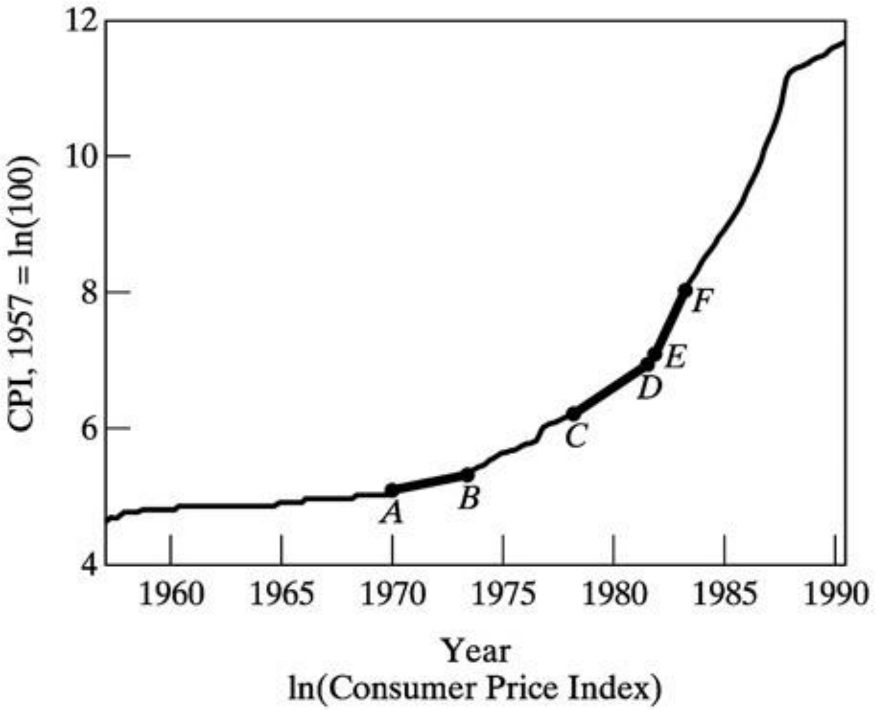
$$\ln(V_0) + rT = \ln(V_T)$$

$$r = \frac{\ln(V_T) - \ln(V_0)}{T}$$

Inflation in Mexico, 1957-1990



(a)



(b)

Cobb-Douglas production

- One special case: $y = 3x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$
- In general:

$$Q = AL^{\alpha} K^{\beta}$$

- Q = real GDP
- L = labor
- K = capital
- α and β are parameters

Cobb-Douglas production

- We measure Q, L, and K at each time:

$$Q_t = A_t L_t^{\alpha} K_t^{\beta}$$

- Taking logs: $\ln Q_t = \ln A_t + \alpha \ln L_t + \beta \ln K_t$

- Nice linear model!
- Can estimate parameters with econometrics
- Using subtraction

$$\ln Q_t - \ln Q_{t-1} = (\ln A_t - \ln A_{t-1}) + \alpha(\ln L_t - \ln L_{t-1}) + \beta(\ln K_t - \ln K_{t-1})$$

- What is $\ln Q_t - \ln Q_{t-1}$?

How long does it take for something to double?

$$V_0 e^{rn} = V_n$$

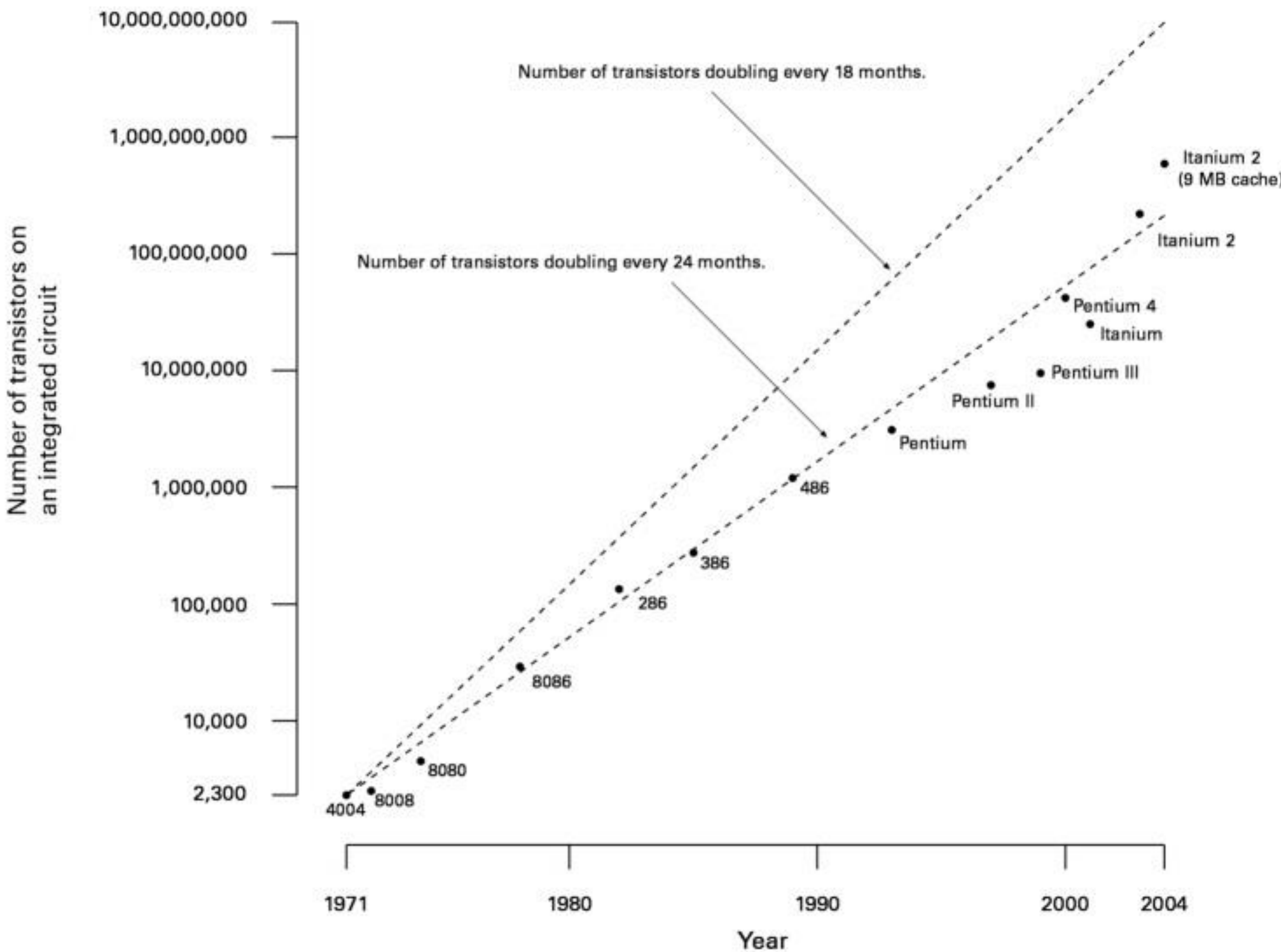
$$e^{rn} = V_n / V_0 = 2$$

$$\ln(e^{rn}) = \ln(2)$$

$$n = \frac{\ln(2)}{r} = \frac{.6931}{r}$$

- With r = 10% it takes 7 years for value to double
- With r = 5% it takes 14 years for value to double
- Moore’s Law of electronics: a doubling every 18 months
  - r = .6931/1.5 = 46%

Moore's Law


Number of transistors on an integrated circuit
10,000,000,000
1,000,000,000
100,000,000
10,000,000
1,000,000
100,000
10,000
2,300
Number of transistors doubling every 18 months.
Number of transistors doubling every 24 months.
Itanium 2
(9 MB cache)
Itanium 2
Pentium 4
Itanium
Pentium III
Pentium II
Pentium
486
386
286
8086
8080
8008
4004
1971
1980
1990
2000
2004
Year

Properties of logarithmic functions

$$\log_b b = 1$$

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^y = y \log_b x$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Simplify

$$10^{\log_{10}(100)} = 100$$

$$\ln e^{x} - e^{\ln x} = x - x = 0$$

$$\log_{10}\frac{1}{x^{5}} = \log_{10} x^{-5} = -5\log_{10} x$$

$$\ln\left(\frac{1}{e^{5}}\left[x^{\alpha} y^{-\beta}\right]^{2}\right) = -5 + 2(\alpha\ln x - \beta\ln y)$$

Problem

- \$10,000 invested at 5% with continuous compounding
- When do you have \$15,000?
- Use formula for future value: $X_{t+n} = X_t e^{rn}$
- $15{,}000 = 10{,}000\ e^{.05n}$
- Solve for n

  $e^{.05n} = 15{,}000/10{,}000 = 1.5$

  $.05\ n = \ln(1.5) \Rightarrow n = \ln(1.5)/.05 = 8.1093$